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DIFFERENTIAL EQUATIONS CONTAINING 2-VARIABLE MIXED-TYPE HERMITE POLYNOMIALS

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Abstract. In this paper, we introduce the 2-variable mixed-type Hermite polynomials and organize some new symmetric identities for these polynomials. We find induced differential equations to give explicit identities of these polynomials from the generating functions of 2-variable mixed-type Hermite polynomials.

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1. Introduction

The ordinary Hermite numbers H_n and Hermite polynomials $H_n(v)$ are usually defined by the generating functions

$$e^{t(2v-t)} = \sum_{n=0}^{\infty} H_n(v)\frac{t^n}{n!}$$

and

$$e^{-t^2} = \sum_{n=0}^{\infty} H_n\frac{t^n}{n!}.$$

Clearly, $H_n = H_n(0)$.

It can be seen that these numbers and polynomials play an important role in various fields of mathematics, applied mathematics, and physics, including number theory, combinations, special functions, and differential equations. Various interesting properties about them are obtained, see [1]-[5]. The ordinary Hermite polynomials $H_n(u)$ satisfy the Hermite differential equation

$$\frac{d^2H(v)}{dv^2} - 2v\frac{dH(v)}{dv} + 2nH(v) = 0, n = 0, 1, 2, \ldots.$$

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We recall that the 2-variable Hermite polynomials $H_n(v,w)$ defined by the generating function(see [2])

$$\sum_{n=0}^{\infty} H_n(v,w)\frac{t^n}{n!} = e^{t(v+wt)} \tag{1}$$

are the solution of heat equation

$$\frac{\partial}{\partial w}H_n(v,w) = \frac{\partial^2}{\partial v^2}H_n(v,w), \quad H_n(v,0) = v^n. \tag{2}$$

Observe that

$$H_n(2v,-1) = H_n(v).$$

Motivated by their importance and potential for applications in certain problems in probability, combinatorics, number theory, differential equations, numerical analysis and other areas of mathematics and physics, several kinds of some special numbers and polynomials were recently studied by many authors, see [1, 2, 3, 4, 5, 6, 7, 8]. Many mathematicians have studied the area of the degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Genocchi polynomials, and degenerate tangent polynomials, see [6, 7, 8, 10]. Mathematicians are recently interested in studying Hermit polynomials through various methods such as Hermit polynomials combined with q-numbers or (p,q)-numbers, degenerated Hermit polynomials, and so on. Furthermore, studies to define a new type of polynomial in which polynomials are combined with polynomials and finding properties and structures of their roots began appearing. A new type of research that combines Hermit polynomials with degenerated Hermit polynomials is an example. Mixed-type Hermit polynomials, which differ from the results of studies of [12], will be introduced. In this paper, we can see the properties related to the mixed-type Hermit polynomials and mixed-type Hermit polynomials is a solution of differential equation of certain initial value problems.

In [13], Hwang and Ryoo proposed the 2-variable degenerate Hermite polynomials $\mathcal{H}_n(v,w,\lambda)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathcal{H}_n(v,w,\lambda)\frac{t^n}{n!} = (1+\lambda)^{\dfrac{vt+wt^2}{\lambda}}. \tag{3}$$

Since $(1+\lambda)^{\frac{t}{\lambda}} \to e^t$ as $\lambda \to 0$, it is evident that (3) reduces to (1). The 2-variable degenerate Hermite polynomials $\mathcal{H}_n(v,w,\lambda)$ in generating function (3) are the solution of equation

$$\begin{aligned} &\left(\frac{\log(1+\lambda)}{\lambda}\right)\frac{\partial}{\partial v}\mathcal{H}_n(v,w,\lambda) = \frac{\partial^2}{\partial v^2}\mathcal{H}_n(v,w,\lambda), \\ &\mathcal{H}_n(v,0,\lambda) = \left(\frac{\lambda}{\log(1+\lambda)}\right)^n v^n. \end{aligned} \tag{4}$$

Since $\dfrac{\lambda}{\log(1+\lambda)} \to 1$ as λ approaches to 0, it is apparent that (4) descends to (2).

The differential equations arising from the generating functions of special numbers and polynomials were also studied, see [10-16]. Hermit polynomials are well known as they play a very important role in mathematical and physical engineering fields. When $\lambda \to 1$, the mixed-type Hermit polynomials become Hermit polynomials. Therefore, the study of mixed-type Hermit polynomials is one of the important studies to understand the characteristics of Hermit polynomials.

There are several important goals of this paper. We construct 2-variable mixed-type Hermit polynomials and check their symmetric properties. We also find the symmetrical properties of these polynomials related to tangent numbers. We introduce some new differential equations from the generating functions of 2-variable mixed-type Hermit polynomials.

This paper is organized as follows. In Section 2, we construct the 2-variable mixed-type Hermite polynomials and obtain basic properties of these polynomials. We derive some symmetric identities for 2-variable mixed-type Hermite polynomials. In Section 3, we introduce the differential equations generated from the generating function of 2-variable mixed-type Hermite polynomials. Using the coefficients of this differential equation, we have explicit identities for the 2-variable mixed-type Hermite polynomials.

2. Basic properties for the 2-variable mixed-type Hermite polynomials

In this section, a new class of the 2-variable mixed-type Hermite polynomials are considered. Furthermore, some properties of these polynomials are also obtained.

We define the 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathbf{H}_n(u,v,\lambda)\frac{t^n}{n!} = e^{ut}(1+\lambda t^2)^{\dfrac{v}{\lambda}}. \tag{5}$$

Since $(1+\lambda t^2)^{\frac{v}{\lambda}} \to e^{vt^2}$ as $\lambda \to 0$, it is evident that (5) reduces to (1). Observe that degenerate Hermite polynomials $\mathcal{H}_n(u,v,\lambda)$ and 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ are completely different.

Now, we recall the λ-analogue of the falling factorial sequences as follows:

$$(w|\lambda)_0 = 1, (w|\lambda)_n = w(w-\lambda)(w-2\lambda)\cdots(w-(n-1)\lambda), (n \geq 1).$$

Note that

$$\lim_{\lambda\to 1}(w|\lambda)_n = w(w-1)(w-2)\cdots(w-(n-1)) = (w)_n, (n \geq 1).$$

For a variable w, we get

$$(1+\lambda t^2)^{\frac{w}{\lambda}} = \sum_{m=0}^{\infty} \left(\frac{w}{\lambda}\right)_m \frac{t^{2m}}{m!} = \sum_{m=0}^{\infty} (w|\lambda)_m \frac{t^{2m}}{m!}. \tag{6}$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [6-13])

$$(w)_n = \sum_{k=0}^{n} S_1(n,k) w^k \text{ and } w^n = \sum_{k=0}^{n} S_2(n,k)(w)_k,$$

respectively. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{7}$$

The generating function (5) is useful for deriving several properties of the 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$.

For example, we have the following expression for these polynomials:

Theorem 2.1. *For any positive integer n, we have*

$$\mathbf{H}_n(u,v,\lambda) = \sum_{k=0}^{[\frac{n}{2}]} u^{n-2k}(v|\lambda)_k \frac{n!}{k!(n-2k)!},$$

where $[\quad]$ denotes the integer part.

Proof. By (6) and (7), we have

$$\sum_{n=0}^{\infty} \mathbf{H}_n(u,v,\lambda)\frac{t^n}{n!} = e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}} = \sum_{k=0}^{\infty} u^k \frac{t^k}{k!} \sum_{l=0}^{\infty} (v|\lambda)_l \frac{t^{2l}}{l!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n}{2}]} u^{n-2k}(v|\lambda)_k \frac{n!}{k!(n-2k)!} \right) \frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$, the expected result of Theorem 2.1 is achieved. □

Since $\lim_{\lambda\to 0}(v|\lambda)_n = v^n, (n \geq 1)$, we get

$$H_n(u,v) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{v^k u^{n-2k}}{k!(n-2k)!}.$$

The following basic properties of the 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ are derived form (5). We, therefore, choose to omit the details involved.

Theorem 2.2. *For any positive integer n, we have*

$$(1)\quad \mathbf{H}_n(u,v,\lambda)=\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{k}S_1(k,l)\lambda^{k-l}v^l u^{n-2k}\frac{n!}{k!(n-2k)!}.$$

$$(2)\quad \mathbf{H}_n(u_1+u_2,v,\lambda)=\sum_{l=0}^{n}\binom{n}{l}u_2^l\mathbf{H}_{n-l}(u_1,v,\lambda).$$

$$(3)\quad \mathbf{H}_n(u,v_1+v_2,\lambda)=\sum_{k=0}^{[\frac{n}{2}]}\mathbf{H}_k(u,v_1,\lambda)(v_2|\lambda)_{n-2k}\frac{n!}{k!(n-2k)!}.$$

$$(4)\quad \mathbf{H}_n(u,v_1+v_2,\lambda)=\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{k}\mathbf{H}_{n-2k}(u,v_1,\lambda)S_1(k,l)v_2^l\lambda^{k-l}.$$

$$(5)\quad \mathbf{H}_n(u_1+u_2,v_1+v_2,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l(u_1,v_1,\lambda)\mathbf{H}_{n-l}(u_2,v_2,\lambda).$$

Theorem 2.3. *For $n=0,1,...,$ 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ with the generating function (5) are the solution of the differential equation*

$$\begin{aligned}&\left(u\lambda\frac{\partial^3}{\partial u^3}-((n-2)\lambda-2v)\frac{\partial^2}{\partial u^2}+u\frac{\partial}{\partial u}-n\right)\mathbf{H}_n(u,v,\lambda)=0,\\ &\mathbf{H}_n(u,0,\lambda)=u^n.\\ &\mathbf{H}_n(0,v,\lambda)=\begin{cases}(v|\lambda)_k\dfrac{(2k)!}{k!}, & \text{if } n=2k,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. We see that

$$\mathfrak{G}(t,u,v|\lambda)=e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}}$$

satisfies

$$\frac{\partial\mathfrak{G}(t,u,v|\lambda)}{\partial t}-\left(u+\frac{2vt}{1+\lambda t^2}\right)\mathfrak{G}(t,u,v|\lambda)=0.$$

By substituting the series (5) for $\mathfrak{G}(t,u,v|\lambda)$, one obtains

$$\begin{aligned}&\mathbf{H}_{n+1}(u,v,\lambda)-u\mathbf{H}_n(x,y,z|\mu)+(n(n-1)\lambda-2vn)\mathbf{H}_{n-1}(u,v,\lambda)\\ &\quad-n(n-1)u\lambda\mathbf{H}_{n-2}(u,v,\lambda)=0, n=2,3,\ldots\end{aligned}\tag{8}$$

We get a recurrence relation for 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ and another recurrence relation which comes from

$$\frac{\partial\mathfrak{G}(t,u,v|\lambda)}{\partial u}-t\mathfrak{G}(t,u,v|\lambda)=0.$$

This implies

$$\begin{aligned}&\frac{\partial \mathbf{H}_n(u,v,\lambda)}{\partial u}-n\mathbf{H}_{n-1}(u,v,\lambda)=0, n=1,2,\ldots,\\&\frac{\partial^2 \mathbf{H}_n(u,v,\lambda)}{\partial u^2}-n(n-1)\mathbf{H}_{n-2}(u,v,\lambda)=0, n=2,3,\ldots.\end{aligned}\tag{9}$$

Eliminate $\mathbf{H}_{n-1}(u,v,\lambda)$ and $\mathbf{H}_{n-2}(u,v\lambda)$ from (8)–(9) in order to obtain

$$\mathbf{H}_{n+1}(u,v,\lambda)-u\mathbf{H}_n(u,v,\lambda)+((n-1)\lambda-2v)\frac{\partial \mathbf{H}_n(u,v\lambda)}{\partial u}-u\lambda\frac{\partial^2 \mathbf{H}_n(u,v,\lambda)}{\partial u^2}=0.\tag{10}$$

Differentiate this equation and use (9) again to get

$$\begin{aligned}&u\lambda\frac{\partial^3 \mathbf{H}_n(u,v,\lambda)}{\partial u^3}-((n-2)-2v)\frac{\partial^2 \mathbf{H}_n(u,v,\lambda)}{\partial u^2}+u\mathbf{H}_n(u,v,\lambda)\\&\quad-n\mathbf{H}_nu,v,\lambda)=0,(n=0,1,\ldots),\end{aligned}$$

thus, we proved the theorem. □

Theorem 2.4. *For $n=0,1,...,$ 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ with the generating function (5) are the solution of the differential equation*

$$\begin{aligned}&\frac{\partial^2}{\partial u\partial v}\mathbf{H}_n(u,v,\lambda)-\sum_{l=0}^{[\frac{n-2}{2}]}\frac{(-1)^l\lambda^l}{(l+1)}\frac{\partial}{\partial v}\mathbf{H}_{n-2l-2}(u,v,\lambda)\frac{n!}{(n-2l-2)!}=0,\\&\frac{\partial \mathbf{H}_n(u,v,\lambda)}{\partial v}-\sum_{k=0}^{[\frac{n-2}{2}]}\frac{(-1)^k\lambda^k n!}{(k+1)(n-2k-2)!}\mathbf{H}_{n-2k-2}(u,v,\lambda)=0,\\&\mathbf{H}_n(u,0,\lambda)=u^n.\\&\mathbf{H}_n(0,v,\lambda)=\begin{cases}(v|\lambda)_k\dfrac{(2k)!}{k!}, & \text{if } n=2k,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. Since

$$\frac{\partial \mathfrak{G}(t,u,v|\lambda)}{\partial v}-\frac{\log(1+\lambda t^2)}{\lambda}\mathfrak{G}(t,u,v|\lambda)=0,$$

we get

$$\frac{\partial \mathbf{H}_n(u,v,\lambda)}{\partial v}=\sum_{k=0}^{[\frac{n-2}{2}]}\frac{(-1)^k\lambda^k n!}{(k+1)(n-2k-2)!}\mathbf{H}_{n-2k-2}(u,v,\lambda).$$

Note that

$$\mathfrak{G}(t,u,v|\lambda)=e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}}$$

satisfies

$$\frac{\partial^2 \mathfrak{G}(t,u,v|\lambda)}{\partial u \partial v} - \frac{\log(1+\lambda t^2)}{\lambda} \frac{\partial \mathfrak{G}(t,u,v|\lambda)}{\partial u} = 0. \tag{11}$$

Substitute the series in (11) for $\mathfrak{G}(t,u,v|\lambda)$ to obtain

$$\frac{\partial^2}{\partial u \partial v} \mathbf{H}_n(u,v,\lambda) - \sum_{k=0}^{[\frac{n-2}{2}]} \frac{(-1)^k \lambda^k n!}{(k+1)(n-2k-2)!} \frac{\partial}{\partial v} \mathbf{H}_{n-2k-2}(u,v,\lambda) = 0.$$

Thus the 2-variable mixed-type degenerate Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$ in generating function (5) are the solution of equation

$$\frac{\partial^2}{\partial u \partial v} \mathbf{H}_n(u,v,\lambda) = \sum_{l=0}^{[\frac{n-2}{2}]} \frac{(-1)^l \lambda^l}{(l+1)} \frac{\partial}{\partial v} \mathbf{H}_{n-2l-2}(u,v,\lambda) \frac{n!}{(n-2l-2)!}.$$

One can obtain the desired results immediately. □

3. Symmetric identities for the 2-variable mixed-type Hermite polynomials

In this section, we share some new symmetric identities for the 2-variable mixed-type Hermite polynomials. We also introduce several explicit formulas and properties for the 2-variable mixed-type Hermite polynomials.

Theorem 3.1. *Let $w_1, w_2 > 0$ and $w_1 \neq w_2$. The following identity holds true:*

$$w_1^m \mathbf{H}_m\left(w_2 u, w_2^2 v, \frac{\lambda}{w_1^2}\right) = w_2^m \mathbf{H}_m\left(w_1 u, w_1^2 v, \frac{\lambda}{w_2^2}\right).$$

Proof. Let $w_1, w_2 > 0$ and $w_1 \neq w_2$. We start with

$$G(t,\lambda) = e^{w_1 w_2 u t}(1+\lambda t^2)^{\frac{w_1^2 w_2^2 v}{\lambda}}.$$

Then the expression for $G(t,\lambda)$ is symmetric in w_1 and w_2

$$G(t,\lambda) = \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_1 u, w_1^2 v, \frac{\lambda}{w_2^2}\right) \frac{(w_2 t)^n}{n!} = \sum_{n=0}^{\infty} w_2^n \mathbf{H}_n\left(w_1 u, w_1^2 v, \frac{\lambda}{w_2^2}\right) \frac{t^n}{n!}.$$

On the similar lines, we obtain that

$$G(t,\lambda) = \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_2 u, w_2^2 v, \frac{\lambda}{w_1^2}\right) \frac{(w_1 t)^n}{n!} = \sum_{n=0}^{\infty} w_1^n \mathbf{H}_n\left(w_2 u, w_2^2 v, \frac{\lambda}{w_1^2}\right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in last two equations, the expected result of Theorem 3.1 is achieved. □

For each integer $k \geq 0$ and $n \geq 1$, the alternating even sum is defined by

$$\tau_k(n) = \sum_{i=0}^{n} (-1)^i (2i)^k$$

and the generating function is

$$\sum_{k=0}^{\infty}\tau_k(n)\frac{t^k}{k!}=\frac{1-(-e^{2t})^{n+1}}{1+e^{2t}}.$$

Ryoo has introduced the tangent polynomials given by the generating function as

$$\sum_{n=0}^{\infty}T_n(u)\frac{t^n}{n!}=\frac{2}{e^{2t}+1}e^{ut}.$$

When $u=0$ and $T_n=T_n(0)$, T_n are called the tangent numbers (see [8, 9]).

Theorem 3.2. *Let $w_1, w_2>0$ and $w_1\neq w_2$. If w_1 and w_2 have the same parity, then we have*

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_1^i w_2^{n-i}T_m\mathbf{H}_{i-m}\left(w_2u, w_2^2v, \frac{\lambda}{w_1^2}\right)\tau_{n-i}(w_1-1)\\ &=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_2^i w_1^{n-i}T_m\mathbf{H}_{i-m}\left(w_1u, w_1^2v, \frac{\lambda}{w_2^2}\right)\tau_{n-i}(w_2-1).\end{aligned}$$

Proof. For integers $n\geq 0, w_1\geq 1$ and $w_2\geq 1$, we now use

$$F(t,\mu)=\frac{2e^{w_1w_2ut}(1+\lambda t^2)^{\frac{w_1^2w_2^2v}{\lambda}}\left(1-(-e^{2t})^{w_1w_2}\right)}{\left(e^{2w_1t}+1\right)\left(e^{2w_2t}+1\right)}.$$

From $F(t,\mu)$, we get the following result:

$$\begin{aligned}&F(t,\mu)\\ &=\frac{2e^{w_1w_2ut}(1+\lambda t^2)^{\frac{w_1^2w_2^2v}{\lambda}}\left(1-(-e^{2t})^{w_1w_2}\right)}{\left(e^{2w_1t}+1\right)\left(e^{2w_2t}+1\right)}\\ &=\frac{2}{\left(e^{2w_1t}+1\right)}e^{w_1w_2ut}(1+\lambda t^2)^{\frac{w_1^2w_2^2v}{\lambda}}\frac{\left(1-(-e^{2t})^{w_1w_2}\right)}{\left(e^{2w_2t}-1\right)}\\ &=\sum_{n=0}^{\infty}T_n\frac{(w_1t)^n}{n!}\sum_{n=0}^{\infty}\mathbf{H}_n\left(w_2u, w_2^2v, \frac{\lambda}{w_1^2}\right)\frac{(w_1t)^n}{n!}\sum_{n=0}^{\infty}\tau_n(w_1-1)\frac{(w_2t)^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_1^i w_2^{n-i}T_m\mathbf{H}_{i-m}\left(w_2u, w_2^2v, \frac{\lambda}{w_1^2}\right)\tau_{n-i}(w_1-1)\right)\frac{t^n}{n!}.\end{aligned}$$

In a similar fashion we have

$$
\begin{aligned}
&F(t,\lambda)\\
&= \frac{2}{(e^{2w_2t}+1)} e^{w_1w_2ut}(1+\lambda t^2)^{\frac{w_1^2w_2^2v}{\lambda}} \frac{\left(1-(-e^{2t})^{w_1w_2}\right)}{(e^{2w_1t}+1)}\\
&= \sum_{n=0}^{\infty} T_n \frac{(w_2t)^n}{n!} \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_1u, w_1^2v, \frac{\lambda}{w_2^2}\right) \frac{(w_2t)^n}{n!} \sum_{n=0}^{\infty} \tau_n\,(w_2-1)\,\frac{(w_1t)^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m} w_2^i w_1^{n-i} T_m \mathbf{H}_{i-m}\left(w_1u, w_1^2v, \frac{\lambda}{w_2^2}\right)\tau_{n-i}(w_2-1)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations gives us the desired identity. $\square$

Corollary 3.3. *Let $w_1, w_2 > 0$ and $w_1 \neq w_2$. Then, the following identity holds true:*

$$
\begin{aligned}
&\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m} w_1^i w_2^{n+1-i} T_m H_{i-m}(w_2u, w_2^2v)\tau_{n-i}(w_1-1)\\
&= \sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m} w_2^i w_1^{n+1-i} T_m H_{i-m}(w_1u, w_1^2v)\tau_{n-i}(w_2-1).
\end{aligned}
$$

Proof. Taking the limit as $\lambda \to 0$ gives the desired result. $\square$

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References

1. L.C. Andrews, *Special Functions for Engineers and Mathematicians*, Macmillan. Co., New York, USA, 1985.
2. P. Appell, J. Hermitt Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynomes d Hermite*, Gauthier-Villars, Paris, France, 1926.
3. A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Krieger, New York, USA, 1981.
4. G.E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, UK, 1999.
5. G. Arfken, *Mathematical Methods for Physicists*, 3rd ed., Academic Press, Orlando, FL, USA, 1985.
6. G. Dattoli, *Generalized polynomials operational identities and their applications*, J. Comput. Appl. Math. **118** (2000), 111-123.
7. Zhiliang Deng, Xiaomei Yang, *Convergence of spectral likelihood approximation based on q-Hermite polynomials for Bayesian inverse problems*, Proc. Amer. Math. Soc. **150** (2022), 4699-4713.

8. L. Carlitz, *Degenerate Stiling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51-88.
9. P.T. Young, *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, J. Number Theorey **128** (2008), 738-758.
10. M. Cenkci, F.T. Howard, *Notes on degenerate numbers*, Discrete Math. **307** (2007), 2395-2375.
11. C. Cesarano, W. Ramirez, S. Khan, *A new class of degenerate Apostol-type Hermite polynomials and applications*, Dolomites Research Notes on Approximation **15** (2022), 1-10.
12. C.S. Ryoo, R.P. Agarwal, J.Y. Kang, *Some properties involving 2-variable modified partially degenerate Hermite polynomials derived from differential equations and distribution of their zeros*, Dynamic Systems and Applications **29** (2020), 248-269.
13. K.W. Hwang, C.S. Ryoo, *Some identities involving two-variable degenerate Hermite polynomials induced from differential equations and structure of their roots*, Mathematics **8** (2020). doi:10.3390/math8040632
14. T. Kim, D.S. Kim, H.I. Kwon, C.S. Ryoo, *Differential equations associated with Mahler and Sheffer-Mahler polynomials*, Nonlinear Funct. Anal. Appl. **24** (2019), 453-462.
15. C.S. Ryoo, *A numerical investigation on the structure of the zeros of the degenerate Euler-tangent mixed-type polynomials*, J. Nonlinear Sci. Appl. **10** (2017), 4474-4484.
16. C.S. Ryoo, *Differential equations associated with tangent numbers*, J. Appl. Math. Inform. **34** (2016), 487-494.
17. C.S. Ryoo, *Some identities involving Hermitt Kampé de Fériet polynomials arising from differential equations and location of their zeros*, Mathematics **7** (2019). doi:10.3390/math7010023
18. V.I. Bogachev, *Chebyshev–Hermite Polynomials and Distributions of Polynomials in Gaussian Random Variables*, Theory of Probability & Its Applications **66** (2022). https://doi.org/10.1137/S0040585X97T990617

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