J. Appl. Math. & Informatics Vol. **41***(2023), No. 3, pp. 657 - 685*
https://doi.org/10.14317/jami.2023.657

PYTHAGOREAN FUZZY SOFT SETS OVER UP-ALGEBRAS[†]

AKARACHAI SATIRAD, RUKCHART PRASERTPONG, PONGPUN JULATHA,
RONNASON CHINRAM AND AIYARED IAMPAN[*]

Abstract. This paper aims to apply the concept of Pythagorean fuzzy soft sets (PFSSs) to UP-algebras. Then we introduce five types of PFSSs over UP-algebras, study their generalization, and provide illustrative examples. In addition, we study the results of four operations of two PFSSs over UP-algebras, namely, the union, the restricted union, the intersection, and the extended intersection. Finally, we will also discuss t-level subsets of PFSSs over UP-algebras to study the relationships between PFSSs and special subsets of UP-algebras.

AMS Mathematics Subject Classification : 03G25, 03E72, 08A72.
Key words and phrases : UP-algebra, pythagorean fuzzy set, pythagorean fuzzy soft set.

1. Introduction and Preliminaries

The concept of fuzzy sets (FSs) was first considered by Zadeh [42] in 1965. Zadeh's and others' FS concepts have found numerous applications in mathematics and other fields. Following the introduction of the concept of FSs, various researchers were interviewed about generalizations of the concept of FSs, including: Atanassov [3] defined a new concept called an intuitionistic fuzzy set (IFS) which is a generalization of a FS, Torra and Narukawa [38, 37] introduced the notion of hesitant fuzzy sets (HFS). Yager [40] introduced a new class of non-standard fuzzy subsets called a Pythagorean fuzzy set (PFS) and the related idea of Pythagorean membership grades.

In 1999, to solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. Uncertainties cannot be handled using

---

Received November 13, 2022. Revised December 19, 2022. Accepted December 20, 2022.
[*] Corresponding author.
[†] This work was supported by the revenue budget in 2022, School of Science, University of Phayao, Thailand (Grant No. PBTSC65020).

traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [17]. In 2001, Maji et al. [16] introduced the concept of fuzzy soft sets as a generalization of the standard soft sets, and presented an application of fuzzy soft sets in a decision making problem. In 2013, Rehman et al. [22] studied properties of fuzzy soft sets and their interrelation with respect to different operations such as union, intersection, restricted union and extended intersection. Then, they illustrate properties of AND and OR operations by giving counter examples. In 2015, Peng et al. [20] introduced the concept of PFSSs and defined the operations such as complement, union, intersection, and, or, addition, multiplication, necessity, and possibility. In 2017, Satirad et al. [32] discussed the relationships among (prime, weakly prime) hesitant fuzzy UP-subalgebras (resp., hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals) and some level subsets of a HFS on UP- algebras. In 2018, Satirad et al. [26] introduced eight types of subsets and of fuzzy sets of fully UP-semigroups, and investigate the algebraic properties of fuzzy sets under the operations of intersection and union. In 2019, Satirad and Iampan [27, 28] introduced ten types of fuzzy soft sets over fully UP-semigroups, and investigate the algebraic properties of fuzzy soft sets under the operations of (extended) intersection and (restricted) union. Jana et al. [14] used Dombi operations to create a few Pythagorean fuzzy Dombi aggregation operators. Additionally, by examining the book [13], they also presented recent research examining the theoretical and practical elements of fuzzy set theory and its actual applications in the disciplines of engineering and science. In 2020, Touqeer [39] introduced the notion of intuitionistic fuzzy soft α-ideals in BCI-algebras, described connections between various types of intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy soft ideals and characterized using the idea of soft (δ, η)-level set. In 2022, Satirad et al. [25, 24] applied the concept of rough sets to PFSs in UP-algebras and studied the relationships between rough Pythagorean fuzzy sets and rough sets in UP-algebras under analyzing t-level subsets of rough Pythagorean fuzzy sets. Palanikumar et al. [19] presented a communication which deals with some new methods to solve multiple attribute decision-making problems based on Pythagorean neutrosophic normal interval-valued set. Jana et al. [10] solved the Pythagorean fuzzy multiple attribute decision making problem by using Pythagorean fuzzy power Dombi weighted averaging and Pythagorean fuzzy power Dombi weighted geometric operators to design an algorithm for the proposed approach.

In this paper, we apply the concept of PFSSs to UP-algebras and introduce five types of them, namely, Pythagorean fuzzy soft UP-subalgebras (PF-SUPSs), Pythagorean fuzzy soft near UP-filters (PFSNUPFs), Pythagorean fuzzy soft UP-filters (PFSUPFs), Pythagorean fuzzy soft UP-ideals (PFSUPIs), and Pythagorean fuzzy soft strong UP-ideals (PFSSUPIs). Then we study the operations on five types of PFSSs such as the union, the restricted union, the extended intersection, and the intersection. Moreover, we investigate t-level subsets of PFSSs over UP-algebras in order to discuss the relationships between PFSSs and special subset of UP-algebras.

First, let's review the definition of UP-algebras.

Definition 1.1. [6] A *UP-algebra* is one that has the algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$ of type $(2, 0)$, where $\mathcal{U}$ is a nonempty set, $\star$ is a binary operation on $\mathcal{U}$, and 0 is a fixed element of $\mathcal{U}$ if it meets the following axioms:

$$(\forall u, v, w \in \mathcal{U})((v \star w) \star ((u \star v) \star (u \star w)) = 0), \tag{UP-1}$$

$$(\forall u \in \mathcal{U})(0 \star u = u), \tag{UP-2}$$

$$(\forall u \in \mathcal{U})(u \star 0 = 0), \tag{UP-3}$$

$$(\forall u, v \in \mathcal{U})(u \star v = 0, v \star u = 0 \Rightarrow u = v). \tag{UP-4}$$

For more examples of UP-algebras, see [1, 2, 4, 7, 9, 18, 30, 31, 29, 33, 34].

According to [6], we know that the concept of UP-algebras is a generalization of KU-algebras (see [21]).

Unless otherwise indicated, we will assume that $\mathcal{U}$ is a UP-algebra $(\mathcal{U}, \star, 0)$.

In $\mathcal{U}$, the following assertions are valid (see [6, 7]).

$$(\forall u \in \mathcal{U})(u \star u = 0), \tag{1.1}$$

$$(\forall u, v, w \in \mathcal{U})(u \star v = 0, v \star w = 0 \Rightarrow u \star w = 0), \tag{1.2}$$

$$(\forall u, v, w \in \mathcal{U})(u \star v = 0 \Rightarrow (w \star u) \star (w \star v) = 0), \tag{1.3}$$

$$(\forall u, v, w \in \mathcal{U})(u \star v = 0 \Rightarrow (v \star w) \star (u \star w) = 0), \tag{1.4}$$

$$(\forall u, v, w \in \mathcal{U})(u \star (v \star u) = 0, \text{ in particular, } (v \star w) \star (u \star (v \star w)) = 0), \tag{1.5}$$

$$(\forall u, v \in \mathcal{U})((v \star u) \star u = 0 \Leftrightarrow u = v \star u), \tag{1.6}$$

$$(\forall u, v \in \mathcal{U})(u \star (v \star v) = 0), \tag{1.7}$$

$$(\forall a, u, v, w \in \mathcal{U})((u \star (v \star w)) \star (u \star ((a \star v) \star (a \star w))) = 0), \tag{1.8}$$

$$(\forall a, u, v, w \in \mathcal{U})((((a \star u) \star (a \star v)) \star w) \star ((u \star v) \star w) = 0), \tag{1.9}$$

$$(\forall u, v, w \in \mathcal{U})(((u \star v) \star w) \star (v \star w) = 0), \tag{1.10}$$

$$(\forall u, v, w \in \mathcal{U})(u \star v = 0 \Rightarrow u \star (w \star v) = 0), \tag{1.11}$$

$$(\forall u, v, w \in \mathcal{U})(((u \star v) \star w) \star (u \star (v \star w)) = 0), \tag{1.12}$$

$$(\forall a, u, v, w \in \mathcal{U})(((u \star v) \star w) \star (v \star (a \star w)) = 0). \tag{1.13}$$

According to [6], the binary relation $\leq$ on $\mathcal{U}$ is defined as follows:

$$(\forall u, v \in \mathcal{U})(u \leq v \Leftrightarrow u \star v = 0).$$

Definition 1.2. [5, 6, 8, 36] A nonempty subset S of $\mathcal{U}$ is called

(1) a *UP-subalgebra* (UPS) of $\mathcal{U}$ if it satisfies the following condition:

$$(\forall u, v \in S)(u \star v \in S), \tag{1.14}$$

(2) a *near UP-filter* (NUPF) of $\mathcal{U}$ if it satisfies the following condition:

$$(\forall u, v \in \mathcal{U})(v \in S \Rightarrow u \star v \in S), \tag{1.15}$$

(3) a *UP-filter* (UPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } \mathcal{U} \text{ is in } S, \tag{1.16}$$

$$(\forall u, v \in \mathcal{U})(u \star v \in S, u \in S \Rightarrow v \in S), \tag{1.17}$$

(4) a *UP-ideal* (UPI) of $\mathcal{U}$ if it satisfies the condition (1.16) and the following condition:

$$(\forall u, v, w \in \mathcal{U})(u \star (v \star w) \in S, v \in S \Rightarrow u \star w \in S), \tag{1.18}$$

(5) a *strong UP-ideal* (SUPI) of $\mathcal{U}$ if it satisfies the condition (1.16) and the following condition:

$$(\forall u, v, w \in \mathcal{U})((w \star v) \star (w \star u) \in S, v \in S \Rightarrow u \in S). \tag{1.19}$$

From [5, 6, 8, 36] allows us to know that the concept of UPSs is a generalization of NUPFs, NUPFs is a generalization of UPFs, UPFs is a generalization of UPIs, and UPIs is a generalization of SUPIs. They also proved that $\mathcal{U}$ is the only SUPI.

Definition 1.3. [42] A *fuzzy set* (FS) F in a nonempty set $\mathcal{U}$ is described by its membership function μ_{F}. To every point $u \in \mathcal{U}$, this function associates a real number $\mu_{\mathrm{F}}(u)$ in the closed interval $[0, 1]$. The real number $\mu_{\mathrm{F}}(u)$ is interpreted for the point as a degree of membership of an object $u \in \mathcal{U}$ to the FS F, that is, $\mathrm{F} := \{(u, \mu_{\mathrm{F}}(u)) \mid u \in \mathcal{U}\}$. We say that a FS F in $\mathcal{U}$ is *constant fuzzy set* if its membership function μ_{F} is constant.

In 2013, Yager [40], and Yager and Abbasov [41] introduced the concept of PFSs for the first time.

Definition 1.4. [40, 41] A *Pythagorean fuzzy set* (PFS) P in a nonempty set $\mathcal{U}$ is described by their membership function μ_{P} and non-membership function ν_{P}. To every point $u \in \mathcal{U}$, these functions associate real numbers $\mu_{\mathrm{P}}(u)$ and $\nu_{\mathrm{P}}(u)$ in the closed interval $[0, 1]$, with the following condition:

$$(\forall u \in \mathcal{U})(0 \leq \mu_{\mathrm{P}}(u)^2 + \nu_{\mathrm{P}}(u)^2 \leq 1). \tag{1.20}$$

The real numbers $\mu_{\mathrm{P}}(u)$ and $\nu_{\mathrm{P}}(u)$ are interpreted for the point as a degree of membership and non-membership of an object $u \in \mathcal{U}$, respectively, to the PFS P, that is, $\mathrm{P} := \{(u, \mu_{\mathrm{P}}(u), \nu_{\mathrm{P}}(u)) \mid x \in \mathcal{U}\}$. For the sake of simplicity, a PFS P is denoted by $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$. We say that a PFS P in $\mathcal{U}$ is *constant Pythagorean fuzzy set* (CPFS) if their membership function μ_{P} and non-membership function ν_{P} are constant.

Definition 1.5. [23] A PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ in $\mathcal{U}$ is called

(1) a *Pythagorean fuzzy UP-subalgebra* (PFUPS) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall u, v \in \mathcal{U})(\mu_{\mathrm{P}}(u \star v) \geq \min\{\mu_{\mathrm{P}}(u), \mu_{\mathrm{P}}(v)\}), \tag{1.21}$$

$$(\forall u, v \in \mathcal{U})(\nu_{\mathrm{P}}(u \star v) \leq \max\{\nu_{\mathrm{P}}(u), \nu_{\mathrm{P}}(v)\}), \tag{1.22}$$

(2) a *Pythagorean fuzzy near UP-filter* (PFNUPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall u, v \in \mathcal{U})(\mu_{\mathrm{P}}(u \star v) \geq \mu_{\mathrm{P}}(v)), \tag{1.23}$$

$$(\forall u, v \in \mathcal{U})(\nu_{\mathrm{P}}(u \star v) \leq \nu_{\mathrm{P}}(v)), \tag{1.24}$$

(3) a *Pythagorean fuzzy UP-filter* (PFUPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall u \in \mathcal{U})(\mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(u)), \tag{1.25}$$

$$(\forall u \in \mathcal{U})(\nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(u)), \tag{1.26}$$

$$(\forall u, v \in \mathcal{U})(\mu_{\mathrm{P}}(v) \geq \min\{\mu_{\mathrm{P}}(u \star v), \mu_{\mathrm{P}}(u)\}), \tag{1.27}$$

$$(\forall u, v \in \mathcal{U})(\nu_{\mathrm{P}}(v) \leq \max\{\nu_{\mathrm{P}}(u \star v), \nu_{\mathrm{P}}(u)\}), \tag{1.28}$$

(4) a *Pythagorean fuzzy UP-ideal* (PFUPI) of $\mathcal{U}$ if it satisfies the conditions (1.25) and (1.26) and the following conditions:

$$(\forall u, v, w \in \mathcal{U})(\mu_{\mathrm{P}}(u \star w) \geq \min\{\mu_{\mathrm{P}}(u \star (v \star w)), \mu_{\mathrm{P}}(v)\}), \tag{1.29}$$

$$(\forall u, v, w \in \mathcal{U})(\nu_{\mathrm{P}}(u \star w) \leq \max\{\nu_{\mathrm{P}}(u \star (v \star w)), \nu_{\mathrm{P}}(v)\}), \tag{1.30}$$

(5) a *Pythagorean fuzzy strong UP-ideal* (PFSUPI) of $\mathcal{U}$ if it satisfies the conditions (1.25) and (1.26) and the following conditions:

$$(\forall u, v, w \in \mathcal{U})(\mu_{\mathrm{P}}(u) \geq \min\{\mu_{\mathrm{P}}((w \star v) \star (w \star u)), \mu_{\mathrm{P}}(v)\}), \tag{1.31}$$

$$(\forall u, v, w \in \mathcal{U})(\nu_{\mathrm{P}}(u) \leq \max\{\nu_{\mathrm{P}}((w \star v) \star (w \star u)), \nu_{\mathrm{P}}(v)\}). \tag{1.32}$$

Satirad et al. [23] proved that the concept of PFUPSs is a generalization of PFNUPFs, PFNUPFs is a generalization of PFUPFs, PFUPFs is a generalization of PFUPIs, and PFUPIs is a generalization of PFSUPIs. Furthermore, they proved that PFSUPIs and constant PFSs coincide in $\mathcal{U}$.

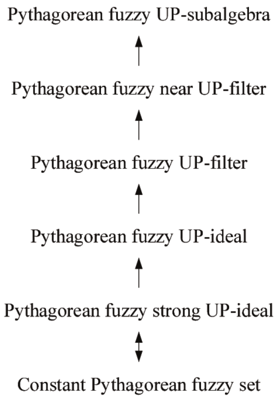

FIGURE 1. PFSs in UP-algebras

Definition 1.6. [36] Let F be a FS with the membership function μ_{F} in $\mathcal{U}$. The sets

$$\begin{aligned} U(\mu_{\mathrm{F}}, t) &= \{u \in \mathcal{U} \mid \mu_{\mathrm{F}}(u) \geq t\}, \\ U^{+}(\mu_{\mathrm{F}}, t) &= \{u \in \mathcal{U} \mid \mu_{\mathrm{F}}(u) > t\}, \\ L(\mu_{\mathrm{F}}, t) &= \{u \in \mathcal{U} \mid \mu_{\mathrm{F}}(u) \leq t\}, \\ L^{-}(\mu_{\mathrm{F}}, t) &= \{u \in \mathcal{U} \mid \mu_{\mathrm{F}}(u) < t\}, \\ E(\mu_{\mathrm{F}}, t) &= \{u \in \mathcal{U} \mid \mu_{\mathrm{F}}(u) = t\} \end{aligned}$$

are referred to as an *upper t-level subset*, an *upper t-strong level subset*, a *lower t-level subset*, a *lower t-strong level subset*, and an *equal t-level subset* of F, respectively, for any $t \in [0, 1]$.

The following three theorems are proved in [25].

Theorem 1.7. P *is a PFUPS (resp., PFNUPF, PFUPF, PFUPI, PFSUPI) of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPSs (resp., NUPFs, UPFs, UPIs, SUPIs) of* $\mathcal{U}$ *for every* $t \in [0, 1]$.

Theorem 1.8. P *is a PFUPS (resp., PFNUPF, PFUPF, PFUPI, PFSUPI) of* $\mathcal{U}$ *if and only if* $U^{+}(\mu_{\mathrm{P}}, t)$ *and* $L^{-}(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPSs (resp., NUPFs, UPFs, UPIs, SUPIs) of* $\mathcal{U}$ *for every* $t \in [0, 1]$.

Theorem 1.9. P *is a PFSUPI of* $\mathcal{U}$ *if and only if* $E(\mu_{\mathrm{P}}, \mu_{\mathrm{P}}(0))$ *and* $E(\nu_{\mathrm{P}}, \nu_{\mathrm{P}}(0))$ *are SUPIs of* $\mathcal{U}$.

Definition 1.10. [40] Let $\{\mathrm{P}_i = (\mu_{\mathrm{P}i}, \nu_{\mathrm{P}i})\}_{i \in I}$ be a nonempty family of PFSs in a nonempty set $\mathcal{U}$ where I is an arbitrary index set. The *intersection* of P_i, denoted by $\bigwedge_{i \in I} \mathrm{P}_i$, is described by theirs membership function $\mu_{\bigwedge_{i \in I} \mathrm{P}_i}$ and non-membership function $\nu_{\bigwedge_{i \in I} \mathrm{P}_i}$ which defined as follows:

$$\begin{aligned} &(u \in \mathcal{U})(\mu_{\bigwedge_{i \in I} \mathrm{P}_i}(u) = \inf\{\mu_{\mathrm{P}i}(u)\}_{i \in I}), \\ &(u \in \mathcal{U})(\nu_{\bigwedge_{i \in I} \mathrm{P}_i}(u) = \sup\{\nu_{\mathrm{P}i}(u)\}_{i \in I}). \end{aligned}$$

The *union* of P_i, denoted by $\bigvee_{i \in I} \mathrm{P}_i$, is described by theirs membership function $\mu_{\bigvee_{i \in I} \mathrm{P}_i}$ and non-membership function $\nu_{\bigvee_{i \in I} \mathrm{P}_i}$ which defined as follows:

$$\begin{aligned} &(u \in \mathcal{U})(\mu_{\bigvee_{i \in I} \mathrm{P}_i}(u) = \sup\{\mu_{\mathrm{P}i}(u)\}_{i \in I}), \\ &(u \in \mathcal{U})(\nu_{\bigvee_{i \in I} \mathrm{P}_i}(u) = \inf\{\nu_{\mathrm{P}i}(u)\}_{i \in I}). \end{aligned}$$

In particular, if $I = \{1, 2, \ldots, n\}$, the intersection of $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$, denoted by $\mathrm{P}_1 \wedge \mathrm{P}_2 \wedge \ldots \wedge \mathrm{P}_n$, is described by theirs membership function $\mu_{\mathrm{P}_1 \wedge \mathrm{P}_2 \wedge \ldots \wedge \mathrm{P}_n}$ and non-membership function $\nu_{\mathrm{P}_1 \wedge \mathrm{P}_2 \wedge \ldots \wedge \mathrm{P}_n}$ which defined as follows:

$$\begin{aligned} &(u \in \mathcal{U})(\mu_{\mathrm{P}_1 \wedge \mathrm{P}_2 \wedge \ldots \wedge \mathrm{P}_n}(u) = \min\{\mu_{\mathrm{P}1}(u), \mu_{\mathrm{P}2}(u), \ldots, \mu_{\mathrm{P}n}(u)\}), \\ &(u \in \mathcal{U})(\nu_{\mathrm{P}_1 \wedge \mathrm{P}_2 \wedge \ldots \wedge \mathrm{P}_n}(u) = \max\{\nu_{\mathrm{P}1}(u), \nu_{\mathrm{P}2}(u), \ldots, \nu_{\mathrm{P}n}(u)\}). \end{aligned}$$

The union of $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$, denoted by $\mathrm{P}_1 \vee \mathrm{P}_2 \vee \ldots \vee \mathrm{P}_n$, is described by theirs membership function $\mu_{\mathrm{P}_1 \vee \mathrm{P}_2 \vee \ldots \vee \mathrm{P}_n}$ and non-membership function $\nu_{\mathrm{P}_1 \vee \mathrm{P}_2 \vee \ldots \vee \mathrm{P}_n}$ which defined as follows:

$$(u \in \mathcal{U})(\mu_{\mathrm{P}_1 \vee \mathrm{P}_2 \vee \ldots \vee \mathrm{P}_n}(u) = \max\{\mu_{\mathrm{P}1}(u), \mu_{\mathrm{P}2}(u), \ldots, \mu_{\mathrm{P}n}(u)\}),$$
$$(u \in \mathcal{U})(\nu_{\mathrm{P}_1 \vee \mathrm{P}_2 \vee \ldots \vee \mathrm{P}_n}(u) = \min\{\nu_{\mathrm{P}1}(u), \nu_{\mathrm{P}2}(u), \ldots, \nu_{\mathrm{P}n}(u)\}).$$

Theorem 1.11. *The intersection of any nonempty family of PFUPSs of $\mathcal{U}$ is also a PFUPS.*

Proof. Assume that P_i is a PFUPS of $\mathcal{U}$ for all $i \in I$. Let $u, v \in \mathcal{U}$. Then

$$\begin{aligned}
\mu_{\bigwedge_{i\in I} \mathrm{P}_i}(u \star v) &= \inf\{\mu_{\mathrm{P}i}(u \star v)\}_{i\in I} \\
&\geq \inf\{\min\{\mu_{\mathrm{P}i}(u), \mu_{\mathrm{P}i}(v)\}\}_{i\in I} \\
&= \min\{\inf\{\mu_{\mathrm{P}i}(u)\}_{i\in I}, \inf\{\mu_{\mathrm{P}i}(v)\}_{i\in I}\} \\
&= \min\{\mu_{\bigwedge_{i\in I} \mathrm{P}_i}(u), \mu_{\bigwedge_{i\in I} \mathrm{P}_i}(v)\} \text{ and} \\
\nu_{\bigwedge_{i\in I} \mathrm{P}_i}(u \star v) &= \sup\{\nu_{\mathrm{P}i}(u \star v)\}_{i\in I} \\
&\leq \sup\{\max\{\nu_{\mathrm{P}i}(u), \nu_{\mathrm{P}i}(v)\}\}_{i\in I} \\
&= \max\{\sup\{\nu_{\mathrm{P}i}(u)\}_{i\in I}, \sup\{\nu_{\mathrm{P}i}(v)\}_{i\in I}\} \\
&= \max\{\nu_{\bigwedge_{i\in I} \mathrm{P}_i}(u), \nu_{\bigwedge_{i\in I} \mathrm{P}_i}(v)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{P}_i$ is a PFUPS of $\mathcal{U}$. □

The following example show that the union of two PFUPSs of UP-algebra may be not a PFUPS.

Example 1.12. Let $\mathcal{U} = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $\star$ defined by the following Cayley table:

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	1	0

We define two PFSs $\mathrm{P}_1 = (\mu_{\mathrm{P}_1}, \nu_{\mathrm{P}_1})$ and $\mathrm{P}_2 = (\mu_{\mathrm{P}_2}, \nu_{\mathrm{P}_2})$ as follows:

$\mathcal{U}$	0	1	2	3
μ_{P_1}	0.8	0.3	0.8	0.2
ν_{P_1}	0.2	0.5	0.2	0.6
μ_{P_2}	0.8	0.2	0.1	0.6
ν_{P_2}	0.2	0.8	0.9	0.7

Then P_1 and P_2 are PFUPSs of $\mathcal{U}$. Since $\mu_{\mathrm{P}_1 \vee \mathrm{P}_2}(3 \cdot 2) = \mu_{\mathrm{P}_1 \vee \mathrm{P}_2}(1) = 0.3 \ngeq 0.6 = \min\{0.6, 0.8\} = \min\{\mu_{\mathrm{P}_1 \vee \mathrm{P}_2}(3), \mu_{\mathrm{P}_1 \vee \mathrm{P}_2}(2)\}$, we have $\mathrm{P}_1 \vee \mathrm{P}_2$ is not a PFUPS of $\mathcal{U}$.

Theorem 1.13. *The intersection of any nonempty family of PFNUPFs of $\mathcal{U}$ is also a PFNUPF.*

Proof. Assume that P_i is a PFNUPF of $\mathcal{U}$ for all $i \in I$. Then

$$\mu_{\bigwedge_{i\in I} \mathrm{P}_i}(u \star v) = \inf\{\mu_{\mathrm{P}i}(u \star v)\}_{i\in I}$$

$$\begin{aligned}
&\geq \inf\{\mu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&= \mu_{\bigwedge_{i\in I}\mathrm{P}_i}(v) \text{ and}\\
\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star v) &= \sup\{\nu_{\mathrm{P}_i}(u\star v)\}_{i\in I}\\
&\leq \sup\{\nu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&= \nu_{\bigwedge_{i\in I}\mathrm{P}_i}(v).
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{P}_i$ is a PFNUPF of $\mathcal{U}$. □

Theorem 1.14. *The union of any nonempty family of PFNUPFs of $\mathcal{U}$ is also a PFNUPF.*

Proof. Assume that P_i is a PFNUPF of $\mathcal{U}$ for all $i\in I$. Then

$$\begin{aligned}
\mu_{\bigvee_{i\in I}\mathrm{P}_i}(u\star v) &= \sup\{\mu_{\mathrm{P}_i}(u\star v)\}_{i\in I}\\
&\geq \sup\{\mu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&= \mu_{\bigvee_{i\in I}\mathrm{P}_i}(v) \text{ and}\\
\nu_{\bigvee_{i\in I}\mathrm{P}_i}(u\star v) &= \inf\{\nu_{\mathrm{P}_i}(u\star v)\}_{i\in I}\\
&\leq \inf\{\nu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&= \nu_{\bigvee_{i\in I}\mathrm{P}_i}(v).
\end{aligned}$$

Hence, $\bigvee_{i\in I}\mathrm{P}_i$ is a PFNUPF of $\mathcal{U}$. □

Theorem 1.15. *The intersection of any nonempty family of PFUPFs of $\mathcal{U}$ is also a PFUPF.*

Proof. Asusume that P_i be a PFUPF of $\mathcal{U}$ for all $i\in I$. Then

$$\begin{aligned}
\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(0) &= \inf\{\mu_{\mathrm{P}_i}(0)\}_{i\in I}\\
&\geq \inf\{\mu_{\mathrm{P}_i}(u)\}_{i\in I}\\
&= \mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u),\\
\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(v) &= \inf\{\mu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&\geq \inf\{\min\{\mu_{\mathrm{P}_i}(u\star v), \mu_{\mathrm{P}_i}(u)\}\}_{i\in I}\\
&= \min\{\inf\{\mu_{\mathrm{P}_i}(u\star v)\}_{i\in I}, \inf\{\mu_{\mathrm{P}_i}(u)\}_{i\in I}\}\\
&= \min\{\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star v), \mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u)\},\\
\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(0) &= \sup\{\nu_{\mathrm{P}_i}(0)\}_{i\in I}\\
&\leq \sup\{\nu_{\mathrm{P}_i}(u)\}_{i\in I}\\
&= \nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u), \text{ and}\\
\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(v) &= \sup\{\nu_{\mathrm{P}_i}(v)\}_{i\in I}\\
&\leq \sup\{\max\{\nu_{\mathrm{P}_i}(u\star v), \nu_{\mathrm{P}_i}(u)\}\}_{i\in I}\\
&= \max\{\sup\{\nu_{\mathrm{P}_i}(u\star v)\}_{i\in I}, \inf\{\nu_{\mathrm{P}_i}(u)\}_{i\in I}\}\\
&= \max\{\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star v), \nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{P}_i$ is a PFUPF of $\mathcal{U}$. □

The following example show that the union of two PFUPFs of UP-algebra may be not a PFUPF.

Example 1.16. Let $\mathcal{U} = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $\star$ defined by the following Cayley table:

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

We define two PFSs $\mathrm{P}_1 = (\mu_{\mathrm{P}_1}, \nu_{\mathrm{P}_1})$ and $\mathrm{P}_2 = (\mu_{\mathrm{P}_2}, \nu_{\mathrm{P}_2})$ as follows:

$\mathcal{U}$	0	1	2	3
μ_{P_1}	0.7	0.7	0.4	0.4
ν_{P_1}	0.2	0.2	0.5	0.5
μ_{P_2}	0.8	0.2	0.5	0.2
ν_{P_2}	0.2	0.6	0.3	0.6

Then P_1 and P_2 are PFUPFs of $\mathcal{U}$. Since $\mu_{\mathrm{P}_1\vee\mathrm{P}_2}(3) = 0.4 \ngeq 0.5 = \min\{0.5, 0.7\} = \min\{\mu_{\mathrm{P}_1\vee\mathrm{P}_2}(2) =, \mu_{\mathrm{P}_1\vee\mathrm{P}_2}(1)\} = \min\{\mu_{\mathrm{P}_1\vee\mathrm{P}_2}(1\cdot 3), \mu_{\mathrm{P}_1\vee\mathrm{P}_2}(1)\}$, we have $\mathrm{P}_1 \vee \mathrm{P}_2$ is not a PFUPF of $\mathcal{U}$.

Theorem 1.17. *The intersection of any nonempty family of PFUPIs of $\mathcal{U}$ is also a PFUPI.*

Proof. Asusume that P_i be a PFUPI of $\mathcal{U}$ for all $i \in I$. Then

$$\begin{aligned}
\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(0) &= \inf\{\mu_{\mathrm{P}_i}(0)\}_{i\in I}\\
&\geq \inf\{\mu_{\mathrm{P}_i}(u)\}_{i\in I}\\
&= \mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u),\\
\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star w) &= \inf\{\mu_{\mathrm{P}_i}(u\star w)\}_{i\in I}\\
&\geq \inf\{\min\{\mu_{\mathrm{P}_i}(u\star(v\star w)), \mu_{\mathrm{P}_i}(v)\}\}_{i\in I}\\
&= \min\{\inf\{\mu_{\mathrm{P}_i}(u\star(v\star w))\}_{i\in I}, \inf\{\mu_{\mathrm{P}_i}(v)\}_{i\in I}\}\\
&= \min\{\mu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star(v\star w)), \mu_{\bigwedge_{i\in I}\mathrm{P}_i}(v)\},\\
\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(0) &= \sup\{\nu_{\mathrm{P}_i}(0)\}_{i\in I}\\
&\leq \sup\{\nu_{\mathrm{P}_i}(u)\}_{i\in I}\\
&= \nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u), \text{ and}\\
\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star w) &= \sup\{\nu_{\mathrm{P}_i}(u\star w)\}_{i\in I}\\
&\leq \sup\{\max\{\nu_{\mathrm{P}_i}(u\star(v\star w)), \nu_{\mathrm{P}_i}(v)\}\}_{i\in I}\\
&= \max\{\sup\{\nu_{\mathrm{P}_i}(u\star(v\star w))\}_{i\in I}, \inf\{\nu_{\mathrm{P}_i}(v)\}_{i\in I}\}\\
&= \max\{\nu_{\bigwedge_{i\in I}\mathrm{P}_i}(u\star(v\star w)), \nu_{\bigwedge_{i\in I}\mathrm{P}_i}(v)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{P}_i$ is a PFUPI of $\mathcal{U}$. $\square$

The following example show that the union of two PFUPIs of UP-algebra may be not a PFUPI.

Example 1.18. In Example 1.16 We define two PFSs $\mathrm{P}_1 = (\mu_{\mathrm{P}_1}, \nu_{\mathrm{P}_1})$ and $\mathrm{P}_2 = (\mu_{\mathrm{P}_2}, \nu_{\mathrm{P}_2})$ as follows:

$\mathcal{U}$	0	1	2	3
μ_{P_1}	1	0.4	0.7	0.4
ν_{P_1}	0	0.5	0.3	0.5
μ_{P_2}	0.9	0.7	0.1	0.1
ν_{P_2}	0.2	0.4	0.9	0.9

Then P_1 and P_2 are PFUPIs of $\mathcal{U}$. Since $\mu_{\mathrm{P}_1\vee \mathrm{P}_2}(0\cdot 3) = \mu_{\mathrm{P}_1\vee \mathrm{P}_2}(3) = 0.4 \ngeq 0.7 = \min\{0.7, 0.7\} = \min\{\mu_{\mathrm{P}_1\vee \mathrm{P}_2}(1), \mu_{\mathrm{P}_1\vee \mathrm{P}_2}(2)\} = \min\{\mu_{\mathrm{P}_1\vee \mathrm{P}_2}(0\cdot(2\cdot 3)) = , \mu_{\mathrm{P}_1\vee \mathrm{P}_2}(2)\}$, we have $\mathrm{P}_1 \vee \mathrm{P}_2$ is not a PFUPI of $\mathcal{U}$.

Theorem 1.19. *The intersection of any nonempty family of PFSUPIs of* $\mathcal{U}$ *is also a PFSUPI. Moreover, the union of any nonempty family of PFSUPIs of* $\mathcal{U}$ *is also a PFSUPI.*

2. PFSSs over UP-algebras

From now on, we shall let E be a set of parameters. Let $\mathrm{PF}(\mathcal{U})$ be the set of all PFSs in $\mathcal{U}$. A subset A of E is called a set of statistics.

Definition 2.1. Let $A \subseteq E$. A pair $(\widetilde{\mathrm{P}}, A)$ is called a *Pythagorean fuzzy soft set* (PFSS) over $\mathcal{U}$ if $\widetilde{\mathrm{P}}$ is a mapping given by $\widetilde{\mathrm{P}}\colon A \to \mathrm{PF}(\mathcal{U})$, that is, a PFSS is a statistic family of PFSs in $\mathcal{U}$. In general, for every $a \in A$, $\widetilde{\mathrm{P}}[a] := \{(u, \mu_{\widetilde{\mathrm{P}}[a]}(u), \nu_{\widetilde{\mathrm{P}}[a]}(u)) \mid u \in \mathcal{U}\}$ is a PFS in $\mathcal{U}$ and it is called a *Pythagorean fuzzy value set* of statistic a.

We call a PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ that is a *constant Pythagorean fuzzy soft set* (CPFSS) based on the element $a \in A$ (we shortly call an *a-constant Pythagorean fuzzy soft set* (a-CPFSS)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a CPFS. If $(\widetilde{\mathrm{P}}, A)$ is an a-CPFSS of $\mathcal{U}$ for all $a \in A$, we say that $(\widetilde{\mathrm{P}}, A)$ is a *CPFSS* of $\mathcal{U}$.

By Definition 2.1, we can find an example of PFSSs over $\mathcal{U}$ as follows:

Example 2.2. Let $\mathcal{U} = \{0, 1, 2, 3\}$ be a set which represents a collection of 4 Thai paintings. Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley tables:

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	0	3
2	0	1	0	3
3	0	1	2	0

Then $\mathcal{U} = (\mathcal{U}, \star, 0)$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}, A)$ be a PFSS over $\mathcal{U}$ where

$$A = \{\text{identity}, \text{beauty}, \text{skill}\}.$$

Then $\widetilde{\mathrm{P}}[\text{identity}], \widetilde{\mathrm{P}}[\text{beauty}]$, and $\widetilde{\mathrm{P}}[\text{skill}]$ are three PFSs in $\mathcal{U}$. We define them as follows:

$\widetilde{\mathrm{P}}$	0	1	2	3
identity	$(0.4, 0.5)$	$(0.3, 0.3)$	$(0.1, 0.6)$	$(0.8, 0.2)$
beauty	$(0.9, 0.3)$	$(0.2, 0.5)$	$(0.1, 0.2)$	$(0.8, 0.4)$
skill	$(0.3, 0.5)$	$(0.3, 0.7)$	$(0.5, 0.6)$	$(0.7, 0.7)$

Definition 2.3. [20] Let $A, B \subseteq E$ and $(\widetilde{\mathrm{P}}, A), (\widetilde{\mathrm{Q}}, B)$ be two PFSSs over $\mathcal{U}$. If $(\widetilde{\mathrm{P}}, A)$ and $(\widetilde{\mathrm{Q}}, B)$ satisfy the following two conditions:

(1) $B \subseteq A$,
(2) $(\forall b \in B, u \in \mathcal{U})(\mu_{\widetilde{\mathrm{Q}}[b]}(u) \leq \mu_{\widetilde{\mathrm{P}}[b]}(u), \nu_{\widetilde{\mathrm{Q}}[b]}(u) \geq \nu_{\widetilde{\mathrm{P}}[b]}(u))$,

then we call $(\widetilde{\mathrm{Q}}, B)$ the Pythagorean fuzzy soft subset of $(\widetilde{\mathrm{P}}, A)$, denoted by $(\widetilde{\mathrm{Q}}, B) \widetilde{\subseteq} (\widetilde{\mathrm{P}}, A)$

Definition 2.4. [20] Let $A, B \subseteq E$ and $(\widetilde{\mathrm{P}}, A), (\widetilde{\mathrm{Q}}, B)$ be two PFSSs over $\mathcal{U}$. If $(\widetilde{\mathrm{Q}}, B) \widetilde{\subseteq} (\widetilde{\mathrm{P}}, A)$ and $(\widetilde{\mathrm{P}}, A) \widetilde{\subseteq} (\widetilde{\mathrm{Q}}, B)$, then we call $(\widetilde{\mathrm{P}}, A)$ *equal* $(\widetilde{\mathrm{Q}}, B)$, denoted by $(\widetilde{\mathrm{Q}}, B) \widetilde{=} (\widetilde{\mathrm{P}}, A)$, meaning, $A = B$ and $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{Q}}[a]$ for all $a \in A$.

Definition 2.5. [20] Let $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ be two PFSSs over $\mathcal{U}$. The *union* of $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ is defined to be the PFSS $(\widetilde{\mathrm{P}}_1, A_1) \widetilde{\cup} (\widetilde{\mathrm{P}}_2, A_2) = (\widetilde{\mathrm{P}}, A)$ satisfying the following conditions:

(1) $A = A_1 \cup A_2$ and
(2) for all $a \in A$,

$$\widetilde{\mathrm{P}}[a] = \begin{cases} \widetilde{\mathrm{P}}_1[a] & \text{if } a \in A_1 \setminus A_2 \\ \widetilde{\mathrm{P}}_2[a] & \text{if } a \in A_2 \setminus A_1 \\ \widetilde{\mathrm{P}}_1[a] \vee \widetilde{\mathrm{P}}_2[a] & \text{if } a \in A_1 \cap A_2. \end{cases}$$

The *restricted union* of $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ is defined to be the PFSS $(\widetilde{\mathrm{P}}_1, A_1) \Cup (\widetilde{\mathrm{P}}_2, A_2) = (\widetilde{\mathrm{P}}, A)$ satisfying the following conditions:

(1) $A = A_1 \cap A_2 \neq \emptyset$ and
(2) $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \vee \widetilde{\mathrm{P}}_2[a]$ for all $a \in A$.

Definition 2.6. Let $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ be two PFSSs over $\mathcal{U}$. The *extended intersection* of $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ is defined to be the PFSS $(\widetilde{\mathrm{P}}_1, A_1) \widetilde{\cap} (\widetilde{\mathrm{P}}_2, A_2) = (\widetilde{\mathrm{P}}, A)$ satisfying the following conditions:

(1) $A = A_1 \cup A_2$ and
(2) for all $a \in A$,

$$\widetilde{\mathrm{P}}[a] = \begin{cases} \widetilde{\mathrm{P}}_1[a] & \text{if } a \in A_1 \setminus A_2 \\ \widetilde{\mathrm{P}}_2[a] & \text{if } a \in A_2 \setminus A_1 \\ \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a] & \text{if } a \in A_1 \cap A_2. \end{cases}$$

The *intersection* [20] of $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\Cap}(\widetilde{\mathrm{P}}_2, A_2) = (\widetilde{\mathrm{P}}, A)$ satisfying the following conditions:

(1) $A = A_1 \cap A_2 \neq \emptyset$ and

(2) $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ for all $a \in A$.

2.1. Pythagorean Fuzzy Soft UP-Subalgebras.

Definition 2.7. A PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ is called a *Pythagorean fuzzy soft UP-subalgebra* (PFSUPS) based on the element $a \in A$ (we shortly call an *a-Pythagorean fuzzy soft UP-subalgebra* (a-PFSUPS)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a PFUPS. If $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPS of $\mathcal{U}$ for all $a \in A$, we say that $(\widetilde{\mathrm{P}}, A)$ is a *PFSUPS* of $\mathcal{U}$.

Theorem 2.8. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$ if and only if $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPSs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPS of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U(\mu_{\widetilde{\mathrm{P}}[a]}, t), L(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.7, we have $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPSs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPSs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.7, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPS of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$. □

Theorem 2.9. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$ if and only if $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPSs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPS of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t), L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.8, we have $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPSs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPSs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.8, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPS of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$. □

The proof of the following theorem can be verified easily.

Theorem 2.10. *If $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$ and $\emptyset \neq B \subseteq A$, then $(\widetilde{\mathrm{P}}|_B, B)$ is a PFSUPS of $\mathcal{U}$.*

The following example shows that there exists a nonempty subset B of A such that $(\widetilde{\mathrm{P}}|_B, B)$ is a PFSUPS of $\mathcal{U}$, but $(\widetilde{\mathrm{P}}, A)$ is not a PFSUPS of $\mathcal{U}$.

Example 2.11. By Example 2.2, we have $\widetilde{\mathrm{P}}[\text{beauty}]$ is a PFUPS of $\mathcal{U}$. But $\widetilde{\mathrm{P}}[\text{identity}]$ and $\widetilde{\mathrm{P}}[\text{skill}]$ are not PFUPSs of $\mathcal{U}$. Indeed, $\nu_{\widetilde{\mathrm{P}}[\text{identity}]}(1 \star 1) =$

$\nu_{\widetilde{\mathrm{P}}[\text{identity}]}(0) = 0.5 \nleq 0.3 = \min\{0.3, 0.3\} = \min\{\nu_{\widetilde{\mathrm{P}}[\text{identity}]}(1), \nu_{\widetilde{\mathrm{P}}[\text{identity}]}(1)\}$ and $\mu_{\widetilde{\mathrm{P}}[\text{skill}]}(2 \star 2) = \mu_{\widetilde{\mathrm{P}}[\text{skill}]}(0) = 0.3 \ngeq 0.5 = \min\{0.5, 0.5\} = \min\{\mu_{\widetilde{\mathrm{P}}[\text{skill}]}(2), \mu_{\widetilde{\mathrm{P}}[\text{skill}]}(2)\}$. Hence, $(\widetilde{\mathrm{P}}, A)$ is not a PFSUPS over $\mathcal{U}$. We take $B = \{\text{beauty}\}$. Thus $(\widetilde{\mathrm{P}}|_B, B)$ is a PFSUPS of $\mathcal{U}$.

Theorem 2.12. *The extended intersection of two PFSUPSs of $\mathcal{U}$ is also a PFSUPS. Moreover, the intersection of two PFSUPSs of $\mathcal{U}$ is also a PFSUPS.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPSs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cap}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPS of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPS of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.11, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ is a PFUPS of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPS of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$. □

Theorem 2.13. *The union of two PFSUPSs of $\mathcal{U}$ is also a PFSUPS if sets of statistics of two PFSUPSs are disjoint.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPSs of $\mathcal{U}$ such that $A_1 \cap A_2 = \emptyset$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Since $A_1 \cap A_2 = \emptyset$, we have $a \in A_1 \setminus A_2$ or $a \in A_2 \setminus A_1$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPS of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPS of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPS of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPS of $\mathcal{U}$. □

The following example shows that Theorem 2.13 is not valid if sets of statistics of two PFSUPSs are not disjoint.

Example 2.14. Let $\mathcal{U}$ be a set of four Thai foods, that is,

$$\mathcal{U} = \{\text{Pad Thai, Som Tam, Laab, Tom Yum Goong}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	Pad Thai	Som Tam	Laab	Tom Yum Goong
Pad Thai	Pad Thai	Som Tam	Laab	Tom Yum Goong
Som Tam	Pad Thai	Pad Thai	Som Tam	Tom Yum Goong
Laab	Pad Thai	Pad Thai	Pad Thai	Tom Yum Goong
Tom Yum Goong	Pad Thai	Pad Thai	Som Tam	Pad Thai

Then $\mathcal{U} = (\mathcal{U}, \star, \text{Pad Thai})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are PFSSs over $\mathcal{U}$ where

$$A_1 := \{\text{popularity, aroma}\}$$

and

$$A_2 := \{\text{popularity, deliciousness}\}$$

with $\widetilde{\mathrm{P}}_1[\text{popularity}], \widetilde{\mathrm{P}}_1[\text{aroma}]$, $\widetilde{\mathrm{P}}_2[\text{popularity}]$, and $\widetilde{\mathrm{P}}_2[\text{deliciousness}]$ are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}_1$	Pad Thai	Som Tam	Laab	Tom Yum Goong
popularity	$(0.9, 0)$	$(0.5, 0.4)$	$(0.9, 0)$	$(0.3, 0.5)$
aroma	$(0.5, 0.4)$	$(0.4, 0.8)$	$(0.4, 0.8)$	$(0.4, 0.8)$

$\widetilde{\mathrm{P}}_2$	Pad Thai	Som Tam	Laab	Tom Yum Goong
popularity	$(0.9, 0.1)$	$(0.3, 0.7)$	$(0.2, 0.8)$	$(0.7, 0.2)$
deliciousness	$(0.5, 0.5)$	$(0.3, 0.7)$	$(0.2, 0.8)$	$(0.1, 0.9)$

Then $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are PFSUPSs of $\mathcal{U}$. Since popularity $\in A_1 \cap A_2$, we have

$$\begin{aligned}
&\mu_{\widetilde{\mathrm{P}}_1[\text{popularity}]\vee\widetilde{\mathrm{P}}_2[\text{popularity}]}(\text{Tom Yum Goong} \star \text{Laab})\\
&= \mu_{\widetilde{\mathrm{P}}_1[\text{popularity}]\vee\widetilde{\mathrm{P}}_2[\text{popularity}]}(\text{Som Tam})\\
&= 0.5\\
&\ngeq 0.7\\
&= \min\{0.7, 0.9\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}_1[\text{popularity}]\vee\widetilde{\mathrm{P}}_2[\text{popularity}]}(\text{Tom Yum Goong}),\\
&\mu_{\widetilde{\mathrm{P}}_1[\text{popularity}]\vee\widetilde{\mathrm{P}}_2[\text{popularity}]}(\text{Laab})\}.
\end{aligned}$$

Thus $\widetilde{\mathrm{P}}_1[\text{popularity}] \vee \widetilde{\mathrm{P}}_2[\text{popularity}]$ is not a PFUPS of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a popularity-PFSUPS of $\mathcal{U}$. Hence, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPS of $\mathcal{U}$. Moreover, $(\widetilde{\mathrm{P}}_1, A_1)\Cup(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPS of $\mathcal{U}$.

2.2. Pythagorean Fuzzy Soft Near UP-Filters.

Definition 2.15. A PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ is called a *Pythagorean fuzzy soft near UP-filter* (PFSNUPF) based on $a \in A$ (we shortly call an *a-Pythagorean fuzzy soft near UP-filter* (a-PFSNUPF)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a PFNUPF. If $(\widetilde{\mathrm{P}}, A)$ is an a-PFSNUPF of $\mathcal{U}$ for all $a \in A$, we say that $(\widetilde{\mathrm{P}}, A)$ is a *PFSNUPF* of $\mathcal{U}$.

Theorem 2.16. *$(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$ if and only if $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, NUPFs for every $a \in A, t \in [0, 1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFNUPF of $\mathcal{U}$ for all $a \in A$. Let $t \in [0, 1]$ be such that $U(\mu_{\widetilde{\mathrm{P}}[a]}, t), L(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.7, we have $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are NUPFs of $\mathcal{U}$ for all $a \in A, t \in [0, 1]$.

Conversely, assume for all $a \in A, t \in [0,1], U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are NUPFs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.7, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFNUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$. □

Theorem 2.17. *$(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$ if and only if $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, NUPFs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFNUPF of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t), L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.8, we have $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are NUPFs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are NUPFs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.8, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFNUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$. □

The proof of the following theorem can be verified easily.

Theorem 2.18. *If $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$ and $\emptyset \neq B \subseteq A$, then $(\widetilde{\mathrm{P}}|_B, B)$ is a PFSNUPF of $\mathcal{U}$.*

From Figure 1, we have the following theorem.

Theorem 2.19. *Every a-PFSNUPF of $\mathcal{U}$ is an a-PFSUPS. Moreover, every PFSNUPF of $\mathcal{U}$ is a PFSUPS.*

The following example shows that the converse of Theorem 2.19 is not true.

Example 2.20. Let $\mathcal{U}$ be a set of four drinks, that is,

$$\mathcal{U} = \{\text{Chocolate, Thai tea, Latte, Espresso}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	Chocolate	Thai tea	Latte	Espresso
Chocolate	Chocolate	Thai tea	Latte	Espresso
Thai tea	Chocolate	Chocolate	Thai tea	Espresso
Latte	Chocolate	Chocolate	Chocolate	Espresso
Espresso	Chocolate	Thai tea	Thai tea	Chocolate

Then $\mathcal{U} = (\mathcal{U}, \star, \text{Chocolate})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}, A)$ be a PFSS over $\mathcal{U}$ where

$$A := \{\text{child, teen, adult}\}$$

with $\widetilde{\mathrm{P}}[\text{child}], \widetilde{\mathrm{P}}[\text{teen}]$, and $\widetilde{\mathrm{P}}[\text{adult}]$ are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}$	Chocolate	Thai tea	Latte	Espresso
child	$(1,0)$	$(0.3,0.4)$	$(0.9,0.2)$	$(0.2,0.5)$
teen	$(0.9,0.1)$	$(0.8,0.2)$	$(0.6,0.4)$	$(0.7,0.4)$
adult	$(0.7,0.4)$	$(0.6,0.4)$	$(0.1,0.6)$	$(0.6,0.8)$

Then $(\widetilde{\mathrm{P}}, A)$ is a child-PFSUPS of $\mathcal{U}$. But $(\widetilde{\mathrm{P}}, A)$ is not a child-PFSNUPF of $\mathcal{U}$ since

$$\begin{aligned}
\mu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Thai tea} \star \text{Latte}) &= \mu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Thai tea}) \\
&= 0.3 \\
&\not\geq 0.9 \\
&= \mu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Latte})
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Thai tea} \star \text{Latte}) &= \nu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Thai tea}) \\
&= 0.4 \\
&\not\leq 0.2 \\
&= \nu_{\widetilde{\mathrm{P}}[\text{child}]}(\text{Latte}).
\end{aligned}$$

Hence, $\widetilde{\mathrm{P}}[\text{child}]$ is not a PFNUPF of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}, A)$ is not a child-PFSNUPF of $\mathcal{U}$.

Theorem 2.21. *The extended intersection of two PFSNUPFs of $\mathcal{U}$ is also a PFSNUPF. Moreover, the intersection of two PFSNUPFs of $\mathcal{U}$ is also a PFSNUPF.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSNUPFs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cap}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFNUPF of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFNUPF of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.13, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ is a PFNUPF of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSNUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$. □

Theorem 2.22. *The union of two PFSNUPFs of $\mathcal{U}$ is also a PFSNUPF. Moreover, the restricted union of two PFSNUPFs of $\mathcal{U}$ is also a PFSNUPF.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSNUPFs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFNUPF of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFNUPF of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.14, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \vee \widetilde{\mathrm{P}}_2[a]$ is a PFNUPF of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSNUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSNUPF of $\mathcal{U}$. □

2.3. Pythagorean Fuzzy Soft UP-Filters.

Definition 2.23. A PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ is called a *Pythagorean fuzzy soft UP-filter* (PFSUPF) based on $a \in A$ (we shortly call an *a-Pythagorean fuzzy soft UP-filter* (a-PFSUPF)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a PFUPF. If $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPF of $\mathcal{U}$ for all $a \in A$, we say that $(\widetilde{\mathrm{P}}, A)$ is a *PFSUPF* of $\mathcal{U}$.

Theorem 2.24. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$ if and only if $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPFs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPF of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U(\mu_{\widetilde{\mathrm{P}}[a]}, t), L(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.7, we have $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPFs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPFs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.7, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$. $\square$

Theorem 2.25. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$ if and only if $U^{+}(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^{-}(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPFs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPF of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U^{+}(\mu_{\widetilde{\mathrm{P}}[a]}, t), L^{-}(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.8, we have $U^{+}(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^{-}(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPFs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U^{+}(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^{-}(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPFs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.8, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$. $\square$

The proof of the following theorem can be verified easily.

Theorem 2.26. *If $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$ and $\emptyset \neq B \subseteq A$, then $(\widetilde{\mathrm{P}}|_{B}, B)$ is a PFSUPF of $\mathcal{U}$.*

From Figure 1, we have the following theorem.

Theorem 2.27. *Every a-PFSUPF of $\mathcal{U}$ is an a-PFSNUPF. Moreover, every PFSUPF of $\mathcal{U}$ is a PFSNUPF.*

The following example shows that the converse of Theorem 2.27 is not true.

Example 2.28. Let $\mathcal{U}$ be a set of four Apple's product, that is,

$$\mathcal{U} = \{\text{iPhone, iPad, Mac, Watch}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	iPhone	iPad	Mac	Watch
iPhone	iPhone	iPad	Mac	Watch
iPad	iPhone	iPhone	Mac	Watch
Mac	iPhone	iPhone	iPhone	Watch
Watch	iPhone	iPhone	iPhone	iPhone

Then $\mathcal{U} = (\mathcal{U}, \star, \text{iPhone})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}, A)$ be a PFSS over $\mathcal{U}$ where

$$A := \{\text{student, athlete, programmer}\}$$

with $\widetilde{\mathrm{P}}[\text{student}], \widetilde{\mathrm{P}}[\text{athlete}]$, and $\widetilde{\mathrm{P}}[\text{programmer}]$ are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}$	iPhone	iPad	Mac	Watch
student	$(0.9, 0.1)$	$(0.7, 0.4)$	$(0.8, 0.2)$	$(0.2, 0.6)$
athlete	$(0.7, 0.4)$	$(0.6, 0.5)$	$(0.7, 0.4)$	$(0.2, 0.6)$
programmer	$(0.8, 0.2)$	$(0.5, 0.7)$	$(0.6, 0.5)$	$(0.8, 0.2)$

Then $(\widetilde{\mathrm{P}}, A)$ is a programmer-PFSNUPF of $\mathcal{U}$. But $(\widetilde{\mathrm{P}}, A)$ is not a programmer-PFSUPF of $\mathcal{U}$ since

$$\begin{aligned}
\mu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{iPad}) &= 0.5 \\
&\ngeq 0.6 \\
&= \min\{0.8, 0.6\} \\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{iPhone}), \mu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac})\} \\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac} \star \text{iPad}), \mu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac})\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{iPad}) &= 0.7 \\
&\nleq 0.5 \\
&= \max\{0.2, 0.5\} \\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{iPhone}), \nu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac})\} \\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac} \star \text{iPad}), \nu_{\widetilde{\mathrm{P}}[\text{programmer}]}(\text{Mac})\}.
\end{aligned}$$

Hence, $\widetilde{\mathrm{P}}[\text{programmer}]$ is not a PFUPF of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}, A)$ is not a programmer-PFSUPF of $\mathcal{U}$.

Theorem 2.29. *The extended intersection of two PFSUPFs of $\mathcal{U}$ is also a PFSUPF. Moreover, the intersection of two PFSUPFs of $\mathcal{U}$ is also a PFSUPF.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPFs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1) \widetilde{\cap} (\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPF of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPF of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.15, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ is a PFUPF of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$. □

Theorem 2.30. *The union of two PFSUPFs of $\mathcal{U}$ is also a PFSUPF if sets of statistics of two PFSUPFs are disjoint.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPFs of $\mathcal{U}$ such that $A_1 \cap A_2 = \emptyset$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Since $A_1 \cap A_2 = \emptyset$, we have $a \in A_1 \setminus A_2$ or $a \in A_2 \setminus A_1$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPF of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPF of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPF of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPF of $\mathcal{U}$. □

The following example shows that Theorem 2.30 is not valid if sets of statistics of two PFSUPFs are not disjoint.

Example 2.31. Let $\mathcal{U}$ be a set of four seasons, that is,

$$\mathcal{U} = \{\text{Spring, Rains, Summer, Winter}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	Winter	Rains	Spring	Summer
Winter	Winter	Rains	Spring	Summer
Rains	Winter	Winter	Spring	Spring
Spring	Winter	Rains	Winter	Rains
Summer	Winter	Winter	Winter	Winter

Then $\mathcal{U} = (\mathcal{U}, \star, \text{Winter})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are PFSSs over $\mathcal{U}$ where

$$A_1, := \{\text{coldness, moisture}\}$$

and

$$A_2 := \{\text{moisture, excitement, warmth}\}$$

with $\widetilde{\mathrm{P}}_1[\text{coldness}], \widetilde{\mathrm{P}}_1[\text{moisture}], \widetilde{\mathrm{P}}_2[\text{moisture}], \widetilde{\mathrm{P}}_2[\text{excitement}]$, and $\widetilde{\mathrm{P}}_2[\text{warmth}]$ are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}_1$	Winter	Rains	Spring	Summer
coldness	$(0.9, 0.4)$	$(0.2, 0.7)$	$(0.2, 0.7)$	$(0.2, 0.7)$
moisture	$(0.8, 0.2)$	$(0.8, 0.2)$	$(0.3, 0.4)$	$(0.3, 0.4)$

$\widetilde{\mathrm{P}}_2$	Winter	Rains	Spring	Summer
moisture	$(0.9, 0.1)$	$(0.1, 0.7)$	$(0.5, 0.4)$	$(0.1, 0.7)$
excitement	$(0.6, 0.5)$	$(0.3, 0.8)$	$(0.6, 0.5)$	$(0.3, 0.8)$
warmth	$(0.5, 0.5)$	$(0.5, 0.5)$	$(0.5, 0.5)$	$(0.5, 0.5)$

Then $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are PFSUPFs of $\mathcal{U}$. Since moisture $\in A_1 \cap A_2$, we have

$$\begin{aligned}
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Summer}) \\
&= 0.3 \\
&\ngeq 0.5 \\
&= \min\{0.5, 0.8\} \\
&= \min\{\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Spring}), \\
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Rains})\} \\
&= \min\{\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Rains} \star \text{Summer}), \\
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Rains})\}.
\end{aligned}$$

Thus $\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]$ is not a PFUPF of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a moisture-PFSUPF of $\mathcal{U}$. Hence, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPF of $\mathcal{U}$. Moreover, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\Cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPF of $\mathcal{U}$.

2.4. Pythagorean Fuzzy Soft UP-Ideals.

Definition 2.32. A PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ is called a *Pythagorean fuzzy soft UP-ideal* (PFSUPI) based on $a \in A$ (we shortly call an *a-Pythagorean fuzzy soft UP-ideal* (a-PFSUPI)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a PFUPI. If $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPI of $\mathcal{U}$ for all $a \in A$, we say that $(\widetilde{\mathrm{P}}, A)$ is a *PFSUPI* of $\mathcal{U}$.

Theorem 2.33. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$ if and only if $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPIs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPI of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U(\mu_{\widetilde{\mathrm{P}}[a]}, t), L(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.7, we have $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPIs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPIs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.7, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$. □

Theorem 2.34. *$(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$ if and only if $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are, if the sets are nonempty, UPIs for every $a \in A, t \in [0,1]$.*

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPI of $\mathcal{U}$ for all $a \in A$. Let $t \in [0,1]$ be such that $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t), L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.8, we have $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPIs of $\mathcal{U}$ for all $a \in A, t \in [0,1]$.

Conversely, assume for all $a \in A, t \in [0,1], U^{+}(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^{-}(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are UPIs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.8, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$. $\square$

The proof of the following theorem can be verified easily.

Theorem 2.35. *If* $(\widetilde{\mathrm{P}}, A)$ *is a PFSUPI of* $\mathcal{U}$ *and* $\emptyset \neq B \subseteq A$*, then* $(\widetilde{\mathrm{P}}|_B, B)$ *is a PFSUPI of* $\mathcal{U}$*.*

From Figure 1, we have the following theorem.

Theorem 2.36. *Every a-PFSUPI of* $\mathcal{U}$ *is an a-PFSUPF. Moreover, every PFSUPI of* $\mathcal{U}$ *is a PFSUPF.*

The following example shows that the converse of Theorem 2.36 is not true.

Example 2.37. Let $\mathcal{U}$ be a set of four types of film, that is,

$$\mathcal{U} = \{\text{Fantasy, Horror, Comedy, Action}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	Comedy	Fantasy	Horror	Action
Comedy	Comedy	Fantasy	Horror	Action
Fantasy	Comedy	Comedy	Horror	Horror
Horror	Comedy	Fantasy	Comedy	Horror
Action	Comedy	Fantasy	Comedy	Comedy

Then $\mathcal{U} = (\mathcal{U}, \star, \text{Comedy})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}, A)$ be a PFSS over $\mathcal{U}$ where

$$A := \{\text{variety, violence, entertainment}\}$$

with $\widetilde{\mathrm{P}}[\text{variety}], \widetilde{\mathrm{P}}[\text{violence}]$, and $\widetilde{\mathrm{P}}[\text{entertainment}]$ are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}$	Comedy	Fantasy	Horror	Action
variety	$(0.7, 0.3)$	$(0.3, 0.5)$	$(0.2, 0.9)$	$(0.2, 0.9)$
violence	$(0.5, 0.5)$	$(0.2, 0.7)$	$(0.7, 0.7)$	$(0.4, 0.8)$
entertainment	$(0.8, 0.2)$	$(0.5, 0.7)$	$(0.6, 0.5)$	$(0.6, 0.5)$

Then $(\widetilde{\mathrm{P}}, A)$ is a variety-PFSUPF of $\mathcal{U}$. But $(\widetilde{\mathrm{P}}, A)$ is not a variety-PFSUPI of $\mathcal{U}$ since

$$\begin{aligned}
&\mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror} \star \text{Action})\\
&= \mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror})\\
&= 0.2\\
&\ngeq 0.3\\
&= \min\{0.7, 0.3\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Comedy}), \mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Fantasy})\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror} \star (\text{Fantasy} \star \text{Action})), \mu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Fantasy})\}
\end{aligned}$$

and

$$
\begin{aligned}
&\nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror} \star \text{Action})\\
&= \nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror})\\
&= 0.9\\
&\not\leq 0.5\\
&= \max\{0.3, 0.5\}\\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Comedy}), \nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Fantasy})\}\\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Horror} \star (\text{Fantasy} \star \text{Action})), \nu_{\widetilde{\mathrm{P}}[\text{variety}]}(\text{Fantasy})\}.
\end{aligned}
$$

Hence, $\widetilde{\mathrm{P}}[\text{variety}]$ is not a PFUPI of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}, A)$ is not a variety-PFSUPI of $\mathcal{U}$.

Theorem 2.38. *The extended intersection of two PFSUPIs of $\mathcal{U}$ is also a PFSUPI. Moreover, the intersection of two PFSUPIs of $\mathcal{U}$ is also a PFSUPI.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPIs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cap}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPI of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPI of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.17, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ is a PFUPI of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$. □

Theorem 2.39. *The union of two PFSUPIs of $\mathcal{U}$ is also a PFSUPI if sets of statistics of two PFSUPIs are disjoint.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSUPIs of $\mathcal{U}$ such that $A_1 \cap A_2 = \emptyset$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Since $A_1 \cap A_2 = \emptyset$, we have $a \in A_1 \setminus A_2$ or $a \in A_2 \setminus A_1$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFUPI of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFUPI of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSUPI of $\mathcal{U}$. □

The following example shows that Theorem 2.39 is not valid if sets of statistics of two PFSUPIs are not disjoint.

Example 2.40. In Example 2.31, we have $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are PFSUPIs of $\mathcal{U}$. Since moisture $\in A_1 \cap A_2$, we have

$$
\begin{aligned}
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Winter} \star \text{Summer})\\
&= \mu_{\widetilde{\mathrm{P}}_1[\text{moisture}] \vee \widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Summer})\\
&= 0.3
\end{aligned}
$$

$$\begin{aligned}
&\ngeq 0.5\\
&= \min\{0.8, 0.5\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}]\vee\widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Rains}),\\
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}]\vee\widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Spring})\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}]\vee\widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Winter} \star (\text{Spring} \star \text{Summer})),\\
&\mu_{\widetilde{\mathrm{P}}_1[\text{moisture}]\vee\widetilde{\mathrm{P}}_2[\text{moisture}]}(\text{Spring})\}.
\end{aligned}$$

Thus $\widetilde{\mathrm{P}}_1[\text{moisture}]\vee\widetilde{\mathrm{P}}_2[\text{moisture}]$ is not a PFUPI of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a moisture-PFSUPI of $\mathcal{U}$. Hence, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPI of $\mathcal{U}$. Moreover, $(\widetilde{\mathrm{P}}_1, A_1)\widetilde{\Cup}(\widetilde{\mathrm{P}}_2, A_2)$ is not a PFSUPI of $\mathcal{U}$.

2.5. Pythagorean Fuzzy Soft Strong UP-Ideals.

Definition 2.41. A PFSS $(\widetilde{\mathrm{P}}, A)$ over $\mathcal{U}$ is called a *Pythagorean fuzzy soft strong UP-ideal* (PFSSUPI) based on $a \in A$ (we shortly call an *a-Pythagorean fuzzy soft strong UP-ideal* (a-PFSSUPI)) of $\mathcal{U}$ if a PFS $\widetilde{\mathrm{P}}[a]$ in $\mathcal{U}$ is a PFSUPI. If $\widetilde{\mathrm{P}}[a]$ is an a-PFSSUPI of $\mathcal{U}$ for all $a \in A$, we say that $\widetilde{\mathrm{P}}[a]$ is a *PFSSUPI* of $\mathcal{U}$.

Theorem 2.42. $(\widetilde{\mathrm{P}}, A)$ *is a PFSSUPI of* $\mathcal{U}$ *if and only if* $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ *and* $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ *are, if the sets are nonempty, SUPIs for every* $a \in A, t \in [0, 1]$.

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. Let $t \in [0, 1]$ be such that $U(\mu_{\widetilde{\mathrm{P}}[a]}, t), L(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.7, we have $U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are SUPIs of $\mathcal{U}$ for all $a \in A, t \in [0, 1]$.

Conversely, assume for all $a \in A, t \in [0, 1], U(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are SUPIs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.7, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$. □

Theorem 2.43. $(\widetilde{\mathrm{P}}, A)$ *is a PFSSUPI of* $\mathcal{U}$ *if and only if* $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ *and* $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ *are, if the sets are nonempty, SUPIs for every* $a \in A, t \in [0, 1]$.

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. Let $t \in [0, 1]$ be such that $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t), L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t) \neq \emptyset$. By Theorem 1.8, we have $U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are SUPIs of $\mathcal{U}$ for all $a \in A, t \in [0, 1]$.

Conversely, assume for all $a \in A, t \in [0, 1], U^+(\mu_{\widetilde{\mathrm{P}}[a]}, t)$ and $L^-(\nu_{\widetilde{\mathrm{P}}[a]}, t)$ are SUPIs of $\mathcal{U}$ if the sets are nonempty. By Theorem 1.8, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$. □

Theorem 2.44. $(\widetilde{\mathrm{P}}, A)$ *is a PFSSUPI of* $\mathcal{U}$ *if and only if* $E(\mu_{\widetilde{\mathrm{P}}[a]}, \mu_{\widetilde{\mathrm{P}}[a]}(0))$ *and* $E(\nu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]}(0))$ *are SUPIs of* $\mathcal{U}$.

Proof. Assume $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$, that is, $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. By Theorem 1.9, we have $E(\mu_{\widetilde{\mathrm{P}}[a]}, \mu_{\widetilde{\mathrm{P}}[a]}(0))$ and $E(\nu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]}(0))$ are SUPIs of $\mathcal{U}$.

Conversely, assume for all $a \in A, E(\mu_{\widetilde{\mathrm{P}}[a]}, \mu_{\widetilde{\mathrm{P}}[a]}(0))$ and $E(\nu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]}(0))$ are SUPIs of $\mathcal{U}$. By Theorem 1.9, we have $\widetilde{\mathrm{P}}[a] = (\mu_{\widetilde{\mathrm{P}}[a]}, \nu_{\widetilde{\mathrm{P}}[a]})$ is a PFSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$. □

The proof of the following theorem can be verified easily.

Theorem 2.45. *If $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$ and $\emptyset \neq B \subseteq A$, then $(\widetilde{\mathrm{P}}|_B, B)$ is a PFSSUPI of $\mathcal{U}$.*

From Figure 1, we have the following theorems.

Theorem 2.46. *a-PFSSUPI and a-CPFSS coincide in $\mathcal{U}$. Moreover, PFSSUPI and CPFSS coincide in $\mathcal{U}$.*

Theorem 2.47. *Every a-PFSSUPI of $\mathcal{U}$ is an a-PFSUPI. Moreover, every PFSSUPI of $\mathcal{U}$ is a PFSUPI.*

The following example shows that the converse of Theorem 2.47 is not true.

Example 2.48. Let $\mathcal{U}$ be a set of four games of E-sports, that is,

$$\mathcal{U} = \{\text{DOTA, Pokemon, Call of Duty, FIFA}\}.$$

Define binary operation $\star$ on $\mathcal{U}$ as the following Cayley table:

$\star$	DOTA	FIFA	Call of Duty	Pokemon
DOTA	DOTA	FIFA	Call of Duty	Pokemon
Pokemon	DOTA	DOTA	FIFA	Pokemon
Call of Duty	DOTA	DOTA	DOTA	Pokemon
FIFA	DOTA	FIFA	Call of Duty	DOTA

Then $\mathcal{U} = (\mathcal{U}, \star, \text{DOTA})$ is a UP-algebra. Let $(\widetilde{\mathrm{P}}, A)$ be a PFSS over $\mathcal{U}$ where

$$A := \{\text{pressure, planning, relaxation}\}$$

with $\widetilde{\mathrm{P}}$[pressure], $\widetilde{\mathrm{P}}$[planning], and $\widetilde{\mathrm{P}}$[relaxation] are PFSs in $\mathcal{U}$ defined as follows:

$\widetilde{\mathrm{P}}$	DOTA	FIFA	Call of Duty	Pokemon
pressure	$(1, 0)$	$(0.7, 0.3)$	$(0.7, 0.3)$	$(0.2, 0.8)$
planning	$(0.8, 0.4)$	$(0.6, 0.6)$	$(0.6, 0.6)$	$(0.3, 0.9)$
relaxation	$(0.2, 0.4)$	$(0.3, 0.4)$	$(0.3, 0.6)$	$(0.6, 0.4)$

Then $(\widetilde{\mathrm{P}}, A)$ is a planning-PFSUPI of $\mathcal{U}$. But $(\widetilde{\mathrm{P}}, A)$ is not a planning-PFSSUPI of $\mathcal{U}$ since

$\mu_{\widetilde{\mathrm{P}}[\text{planning}]}$
(Call of Duty)
$= 0.6$

$$
\begin{aligned}
&\not\geq 0.8\\
&= \min\{0.8, 0.8\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA}), \mu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA})\}\\
&= \min\{\mu_{\widetilde{\mathrm{P}}[\text{planning}]}((\text{Call of Duty} \star \text{DOTA}) \star (\text{Call of Duty} \star \text{Call of Duty})),\\
&\quad \mu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA})\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu_{\widetilde{\mathrm{P}}[\text{planning}]}\\
&(\text{Call of Duty})\\
&= 0.6\\
&\not\leq 0.4\\
&= \max\{0.4, 0.4\}\\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA}), \nu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA})\}\\
&= \max\{\nu_{\widetilde{\mathrm{P}}[\text{planning}]}((\text{Call of Duty} \star \text{DOTA}) \star (\text{Call of Duty} \star \text{Call of Duty})),\\
&\quad \nu_{\widetilde{\mathrm{P}}[\text{planning}]}(\text{DOTA})\}.
\end{aligned}
$$

Hence, $\widetilde{\mathrm{P}}[\text{planning}]$ is not a PFSUPI of $\mathcal{U}$, that is, $(\widetilde{\mathrm{P}}, A)$ is not a planning-PFSSUPI of $\mathcal{U}$.

Theorem 2.49. *The extended intersection of two PFSSUPIs of $\mathcal{U}$ is also a PFSSUPI. Moreover, the intersection of two PFSSUPIs of $\mathcal{U}$ is also a PFSSUPI.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSSUPIs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cap}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFSUPI of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFSUPI of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.19, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \wedge \widetilde{\mathrm{P}}_2[a]$ is a PFSUPI of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$. □

Theorem 2.50. *The union of two PFSSUPIs of $\mathcal{U}$ is also a PFSSUPI. Moreover, the restricted union of two PFSSUPIs of $\mathcal{U}$ is also a PFSSUPI.*

Proof. Assume that $(\widetilde{\mathrm{P}}_1, A_1)$ and $(\widetilde{\mathrm{P}}_2, A_2)$ are two PFSSUPIs of $\mathcal{U}$. We denote $(\widetilde{\mathrm{P}}, A_1)\widetilde{\cup}(\widetilde{\mathrm{P}}_2, A_2)$ by $(\widetilde{\mathrm{P}}, A)$ where $A = A_1 \cup A_2$. Next, let $a \in A$.

Case 1: $a \in A_1 \setminus A_2$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a]$ is a PFSUPI of $\mathcal{U}$.

Case 2: $a \in A_2 \setminus A_1$. Then $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_2[a]$ is a PFSUPI of $\mathcal{U}$.

Case 3: $a \in A_1 \cap A_2$. By Theorem 1.19, we have $\widetilde{\mathrm{P}}[a] = \widetilde{\mathrm{P}}_1[a] \vee \widetilde{\mathrm{P}}_2[a]$ is a PFSUPI of $\mathcal{U}$.

Thus $(\widetilde{\mathrm{P}}, A)$ is an a-PFSSUPI of $\mathcal{U}$ for all $a \in A$. Hence, $(\widetilde{\mathrm{P}}, A)$ is a PFSSUPI of $\mathcal{U}$. □

3. Conclusions and Future Works

In this paper, we introduced five types of PFSSs of UP-algebras and proved that the concept of PFSUPSs is a generalization of PFSNUPFs, PFSNUPFs is a generalization of PFSUPFs, PFSUPFs is a generalization of PFSUPIs, and PFSUPIs is a generalization of PFSSUPIs. Furthermore, they proved that PFSSUPIs and CPFSSs coincide. We got the diagram of generalization of PFSSs over UP-algebras, which is shown with Figure 2.

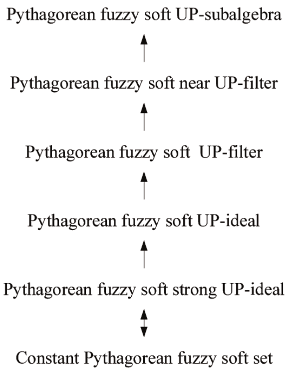

Figure 2. PFSSs over UP-algebras

After, we found that the (extended) intersection of two PFSUPSs (resp., PFSNUPFs, PFSUPFs, PFSUPIs, PFSSUPIs) is also a PFSUPS (resp., PFSNUPF, PFSUPF, PFSUPI, PFSSUPI) but the (restricted) union is not satisfy except PFSNUPFs and PFSSUPIs.

Finally, we connected between PFSSs and special subset of UP-algebras under upper t-level subsets, upper t-strong level subsets, lower t-level subsets, lower t-strong level subsets, and equal t-level subset of PFSs.

Research topics that will expand on this study in the near future include:

(1) to study Fermatean fuzzy sets based on the concept of Senapati and Yager [35],
(2) to introduce the concept of bipolar Pythagorean fuzzy soft sets based on the concept of Jana and Pal [11],
(3) to study Pythagorean fuzzy sets based on Pythagorean fuzzy points and Pythagorean fuzzy numbers according to Jana et al.'s approach [15, 12].

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : This work was supported by the revenue budget in 2022, School of Science, University of Phayao, Thailand (Grant No. PBTSC65020).

References

1. M.A. Ansari, A. Haidar, and A.N.A. Koam, *On a graph associated to UP-algebras*, Math. Comput. Appl. **23** (2018), Article number: 61.
2. M.A. Ansari, A.N.A. Koam, and A. Haider, *Rough set theory applied to UP-algebras*, Ital. J. Pure Appl. Math. **42** (2019), 388-402.
3. K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. **20** (1986), 87-96.
4. N. Dokkhamdang, A. Kesorn, and A. Iampan, *Generalized fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform. **16** (2018), 171-190.
5. T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, *Fuzzy translations of a fuzzy set in UP-algebras*, J. Indones. Math. Soc. **23** (2017), 1-19.
6. A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Top. **5** (2017), 35-54.
7. A. Iampan, *Introducing fully UP-semigroups*, Discuss. Math., Gen. Algebra Appl. **38** (2018), 297-306.
8. A. Iampan, *Multipliers and near UP-filters of UP-algebras*, J. Discrete Math. Sci. Cryptography **24** (2021), 667-680.
9. A. Iampan, M. Songsaeng, and G. Muhiuddin, *Fuzzy duplex UP-algebras*, Eur. J. Pure Appl. Math. **13** (2020), 459-471.
10. C. Jana, H. Garg, and M. Pal, *Multi-attribute decision making for power Dombi operators under Pythagorean fuzzy information with MABAC method*, J. Ambient Intell. Human. Comput. (2022), in press.
11. C. Jana and M. Pal, *Application of bipolar intuitionistic fuzzy soft sets in decision making problem*, Int. J. Fuzzy Syst. Appl. **7** (2018), 32-55.
12. C. Jana and M. Pal, *Extended bipolar fuzzy EDAS approach for multi-criteria group decision-making process*, Comput. Appl. Math. **40** (2021), Article number: 9.
13. C. Jana, T. Senapati, and M. Pal, *Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures*, IGI Global, USA, 2019.
14. C. Jana, T. Senapati, and M. Pal, *Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making*, Int. J. Intell. Syst. **34** (2019), 2019-2038.
15. C. Jana, T. Senapati, K.P. Shum, and M. Pal, *Bipolar fuzzy soft subalgebras and ideals of BCK/BCI-algebras based on bipolar fuzzy points*, J. Intell. Fuzzy Syst. **37** (2019), 2785-2795.
16. P.K. Maji, R. Biswas, and A.R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. **9** (2001), 589-602.
17. D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl. **37** (1999), 19-31.
18. P. Mosrijai and A. Iampan, *A new branch of bialgebraic structures: UP-bialgebras*, J. Taibah Univ. Sci. **13** (2019), 450-459.
19. M. Palanikumar, K. Arulmozhi, and C. Jana, *Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued fuzzy aggregation operators*, Comput. Appl. Math. **41** (2022), Article number: 90.
20. X. Peng, Y. Yang, J. Song, and Y. Jiang, *Pythagorean fuzzy soft set and its application*, Comput. Eng. **41** (2015), 224-229.
21. C. Prabpayak and U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna. **5** (2009), 54-57.
22. A. Rehman, S. Abdullah, M. Aslam, and M.S. Kamran, *A study on fuzzy soft set and its operations*, Ann. Fuzzy Math. Inform. **6** (2013), 339-362.
23. A. Satirad, R. Chinram, and A. Iampan, *Pythagorean fuzzy sets in UP-algebras and approximations*, AIMS Math. **6** (2021), 6002-6032.

24. A. Satirad, R. Chinram, P. Julatha, and A. Iampan, *New types of rough Pythagorean fuzzy UP-filters of UP-algebras*, J. Math. Comput. Sci. **28** (2022), 236-257.
25. A. Satirad, R. Chinram, P. Julatha, and A. Iampan, *Rough Pythagorean fuzzy sets in UP-algebras*, Eur. J. Pure Appl. Math. **15** (2022), 169-198.
26. A. Satirad and A. Iampan, *Fuzzy sets in fully UP-semigroups*, Ital. J. Pure Appl. Math. **42** (2019), 539-558.
27. A. Satirad and A. Iampan, *Fuzzy soft sets over fully UP-semigroups*, Eur. J. Pure Appl. Math. **12** (2019), 294-331.
28. A. Satirad and A. Iampan, *Properties of operations for fuzzy soft sets over fully UP-semigroups*, Int. J. Anal. Appl. **17** (2019), 821-837.
29. A. Satirad and A. Iampan, *Topological UP-algebras*, Discuss. Math., Gen. Algebra Appl. **39** (2019), 231-250.
30. A. Satirad, P. Mosrijai, and A. Iampan, *Formulas for finding UP-algebras*, Int. J. Math. Comput. Sci. **14** (2019), 403-409.
31. A. Satirad, P. Mosrijai, and A. Iampan, *Generalized power UP-algebras*, Int. J. Math. Comput. Sci. **14** (2019), 17-25.
32. A. Satirad, P. Mosrijai, W. Kamti, and A. Iampan, *Level subsets of a hesitant fuzzy set on UP-algebras*, Ann. Fuzzy Math. Inform. **14** (2017), 279-302.
33. T. Senapati, Y.B. Jun, and K.P. Shum, *Cubic set structure applied in UP-algebras*, Discrete Math. Algorithms Appl. **10** (2018), 1850049.
34. T. Senapati, G. Muhiuddin, and K.P. Shum, *Representation of UP-algebras in interval-valued intuitionistic fuzzy environment*, Ital. J. Pure Appl. Math. **38** (2017), 497-517.
35. T. Senapati and R.R. Yager, *Fermatean fuzzy sets*, J. Ambient Intell. Human. Comput. **11** (2020), 663-674.
36. J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform. **12** (2016), 739-756.
37. V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. **25** (2010), 529-539.
38. V. Torra and Y. Narukawa, *On hesitant fuzzy sets and decision*, 18th IEEE Int. Conf. Fuzzy Syst. (2009), 1378-1382.
39. M. Touqeer, *Intuitionistic fuzzy soft set theoretic approaches to α-ideals in BCI-algebras*, Fuzzy inf. Eng. **12** (2020), 150-180.
40. R.R. Yager, *Pythagorean fuzzy subsets*, 2013 Jt. IFSA World Congr. NAFIPS Annu. Meet. (IFSA/NAFIPS), Edmomton, Canada, 2013, 57-61.
41. R.R. Yager and A.M. Abbasov, *Pythagorean member grades, complex numbers, and decision making*, Int. J. Intell. Syst. **28** (2013), 436-452.
42. L.A. Zadeh, *Fuzzy sets*, Inf. Cont. **8** (1965), 338-353.

Akarachai Satirad was born in Phayao, Thailand, in 1995. He is a Ph.D. student at Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. He received his B.Sc. and M.Sc. degrees in Mathematics from University of Phayao, Phayao, Thailand, under the thesis advisor of Associate Professor Dr. Aiyared Iampan. His areas of interest include fuzzy algebraic structures and logical algebras.

Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, Mae Ka, University of Phayao, Phayao 56000, Thailand.
`e-mail: akarachai.sa@gmail.com`

Rukchart Prasertpong is a faculty member of the Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Thailand. His areas of interest include algebraic theory of semigroups and Γ-semigroups, fuzzy algebraic structures, and rough algebraic structures.

Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand.
e-mail: `rukchart.p@nsru.ac.th`

Pongpun Julatha is a faculty member of the Faculty of Science and Technology, Pibulsongkram Rajabhat University, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Thailand. His areas of interest include algebraic theory of semigroups, ternary semigroups, and Γ-semigroups and fuzzy algebraic structures.

Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand.
e-mail: `pongpun.j@psru.ac.th`

Ronnason Chinram was born in Ranong, Thailand, in 1975. He received his M.Sc. and Ph.D. from Chulalongkorn University, Thailand. Since 1997, he has been with Prince of Songkla University, Thailand and now he is an associate professor in Mathematics. Moreover, he is the Head of Division of Computational Science and the Head of Algebra and Applications Research Unit. Now, he has more than 100 research publications in international well reputed journals. His research interests focus on semigroup theory, algebraic systems, fuzzy mathematics and decision-making problems.

Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand.
e-mail: `ronnason.c@psu.ac.th`

Aiyared Iampan was born in Nakhon Sawan, Thailand, in 1979. He is an Associate Professor at Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Phitsanulok, Thailand, under the thesis advisor of Professor Dr. Manoj Siripitukdet. His areas of interest include algebraic theory of semigroups, ternary semigroups, and Γ-semigroups, lattices and ordered algebraic structures, fuzzy algebraic structures, and logical algebras. He was the founder of the Group for Young Algebraists in University of Phayao in 2012 and one of the co-founders of the Fuzzy Algebras and Decision-Making Problems Research Unit in University of Phayao in 2021.

Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, Mae Ka, University of Phayao, Phayao 56000, Thailand.
e-mail: `aiyared.ia@up.ac.th`