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STUDY OF DYNAMICAL MODEL FOR PIEZOELECTRIC CYLINDER IN FRICTIONAL ANTIPLANE CONTACT PROBLEM

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Abstract. We propose a mathematical model which describes the frictional contact between a piezoelectric body and an electrically conductive foundation. The behavior of the material is described with a linearly electro-viscoelastic constitutive law with long term memory. The mechanical process is dynamic and the electrical conductivity coefficient depends on the total slip rate, the friction is modeled with Tresca's law which the friction bound depends on the total slip rate with taking into account the electrical conductivity of the foundation both. The main results of this paper concern the existence and uniqueness of the weak solution of the model; the proof is based on results for second order evolution variational inequalities with a time-dependent hemivariational inequality in Banach spaces.

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1. Introduction

The present paper concerns the study of contact problems involving piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-elastic. Background of the theory of piezoelectric materials can be found in [6, 13, 20]. General models for three-dimensional linear can be found in [1].

Antiplane shear deformations are one of the simplest examples of deformations that solids can undergo: in antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial

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coordinate. Mathematical and mechanical state of the art on contact mechanics can be found in [22, 23].

This work is interested in dynamic contact problem for electro-viscoelastic materials with long term memory in frictional contact with a conductive foundation; (more details for dynamic contact problem see [17]). We assume that the contact is bilateral, i.e. there is no loss of the contact during the process, moreover, the friction bound and the electric conductivity coefficient are assumed to depend on the total slip rate with taking into account the electrical conductivity of the foundation both in the friction law and in the electrical condition on the contact surface.

The present paper represents a continuation of [4], there a mathematical model which describes the antiplane shear deformation of an electro-viscoelastic cylinder with long term memory in frictional contact and a rigid foundation was assumed to be electrically conductive, the process was assumed to be mechanically quasistatic and electrically static, therefore, the variational formulation of this model was given by a system of two coupled an evolutionary variational equality for the displacement field and a time-dependent variational equation for the potential field, which was solved by using results on integro-differential inequality. In this work, we consider a similar physical setting like [4], in which the material behavior is modeled with a linear electro-viscoelastic constitutive law with long term memory, but the mechanical process is assumed to be dynamic and the electrical conductivity coefficient depends on the total slip rate. In a quasistatic version of Tresca's friction law, we assumed that the friction bound depends on the total slip rate and on difference between the potential on the foundation and the body surface; this dependence of friction bound is used in the study of antiplane problems with a total slip rate dependent or since the foundation is supposed to be electrically conductive. We derive a variational formulation of the problem which is of the form of a system coupling a second order evalutionary variational inequality for the displacement field with a time-dependent hemivariational inequality for the electric potential field.

The rest of the paper is structured as follows. In Section 2 we describe the model of frictional contact between an electro-viscoelastic cylinder body with long memory and a conductive foundation. In Section 3 we derive the variational formulation of this problem, which is in the form of a system coupling a class of evolutionary variational inequality for the displacement field with a time dependent hemivariational inequality for the electric potential with integrals terms. In Section 4 we need to present a main result in which we apply it in our proof of theorem 3.1 in section 5; the proof is based on arguments of evolutionary variational and a time-dependent hemivariational inequalities in Banach spaces .

2. Statement of the problem

We consider a piezoelectric body $\mathbf{B}$ identified with a region in $\mathbb{R}^3$, it occupies in a fixed and undistorted reference configuration. We assume that $\mathbf{B}$ is a cylinder with generators parallel to the x_3-axes having a cross section that is a regular region $\Omega \subset \mathbb{R}^2$ in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The effects in the axial direction are negligible since the cylinder is assumed to be sufficiently long, thus, $\mathbf{B} = \Omega \times (-\infty, +\infty)$. We assume that the material is nonhomogeneous and we model its behavior with the linear isotropic version of Tresca's law.

The body which occupies a bounded domain Ω is submitted to the action of body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary.

To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, such that $meas\ \Gamma_1 > 0$. We assume that the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and it is in contact with a rigid foundation on $\Gamma_3 \times (-\infty, +\infty)$.

On the one hand, we consider a partition of $\Gamma_1 \times (-\infty, +\infty) \cup \Gamma_2 \times (-\infty, +\infty)$ into two open parts $\Gamma_a \times (-\infty, +\infty)$ and $\Gamma_b \times (-\infty, +\infty)$, such that $meas\ \Gamma_a > 0$. On the other hand, the cylinder is subjected to time-dependent volume forces of density $\mathbf{f}_0$ in $\Omega \times (-\infty, +\infty)$ and to time-dependent surface tractions of density $\mathbf{f}_2$ on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes in $\Gamma_a \times (-\infty, +\infty)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, +\infty)$.

We are interested in the deformation of this body on the time interval of interest $[0, T]$, with $T > 0$. Everywhere in this paper, the dots represent the derivatives with respect to time, i.e. $\dot{u} = \frac{\partial u}{\partial t}$, $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$ and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable, i.e. $u_{,i} = \frac{\partial u}{\partial x_i}$, $i = 1, 2$.

The indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and " $\cdot$ ", $\|\cdot\|$ will represent the inner products and the Euclidean norm on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \text{for all } \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^3,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

We assume that the forces and the electric charges are given by

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{1}$$
$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2}$$
$$q_0 = q_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{3}$$
$$q_2 = q_2(x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{4}$$

The displacement $\mathbf{u}$ and the electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0,0,u) \quad \text{with } u = u(x_1,x_2,t) : \Omega \times [0,T] \to \mathbb{R}, \tag{5}$$

$$\varphi = \varphi(x_1,x_2,t) : \Omega \times [0,T] \to \mathbb{R}. \tag{6}$$

The strain tensor $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and and the electric field $\mathbf{E}(\varphi) = (E_i(\varphi))$

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}\,.$$

The stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})$ and the electric displacement field $\mathbf{D} = (D_i)$ are modeled the material behavior by a linear isotropic electro-viscoelastic constitutive law with long term memory of the form

$$\begin{aligned} &\boldsymbol{\sigma} = \lambda(tr\varepsilon(\mathbf{u}))\mathbf{I} + 2\mu\varepsilon(\mathbf{u}) + 2\int_0^t \theta(t-s)\varepsilon(\mathbf{u}(s))ds \\ &+\int_0^t \zeta(t-s)tr(\varepsilon(\mathbf{u}(s)))\mathbf{I}\,ds - \mathcal{E}^*\mathbf{E}(\varphi), \end{aligned} \tag{7}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta\mathbf{E}(\varphi), \tag{8}$$

where ζ and θ are relaxation coefficients that are time dependent, λ and μ are the Lamé coefficients, tr $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \tag{9}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon}\cdot\mathbf{v} = \boldsymbol{\varepsilon}\cdot\mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon} \in \mathcal{S}^3$, $\mathbf{v} \in \mathbb{R}^3$, it follows from (9) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{10}$$

In the antiplane context (5), (6), using the constitutive equations (7), (8) and equalities (9), (10) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}$$

$$\begin{cases} \sigma_{13}(t) = \sigma_{31}(t) = \mu u_{,1}(t) + \int_0^t \theta(t-s)u_{,1}(s)ds + e\varphi_{,1}(t), \\ \sigma_{23}(t) = \sigma_{32}(t) = \mu u_{,2}(t) + \int_0^t \theta(t-s)u_{,2}(s)ds + e\varphi_{,2}(t). \end{cases} \tag{11}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}. \tag{12}$$

We assume that the process is mechanically dynamic and electrically static and, therefore, is governed by the balance equations

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{\mathbf{u}}, \quad \operatorname{Div}\mathbf{D} - q_0 = 0 \quad \text{in } \mathcal{B} \times (0,T),$$

where $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$ and ρ denotes the density of mass. Taking into account (11), (12), (5), (6), (1) and (3), the equilibrium equations above reduce to the following scalar equations

$$\operatorname{div}\Big(\mu\nabla u(t) + \int_0^t \theta(t-s)\nabla u(s)ds + e\nabla\varphi(t)\Big) + f_0(t) = \rho\ddot{u}(t), \quad \text{in } \Omega, \forall t \in [0,T], \tag{13}$$

$$\operatorname{div}(e\nabla u(t) - \beta\nabla\varphi(t)) = q_0(t), \quad \text{in } \Omega, \forall t \in [0,T]. \tag{14}$$

During the process the cylinder is clamped on $\Gamma_1 \times (-\infty,+\infty) \times (0,T)$ and the electric potential vanishes on $\Gamma_a \times (-\infty,+\infty) \times (0,T)$; thus, (5) and (6) imply that

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{15}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T). \tag{16}$$

The Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma\nu}(t) = \Big(0,0,\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t)\Big), \mathbf{D}\cdot\boldsymbol{\nu}(t) = e\partial_\nu u(t) - \beta\partial_\nu\varphi(t), \tag{17}$$

where $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty,+\infty)$ defined by

$$\boldsymbol{\nu} = (\nu_1,\nu_2,0) \quad \text{with} \quad \nu_i = \nu_i(x_1,x_2) : \Gamma \to \mathbb{R}, \quad i = 1,2. \tag{18}$$

Taking into account (2), (4) and (17), the traction condition on Γ_2 and the electric conditions on Γ_b are given by

$$\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t) = f_2(t) \quad \text{on } \Gamma_2, \forall t \in [0,T], \tag{19}$$

$$e\partial_\nu u(t) - \beta\partial_\nu\varphi(t) = q_2(t) \quad \text{on } \Gamma_b, \forall t \in [0,T]. \tag{20}$$

The contact is bilateral, so the boundary conditions on $\Gamma_3 \times (-\infty,+\infty)$ are

$$\mathbf{u}_\tau(t) = (0,0,u(t)), \quad \boldsymbol{\sigma}_\tau(t) = \Big(0,0,\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t)\Big). \tag{21}$$

In the antiplane shear context, we assume that the friction is modeled with the dependence of g on both $\mathbf{S}(\dot{u})$ and the electric variables $\varphi - \varphi_F$ (since the foundation is supposed to be electrically conductive) by the following form

$$\begin{cases} |\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu \varphi(t)| \leq g\Big(\int_0^t |\dot{u}(s)|ds, \varphi(t) - \varphi_F\Big), \\ \mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu \varphi(t) = -g\Big(\int_0^t |\dot{u}(s)|ds, \varphi(t) - \varphi_F\Big)\frac{\dot{u}(t)}{|\dot{u}(t)|}, \dot{u}(t) \neq 0, \end{cases} \tag{22}$$

and the electric contact is modeled by

$$e\partial_\nu u(t) - \beta\partial_\nu \varphi(t) = k\big(\int_0^t |\dot{u}(s)|ds\big)(\varphi - \varphi_F) \quad \text{on } \Gamma_3, \forall t \in [0,T], \tag{23}$$

where $S\dot{u}(t) = \int_0^t |\dot{u}(s)|ds$, for all $t \in [0,T]$, represents the total slip rate, φ_F represents the electric potential of the foundation assumed to be given and the electric charges on the contact surface are proportional to the difference of potential $(\varphi - \varphi_F)$ with a total slip rate dependent proportionality coefficient.

Finally, we prescribe the initials displacement and velocity by

$$u(0) = u_0, \ \dot{u}(0) = u_1 \quad \text{in } \Omega, \tag{24}$$

where u_0 and u_1 are given functions in Ω. We have the following problem:
Problem $\mathcal{P}$: Find the displacement $u : \Omega \times [0,T] \to \mathbb{R}$ and the electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \operatorname{div}(\mu\nabla u(t) + \int_0^t \theta(t-s)\nabla u(s)ds + e\nabla\varphi(t)) + f_0(t) = \rho\ddot{u}(t) & \text{in } \Omega \\ \operatorname{div}(e\nabla u(t) + \beta\nabla\varphi(t)) = q_0(t) & \text{in } \Omega \\ u(t) = 0 & \text{on } \Gamma_1 \\ \varphi(t) = 0 & \text{on } \Gamma_a \\ \mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t) = f_2(t) & \text{in } \Gamma_2 \\ e\partial_\nu u(t) - \beta\partial_\nu\varphi(t) = q_2(t) & \text{in } \Gamma_b \\ \begin{cases} |\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t)| \leq g\Big(\int_0^t |\dot{u}(s)|ds, \varphi - \varphi_F\Big) \\ \mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t) = -g\Big(\int_0^t |\dot{u}(s)|ds, \varphi - \varphi_F\Big)\frac{\dot{u}(t)}{|\dot{u}(t)|}, \end{cases} & \text{on } \Gamma_3 \\ e\partial_\nu u(t) - \beta\partial_\nu\varphi(t) = k(S\dot{u}(t))(\varphi(t) - \varphi_F) & \text{on } \Gamma_3 \\ u(0) = u_0, \ \dot{u}(0) = u_1 & \text{in } \Omega \end{cases} \tag{25}$$

for all $t \in [0,T]$.

3. Variational formulation and main results

We introduce the function spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\, \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\}$$

where, here and below, we write w for the trace γw of a function $w \in H^1(\Omega)$ on Γ. Since $\operatorname{meas}\Gamma_1 > 0$ and $\operatorname{meas}\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (\varphi, \psi)_W = \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx \quad \forall \varphi,\, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^2)} \quad \forall v \in V, \quad \|\psi\|_W = \|\nabla \psi\|_{L^2(\Omega;\mathbb{R}^2)} \quad \forall \psi \in W. \tag{26}$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \leq c_V \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq c_W \|\psi\|_W \quad \forall \psi \in W. \tag{27}$$

Given a real Banach space $(X, \|.\|_X)$ we use the standard notations for the Lebesgue space $L^2(0, T; X)$ and $L^\infty(0, T; X)$ as well as the Sobolev space $W^{1,2}(0, T; X)$. We recall that the norms on this spaces are given by

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|^2_X \, dt,$$

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X,$$

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|^2_X \, dt + \int_0^T \|\dot{u}(t)\|^2_X \, dt.$$

We use the notation $W^{1,2}(0, T)$ for the space $W^{1,2}(0, T; \mathbb{R})$ and the notation $\|.\|_{W^{1,2}(0,T)}$ for the norm $\|.\|_{W^{1,2}(0,T;\mathbb{R})}$.

In the study of problem $\mathcal{P}$ we assume that the mass density, the viscosity coefficient, the Lamé coefficient, the electric permittivity coefficient and the piezoelectric coefficient satisfy

$$\mathbf{H}(c): \begin{cases} a)\rho \in L^\infty(\Omega) \text{ and there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \geq \rho^* \\ \quad \text{a.e. } \mathbf{x} \in \Omega. \\ b)\theta \in W^{1,2}(0, T). \\ c)\mu \in L^\infty(\Omega) \text{ and there exists } \mu^* > 0 \text{ such that } \mu(\mathbf{x}) \geq \mu^* \\ \quad \text{a.e. } \mathbf{x} \in \Omega. \\ d)\beta \in L^\infty(\Omega) \text{and there exists } \beta^* > 0 \text{ such that } \beta(\mathbf{x}) \geq \beta^* \\ \quad \text{a.e. } \mathbf{x} \in \Omega. \\ e)e \in L^\infty(\Omega). \end{cases}$$

The forces and surface free charge densities have the regularity

H(f)**:** $f_0 \in W^{1,2}(0,T,L^2(\Omega))$, $\quad f_2 \in W^{1,2}(0,T,L^2(\Gamma_2))$.
H(q)**:** $q_0 \in W^{1,2}(0,T,L^2(\Omega))$, $\quad q_2 \in W^{1,2}(0,T,L^2(\Gamma_b))$.

The functions g, k, j and J satisfy

$$\mathbf{H}(g)\textbf{:}\ \begin{cases} (\mathbf{a}) g : \Gamma_3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+; \\ (\mathbf{b}) \exists L_g \geq 0 \quad \text{such that} \\ \quad |g(x,r_1,s_1) - g(x,r_2,s_2)| \leq L_g ||(r_1,s_1) - (r_2,s_2)||, \\ \qquad \forall r_1, r_2, s_1, s_2 \in \mathbb{R} \quad \text{a.e.} \quad x \in \Gamma_3; \\ (\mathbf{c}) \forall r, s \in \mathbb{R}, \quad g(.,r,s) \text{ is Lebesgue measurable on } \quad \Gamma_3; \\ (\mathbf{d}) g(.,0,0) \in L^2(\Gamma_3). \end{cases}$$

$$\mathbf{H}(k)\textbf{:}\ \begin{cases} (\mathbf{a}) k : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+; \\ (\mathbf{b}) \exists L_k \geq 0 \quad \text{such that} \quad |k(x,r_1) - k(x,r_2)| \leq L_k |r_1 - r_2|, \\ \qquad \forall r_1, r_2 \in \mathbb{R} \quad \text{a.e.} \quad x \in \Gamma_3; \\ (\mathbf{c}) \forall r \in \mathbb{R}, \quad k(.,r) \text{ is Lebesgue measurable on } \quad \Gamma_3; \\ (\mathbf{d}) 0 \leq k(x,r) \leq k_0, \forall r \in \mathbb{R} \quad \text{a.e.} \quad x \in \Gamma_3, \text{ with } \ k_0 > 0. \end{cases}$$

$$\mathbf{H}(j)\textbf{:}\ \begin{cases} j : L^2(\Gamma_3) \times W \times V \to \mathbb{R} \text{ and}, \\ (\mathbf{a}) \forall \eta \in L^2(\Gamma_3), \forall v_1 \in W, j(\eta, v_1, .) \text{ is convex et i.s.c on } V. \\ (\mathbf{b}) \forall \eta \in L^2(\Gamma_3), \forall v_2 \in V, j(\eta, ., v_2) \text{ is convex et i.s.c on } W. \\ (\mathbf{c}) j(0,.,.) \in L^1(\Gamma_3). \end{cases}$$

$$\mathbf{H}(J)\textbf{:}\ \begin{cases} J : L^2(\Gamma_3) \times W^2 \to \mathbb{R} \text{ and}, \\ (\mathbf{a}) \forall \eta \in L^2(\Gamma_3), \forall v_1 \in W, J(\eta, v_1, .) \text{ is convex et i.s.c on } W. \\ (\mathbf{b}) \forall \eta \in L^2(\Gamma_3), \forall v \in W, J(\eta, ., v_2) \text{ is convex et i.s.c on } W. \\ (\mathbf{c}) J(0,.,.) \in L^1(\Gamma_3). \end{cases}$$

The initial data and the electric potential of the foundation are given by

H(0)**:** $(u_0, u_1) \in V \times L^2(\Omega)$.
H(φ_F)**:** $\varphi_F \in L^2(\Gamma_3)$.

Next, we define the bilinear forms $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e^* : W \times V \to \mathbb{R}$, and $a_\beta : W \times W \to \mathbb{R}$, by equalities

$$a_\mu(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, dx, \tag{28}$$

$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi \, dx = a_e^*(\varphi, u), \tag{29}$$

$$a_\beta(\varphi,\psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi \, dx, \tag{30}$$

for all u, $v \in V$ and $\varphi, \psi \in W$.

Assumptions **H**(c) imply that the integrals above are well defined and, using (26) and (27), it follows that the forms a_θ, a_μ, a_e, a_e^* and a_β are continuous; moreover, the form a_μ (resp. a_β) is symmetric and V-elliptic (resp. a_β is symmetric and W-elliptic), since

$$a_\mu(v,v) \geq \mu^* \|v\|_V^2 \quad \forall v \in V, \tag{31}$$

$$a_\beta(\psi,\psi) \geq \beta^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{32}$$

By assumptions $\mathbf{H}(f)$ and $\mathbf{H}(q)$, We can define the mappings $f : [0,T] \to V'$ and $q : [0,T] \to W'$ respectively, by

$$(f(t), v)_{V'\times V} = \int_{\Omega} f_0(t) v \, dx + \int_{\Gamma_2} f_2(t) v \, da, \tag{33}$$

$$(q(t), \psi)_{W'\times W} = \int_{\Omega} q_0(t) \psi \, dx - \int_{\Gamma_b} q_2(t) \psi \, da, \tag{34}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The assumptions $\mathbf{H}(f)$ and $\mathbf{H}(q)$ combined with the trace theorem implies that f and q have the regularity

$$f \in W^{1,2}(0,T;V'), \tag{35}$$

$$q \in W^{1,2}(0,T;W'). \tag{36}$$

We will use a modified inner product on $H = L^2(\Omega)$ besides the canonical inner product $(.,.)_H$. We use assumption $\mathbf{H}(c)$:a) define a modified inner product on H, given by

$$((u,v))_H = (\rho u, v)_{L^2(\Omega)} \quad \forall u, v \in H,$$

that is, it is weighted with ρ, and we let $|.|_H$ be the associated norm, i.e.,

$$|v|_H = (\rho v, v)^{\frac{1}{2}}_{L^2(\Omega)} \quad \forall v \in H.$$

It follows from assumption $\mathbf{H}(c)$:a) that $\|.\|_{L^2(\Omega)}$ and $|.|_H$ are equivalent norms on H, and the inclusion mapping of $(V, \|.\|_V)$ into $(H, |.|_H)$ is continuous and dense. We denote by V' the dual of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \equiv H' \subset V'.$$

Using the notation $(.,.)_{V'\times V}$ to represent the duality pairing between V' and V, we have

$$(u,v)_{V'\times V} = ((u,v))_H = (\rho u, v)_{L^2(\Omega)} \quad \forall u \in H, v \in V. \tag{37}$$

For every $t \in [0,T]$ we need to consider the operator $\mathbf{S}$ defined by

$$\mathbf{S} : W^{1,2}(0,T;V) \to W^{1,2}(0,T;L^2(\Gamma_3)),$$

$$\mathbf{S}v(t) = \int_0^t |v(s)| ds \quad \text{a.e. on} \quad \Gamma_3. \tag{38}$$

From (38), it follows that the for all $v_1, v_2 \in W^{1,2}(0,T;V)$ and $t \in [0,T]$, the following inequality holds

$$\|\mathbf{S}v_1(t) - \mathbf{S}v_2(t)\|_{L^2(\Gamma_3)} \le \int_0^t \|v_1(s) - v_2(s)\|_{L^2(\Gamma_3)} ds. \tag{39}$$

Here and bellow C represents a positive constant whose values may change from line to line. And From (27), the inequality (39) becomes

$$\|\mathbf{S}v_1(t) - \mathbf{S}v_2(t)\|_{L^2(\Gamma_3)} \leq C\int_0^t \|v_1(s) - v_2(s)\|_V ds. \tag{40}$$

We define now the functionals $j : (L^2(\Gamma_3) \times W) \times V \to \mathbb{R}_+$ and $J : L^2(\Gamma_3) \times W^2 \to \mathbb{R}_+$ respectively by

$$j((\eta, \varphi), v) = \int_{\Gamma_3} g(\eta, \varphi - \varphi_F)|v|da, \quad \forall \eta \in L^2(\Gamma_3), \forall \varphi \in W, v \in V. \tag{41}$$

$$J(\eta, \varphi, \psi) = \int_{\Gamma_3} k(\eta)|\varphi - \varphi_F||\psi|da, \quad \forall \eta \in L^2(\Gamma_3), \forall (\varphi, \psi) \in W^2. \tag{42}$$

We assume in what follows that the couple (u, φ) is a smooth solution to problem $\mathcal{P}$ and $t \in [0, T]$ such that $u(t) \in V$ and $\varphi(t) \in W$. Let $v \in V$ and $\psi \in W$. From (13), (15) and (19), we obtain

$$\begin{aligned}
&\int_\Omega \rho\ddot{u}(t)(v - \dot{u}(t))dx + \int_\Omega \Big(\mu\nabla u(t) + \int_0^t \theta(t-s)(\nabla u(s))ds + e\nabla\varphi(t)\Big)\nabla(v - \dot{u}(t))dx \\
&= \int_\Omega f_0(t)(v - \dot{u}(t))dx + \int_{\Gamma_2} f_2(t)(v - \dot{u}(t))da + \int_{\Gamma_3} \Big(\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds \\
&\qquad + e\partial_\nu\varphi(t)\Big)(v - \dot{u}(t))da,
\end{aligned}$$

and from (14), (16), (20) and (23), we obtain

$$\begin{aligned}
\int_\Omega \Big(e\nabla u(t) - \beta\nabla\varphi(t)\Big)\nabla\psi dx &= \int_{\Gamma_b} q_2(t)\psi da - \int_\Omega q_0(t)\psi dx \\
&\quad + \int_{\Gamma_3} k(S\dot{u}(t))(\varphi(t) - \varphi_F)\psi da.
\end{aligned}$$

Using the frictional contact condition (22) on $\Gamma_3 \times (0, T)$, we deduce that

$$\begin{aligned}
&\Big(\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi(t)\Big)(v - \dot{u}(t)) \\
&\qquad \geq g\big(\mathbf{S}\dot{u}(t), \varphi(t) - \varphi_F\big)|\dot{u}(t)| - g\big(\mathbf{S}\dot{u}(t), \varphi(t) - \varphi_F\big)|v|.
\end{aligned} \tag{43}$$

Using the electric contact condition (23) on $\Gamma_3 \times (0, T)$, we deduce that

$$k(S\dot{u}(t))(\varphi(t) - \varphi_F)\psi \leq k\big(\mathbf{S}\dot{u}(t)\big)|\varphi(t) - \varphi_F||\psi|. \tag{44}$$

From (28), (29), (37),(33) and (43) we obtain

$$(\ddot{u}(t), v-\dot{u}(t))_{V'\times V} + a_\mu(u(t), v-\dot{u}(t)) + \Big(\int_0^t \theta(t-s)u(s)ds, v-\dot{u}(t)\Big)_V$$
$$+ a_e^*(\varphi(t), v-\dot{u}(t)) + j((S\dot{u}(t),\varphi(t)), v) - j((S\dot{u}(t),\varphi(t)),\dot{u}(t))$$
$$\geq (f(t), v-\dot{u}(t))_{V'\times V},\ \forall v\in V,\ \text{p.p.}\, t\in[0,T].$$

From (29), (30), (34) and (44) we obtain

$$a_\beta(\varphi(t),\psi) - a_e(u(t),\psi) + J(S\dot{u}(t),\varphi(t),\psi) \geq (q(t),\psi)_{W'\times W},\ \forall\psi\in W,\ \text{p.p.}\, t\in[0,T].$$

We obtained the following variational formulation of problem $\mathcal{P}$:

Problem $\mathcal{P}_V$. Find the displacement $u:[0,T]\to V$ and the electric potential $\varphi:[0,T]\to W$ such that

$$(\ddot{u}(t), v-\dot{u}(t))_{V'\times V} + a_\mu(u(t), v-\dot{u}(t)) + \Big(\int_0^t \theta(t-s)u(s)ds, v-\dot{u}(t)\Big)_V$$
$$+ a_e^*(\varphi(t), v-\dot{u}(t)) + j((S\dot{u}(t),\varphi(t)), v) - j((S\dot{u}(t),\varphi(t)),\dot{u}(t))$$
$$\geq (f(t), v-\dot{u}(t))_{V'\times V}, \forall v\in V,\ t\in[0,T], \tag{45}$$

$$a_\beta(\varphi(t),\psi) - a_e(u(t),\psi) + J(S\dot{u}(t),\varphi(t),\psi) \geq (q(t),\psi)_{W'\times W},\ \forall\psi\in W,\ t\in[0,T]. \tag{46}$$

$$(u(0),\dot{u}(0)) = (u_0,u_1). \tag{47}$$

We have the following existence and uniqueness result.

Theorem 3.1. *Assume that* $\boldsymbol{H}(c)$,$\boldsymbol{H}(f)$,$\boldsymbol{H}(q)$, $\boldsymbol{H}(j)$,$\boldsymbol{H}(J)$,$\boldsymbol{H}(k)$, $\boldsymbol{H}(g)$,$\boldsymbol{H}(\varphi_F)$ *and* $\boldsymbol{H}(0)$ *hold. Then there exists a unique solution of problem* $\mathcal{P}_V$. *Moreover, the solution satisfies*

$$u\in W^{2,2}(0,T;V),\quad \ddot{u}\in L^2(0,T;V')\quad \textit{and}\quad \varphi\in W^{1,2}(0,T;W). \tag{48}$$

The proof of theorem 3.1 will be carried in several steps.

4. First existence and uniqueness result

We consider the following problem.

Problem $\mathcal{P}_V^0$. Find $u:[0,T]\to V$ and $\varphi:[0,T]\to W$ such that

$$(\ddot{u}(t), v-\dot{u}(t))_{V'\times V} + a_\mu(u(t), v-\dot{u}(t)) + j((S\dot{u}(t),\varphi(t)), v)$$
$$- j((S\dot{u}(t),\varphi(t)),\dot{u}(t)) \geq (f(t), v-\dot{u}(t))_{V'\times V}\quad \forall v\in V,\ t\in[0,T], \tag{49}$$

$$a_\beta(\varphi(t),\psi) - a_e(u(t),\psi) + J(S\dot{u}(t),\varphi(t),\psi) \geq (q(t),\psi)_{W'\times W}\ \forall\psi\in W,\ t\in[0,T]. \tag{50}$$

$$(u(0),\dot{u}(0)) = (u_0,u_1). \tag{51}$$

We have the following existence and uniqueness result.

Theorem 4.1. *Under hypotheses* ***H****(c)(a,c,d,e),* ***H****(f),****H****(q),* ***H****(j),****H****(J),****H****(k),* ***H****(g),****H***(φ_F) *and* ***H***(0)*, problem* $\mathcal{P}_V^0$ *admits a unique solution which satisfies*

$$u \in W^{2,2}(0,T;V), \quad \ddot{u} \in L^2(0,T;V') \quad \textit{and} \quad \varphi \in W^{1,2}(0,T;W). \tag{52}$$

The proof of theorem 4.1 will be carried in two steps:

Step.1. Solvability of problem $\mathcal{P}_{V_\xi}^0$.

Problem $\mathcal{P}_{V_\xi}^0$. For all $\xi \in W^{1,2}(0,T;L^2(\Gamma_3))$. Find $u_\xi : [0,T] \to V$ and $\varphi_\xi : [0,T] \to W$ such that

$$\begin{aligned}&(\ddot{u}_\xi(t), v - \dot{u}_\xi(t))_{V'\times V} + a_\mu(u_\xi(t), v - \dot{u}_\xi(t)) + j((\xi(t), \varphi_\xi(t)), v)\\ &\quad - j((\xi(t), \varphi_\xi(t)), \dot{u}_\xi(t)) \geq (f(t), v - \dot{u}_\xi(t))_{V'\times V} \quad \forall v \in V,\ t \in [0,T],\end{aligned} \tag{53}$$

$$a_\beta(\varphi_\xi(t), \psi) - a_e(u_\xi(t), \psi) + J(\xi(t), \varphi_\xi(t), \psi) \geq (q(t), \psi)_{W'\times W}\ \forall \psi \in W,\ t \in [0,T]. \tag{54}$$

$$(u_\xi(0), \dot{u}_\xi(0)) = (u_0, u_1), \tag{55}$$

Theorem 4.2. *For* $\xi \in W^{1,2}(0,T;L^2(\Gamma_3))$*. Under hypotheses* ***H****(c)(a,c,d,e),* ***H****(f),****H****(q),* ***H****(j),****H****(J),****H****(k),* ***H****(g),****H***(φ_F) *and* ***H***(0)*, problem* $\mathcal{P}_{V_\xi}^0$ *admits a unique solution which satisfies*

$$u_\xi \in W^{2,2}(0,T;V), \quad \ddot{u}_\xi \in L^2(0,T;V') \quad \textit{et} \quad \varphi_\xi \in W^{1,2}(0,T;W). \tag{56}$$

Proof. We note that (15) and (16) yield $u_\xi(t) \in V$ and $\varphi_\xi(t) \in W$. For $i = 1,2$, let $\xi_i \in W^{1,2}(0,T,L^2(\Gamma_3))$ and $(u_i(t), \varphi_i(t))$ be the solution to problem $\mathcal{P}_{V_\xi}^0$ for $\xi = \xi_i$ such that

$$\begin{aligned}&(\ddot{u}_1(s) - \ddot{u}_2(s), \dot{u}_1(s) - \dot{u}_2(s))_{V'\times V} + a_\mu(u_1(s) - u_2(s), \dot{u}_1(s) - \dot{u}_2(s)) \leq\\ &+ j((\xi_1(s), \varphi_1(s)), \dot{u}_2(s)) - j((\xi_1(s), \varphi_1(s)), \dot{u}_1(s)) + j((\xi_2(s), \varphi_2(s)), \dot{u}_1(s))\\ &\qquad - j((\xi_2(s), \varphi_2(s)), \dot{u}_2(s)), \quad \forall s \in [0,T],\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{2}|\dot{u}_1(t) - \dot{u}_2(t)|_H^2 + \frac{\mu^*}{2}\|u_1(t) - u_2(t)\|_V^2 \leq \int_0^t j((\xi_1(s), \varphi_1(s)), \dot{u}_2(s))ds\\ &\underbrace{- \int_0^t j((\xi_1(s), \varphi_1(s)), \dot{u}_1(s))ds + \int_0^t j((\xi_2(s), \varphi_2(s)), \dot{u}_1(s))ds - \int_0^t j((\xi_2(s), \varphi_2(s)), \dot{u}_2(s))ds}_{\int_0^t A(s)ds},\end{aligned} \tag{57}$$

for all $t \in [0,T]$. On the other hand, we have

$$j((\xi_1(s), \varphi_1(s)), \dot{u}_2(s)) - j((\xi_1(s), \varphi_1(s)), \dot{u}_1(s)) =$$

$$\int_{\Gamma_3} g(\xi_1(s), \varphi_1(s) - \varphi_F)(|\dot{u}_2(s)| - |\dot{u}_1(s)|)da,$$

$$j((\xi_2(s), \varphi_2(s)), \dot{u}_1(s)) - j((\xi_2(s), \varphi_2(s)), \dot{u}_2(s)) = \int_{\Gamma_3} g(\xi_2(s), \varphi_2(s) - \varphi_F)(|\dot{u}_1(s)| - |\dot{u}_2(s)|)da.$$

Thus

$$\begin{aligned} A(s) &= \int_{\Gamma_3} \Big(g\big(\xi_2(s), \varphi_2(s) - \varphi_F\big) - g\big(\xi_1(s), \varphi_1(s) - \varphi_F\big)\Big)(|\dot{u}_1(s)| - |\dot{u}_2(s)|)da \\ &\leq \int_{\Gamma_3} L_g\Big(|\xi_2(s) - \xi_1(s)| + |\varphi_2(s) - \varphi_1(s)|\Big)|\dot{u}_1(s) - \dot{u}_2(s)|da \\ &\leq C\Big\{\|\xi_1(s) - \xi_2(s)\|^2_{L^2(\Gamma_3)} + \|\varphi_1(s) - \varphi_2(s)\|^2_W + \|\dot{u}_1(s) - \dot{u}_2(s)\|^2_V\Big\}. \end{aligned}$$

Then, (57) becomes

$$\begin{aligned} \frac{1}{2}\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_V + \frac{\mu^*}{2}\|u_1(t) - u_2(t)\|^2_V \leq C\Big\{\int_0^t \|\xi_2(s) - \xi_1(s)\|^2_{L^2(\Gamma_3)}ds \\ + \int_0^t \|\varphi_2(s) - \varphi_1(s)\|^2_W ds + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|^2_V ds\Big\}. \end{aligned} \tag{58}$$

Moreover, if we take $\psi = \psi - \varphi_i$ for $(i = 1, 2)$ in inequality (54), we find

$$\begin{aligned} a_\beta(\varphi_1(t) - \varphi_2(t), \varphi_1(t) - \varphi_1(t)) \leq a_e(u_1(t) - u_2(t), \varphi_1(t) - \varphi_1(t)) \\ + \underbrace{J(\xi_1(t), \varphi_1(t), \varphi_2(t) - \varphi_1(t)) + J(\xi_2(t), \varphi_2(t), \varphi_1(t) - \varphi_2(t))}_{B(t)}, \end{aligned} \tag{59}$$

and

$$\begin{aligned} B(t) &= \int_{\Gamma_3} \Big(k(\xi_1(t))(\varphi_1(t) - \varphi_F) + k(\xi_2(t))(\varphi_2(t) - \varphi_F)\Big)|\varphi_1(t) - \varphi_2(t)|da \\ &\leq \int_{\Gamma_3} \Big(L_k v_0|\xi_1(t) - \xi_2(t)| + 2v_0 k_0\Big)|\varphi_1(t) - \varphi_2(t)|da \\ &\leq C\|\xi_1(t) - \xi_2(t)\|^2_{L^2(\Gamma_3)} + \frac{\beta^*}{2}\|\varphi_1(t) - \varphi_2(t)\|^2_W. \end{aligned} \tag{60}$$

Hence (59) becomes

$$\frac{\beta^*}{4}\|\varphi_1(t) - \varphi_2(t)\|^2_W \leq C\Big\{\|u_1(t) - u_2(t)\|^2_V + \|\xi_1(t) - \xi_2(t)\|^2_{L^2(\Gamma_3)}\Big\}. \tag{61}$$

Integrating between 0 and t with $t \in [0,T]$, we obtain

$$\frac{\beta^*}{4}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|_W^2 ds \le C\Big\{\int_0^t \|u_1(s)-u_2(s)\|_V^2 ds + \int_0^t \|\xi_1(s)-\xi_2(s)\|_{L^2(\Gamma_3)}^2 ds\Big\}. \tag{62}$$

We combine (58) et (62) to obtain

$$\frac{1}{2}\|\dot{u}_1(t)-\dot{u}_2(t)\|_V^2 + \frac{\mu^*}{2}\|u_1(t)-u_2(t)\|_V^2 \le C\Big\{\int_0^t \|\xi_1(s)-\xi_2(s)\|_{L^2(\Gamma_3)}^2 ds + \int_0^t \|u_1(s)-u_2(s)\|_V^2 ds + \int_0^t \|\dot{u}_1(s)-\dot{u}_2(s)\|_V^2 ds\Big\}. \tag{63}$$

Grönwall's inequality implies

$$\|\dot{u}_1(t)-\dot{u}_2(t)\|_V^2 + \|u_1(t)-u_2(t)\|_V^2 \le C\int_0^t \|\xi_1(s)-\xi_2(s)\|_{L^2(\Gamma_3)}^2 ds. \tag{64}$$

Integrating between 0 and T, we find

$$\|\dot{u}_1-\dot{u}_2\|_{L^2(0,T;V)}^2 + \|u_1-u_2\|_{L^2(0,T;V)}^2 \le C\int_0^T \|\xi_2(t)-\xi_1(t)\|_{L^2(\Gamma_3)}^2 dt. \tag{65}$$

From (65) and since $\xi \in W^{1,2}(0,T;L^2(\Gamma_3))$, we deduce that $u \in W^{1,2}(0,T;V)$. And from (61) we have the regularity $\varphi \in L^2(0,T;W)$.

On the other hand, we also have

$$\|\ddot{u}_\xi(t)\|_{V'} = \sup_{\|v\|_V\le 1} \Big|\Big(\frac{\partial}{\partial t}\dot{u}_\xi(t), v\Big)_{V'\times V}\Big|.$$

We integrate between $0, T$ under the initial condition (55), we find

$$\int_0^T \|\ddot{u}_\xi(t)\|_{V'} dt \le \sup_{\|v\|_V\le 1}\int_0^T \Big|\Big(\frac{\partial}{\partial t}\dot{u}_\xi(t), v\Big)_{V'\times V}\Big| dt \le \|\dot{u}_\xi(T)-u_1\|_V,$$

and

$$\int_0^T \|\ddot{u}_\xi\|_{L^2(0,T;V')} dt \le C\int_0^T \operatorname*{ess\,sup}_{t\in[0,T]} \|\ddot{u}_\xi(t)\|_{V'} dt \le C\|\dot{u}_\xi(T)-u_1\|_V,$$

thus

$$\|\ddot{u}_\xi\|_{L^2(0,T;V')} \le \frac{C}{T}\|\dot{u}_\xi(T)-u_1\|_V. \tag{66}$$

Since $\dot{u}_\xi \in L^2(0,T;V)$, we have the regularity $\ddot{u}_\xi \in L^2(0,T;V')$, which implies $u_\xi \in W^{2,2}(0,T;V)$.

On other hand, for all $t \in [0,T]$, we consider $t_1, t_2 \in [0,T]$ and we rewrite the inequality (54) for $t = t_1$ with $\psi - \varphi_\xi(t) = \varphi_\xi(t_2) - \varphi_\xi(t_1)$, and for $t = t_2$ with $\psi - \varphi_\xi(t) = \varphi_\xi(t_1) - \varphi_\xi(t_2)$, then we add the results, we obtain

$$\frac{\beta^*}{4}\|\varphi_\xi(t_1) - \varphi_\xi(t_2)\|_W^2 \leq C\Big\{\|u_\xi(t_1) - u_\xi(t_2)\|_V^2 + \|q(t_1) - q(t_2)\|_{W'}^2 + \|\xi(t_1) - \xi(t_2)\|_{L^2(\Gamma_3)}^2\Big\}. \tag{67}$$

Since $q \in W^{1,2}(0,T;W'), \xi \in W^{1,2}(0,T;L^2(\Gamma_3))$ and $u_\xi \in W^{1,2}(0,T;V)$, then the inequality (67) implies that $\varphi_\xi : [0,T] \to W$ is an absolutely continuous function and verifies

$$\|\dot{\varphi}_\xi(t)\|_W^2 \leq C\Big\{\|\dot{u}_\xi(t)\|_V^2 + \|\dot{q}(t)\|_{W'}^2 + \|\dot{\xi}(t)\|_{L^2(\Gamma_3)}^2\Big\}, \forall t \in [0,T]. \tag{68}$$

and since $u_\xi \in L^2(0,T;V), q \in L^2(0,T;W')$ and $\xi \in L^2(0,T;L^2(\Gamma_3))$, which shows that $\dot{\varphi}_\xi \in L^2(0,T;W)$.

Which proves (56) and we complete the proof of theorem 4.2. □

Step.2. Application of the Banach fixed point theorem. Next, for each $\xi \in W^{1,2}(0,T;L^2(\Gamma_3))$, we denote by (u_ξ, φ_ξ) the solution of problem $\mathcal{P}^0_{V_\xi}$. We define the operator

$$\Lambda : W^{1,2}(0,T,L^2(\Gamma_3)) \longrightarrow W^{1,2}(0,T,L^2(\Gamma_3))$$
$$\Lambda\xi(t) = S\dot{u}_\xi(t). \tag{69}$$

We have the following result.

Proposition 4.3. *The operator Λ has a unique fixed point $\xi^* \in W^{1,2}(0,T;L^2(\Gamma_3))$.*

Proof. Let $\xi \in W^{1,2}(0,T;L^2(\Gamma_3))$. We denote by (u_i, φ_i) the solutions of $\mathcal{P}^0_{V_\xi}$ for $\xi = \xi_i$ (for $i = 1,2$). We have

$$\|\Lambda\xi_1(t) - \Lambda\xi_2(t)\|_{L^2(\Gamma_3)}^2 = \|S\dot{u}_1(t) - S\dot{u}_2(t)\|_{L^2(\Gamma_3)}^2 \leq C\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2 ds. \tag{70}$$

By integrating between 0 and T, we obtain by (65) that

$$\|\Lambda\xi_1 - \Lambda\xi_2\|_{L^2(0,T;L^2(\Gamma_3))}^2 \leq C\int_0^T \|\dot{u}_1(t) - \dot{u}_2(t)\|_V^2 dt \leq C\int_0^T \|\xi_2(t) - \xi_1(t)\|_{L^2(\Gamma_3)}^2 dt, \tag{71}$$

since

$$\xi_1(t) - \xi_2(t) = \int_0^t \Big(\dot{\xi}_1(s) - \dot{\xi}_2(s)\Big) ds,$$

then

$$\|\xi_1(t)-\xi_2(t)\|^2_{L^2(\Gamma_3)} \leq C\int_0^t \|\dot{\xi}_1(s)-\dot{\xi}_2(s)\|^2_{L^2(\Gamma_3)}ds,$$

therefore (71) yields

$$\|\Lambda\xi_1-\Lambda\xi_2\|^2_{L^2(0,T;L^2(\Gamma_3))} \leq C\int_0^T \|\dot{\xi}_2(t)-\dot{\xi}_1(t)\|^2_{L^2(\Gamma_3)}dt. \tag{72}$$

And, we also have

$$\left\|\frac{d}{dt}\Lambda\xi_1(t)-\frac{d}{dt}\Lambda\xi_2(t)\right\|^2_{L^2(\Gamma_3)} = \Big\||\dot{u}_1(t)|-|\dot{u}_2(t)|\Big\|^2_{L^2(\Gamma_3)} \leq \Big\|\dot{u}_1(t)-\dot{u}_2(t)\Big\|^2_{L^2(\Gamma_3)},$$

thus, by (65) we have

$$\left\|\frac{d}{dt}\Lambda\xi_1-\frac{d}{dt}\Lambda\xi_2\right\|^2_{L^2(0,T;L^2(\Gamma_3))} \leq \int_0^T \|\dot{u}_1(t)-\dot{u}_2(t)\|^2_V dt \leq C\int_0^T \|\xi_2(t)-\xi_1(t)\|^2_{L^2(\Gamma_3)}dt. \tag{73}$$

From (72) and (73), we obtain

$$\|\Lambda\xi_1-\Lambda\xi_2\|_{W^{1,2}(0,T;L^2(\Gamma_3))} \leq C\|\xi_2-\xi_1\|_{W^{1,2}(0,T;L^2(\Gamma_3))}. \tag{74}$$

Reiterating the previous inequality p times, we find that

$$\|\Lambda^p\xi_1-\Lambda^p\xi_2\|_{W^{1,2}(0,T;L^2(\Gamma_3))} \leq \sqrt{\frac{C^pT^p}{p!}}\|\xi_2-\xi_1\|_{W^{1,2}(0,T;L^2(\Gamma_3))}. \tag{75}$$

This last inequality shows that for a sufficiently large p, the operator Λ^p is a contraction on the Banach space $W^{1,2}(0,T;L^2(\Gamma_3))$ and, therefore, there exists a unique element $\xi^* \in W^{1,2}(0,T;L^2(\Gamma_3))$ such that $\Lambda\xi^*=\xi^*$, which shows that ξ^* is the unique fixed point of Λ. □

We have now all the ingredients to provide the proof of theorem 4.1.

Proof. **Existence:** Let ξ^* be the fixed point of the operator Λ obtained in preposition 4.3. Since $\xi^*=\Lambda\xi^*=S\dot{u}_{\xi^*}(t)$, it follows from problem $\mathcal{P}^0_{V_\xi}$ that $(u_{\xi^*},\varphi_{\xi^*})$ is a solution of problem $\mathcal{P}^0_V$, which concludes the existence part.
Uniqueness: The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ defined by (69). □

5. Proof of theorem 3.1

In the first step of proof, we consider the following variational problem.
Problem $\mathcal{P}_{V_\eta}$.

For all $\eta \in W^{1,2}(0,T;V')$. Find $u_\eta : [0,T] \to V$ and $\varphi_\eta : [0,T] \to W$ such that

$$\begin{aligned}(\ddot{u}_\eta(t), v - \dot{u}_\eta(t))_{V'\times V} + a_\mu(u_\eta(t), v - \dot{u}_\eta(t)) + (\eta(t), v - \dot{u}_\eta(t))_{V'\times V}\\ + j((S\dot{u}_\eta(t), \varphi_\eta(t)), v) - j((S\dot{u}_\eta(t), \varphi_\eta(t)), \dot{u}_\eta(t))\\ \geq (f(t), v - \dot{u}_\eta(t))_{V'\times V} \quad \forall v \in V,\ t \in [0,T],\end{aligned} \tag{76}$$

$$\begin{aligned}a_\beta(\varphi_\eta(t), \psi) - a_e(u_\eta(t), \psi) + J(S\dot{u}_\eta(t), \varphi_\eta(t), \psi)\\ \geq (q(t), \psi)_{W'\times W},\ \forall \psi \in W,\ t \in [0,T],\end{aligned} \tag{77}$$

$$(u_\eta(0), \dot{u}_\eta(0)) = (u_0, u_1). \tag{78}$$

We have the result.

Theorem 5.1. *Under hypotheses* ***H****(c)(a,c,d,e),* ***H****(f),***H****(q),* ***H****(j),***H****(J),***H****(k),* ***H****(g),***H****(φ_F) and* ***H****(0), problem $\mathcal{P}_{V_\eta}$ admits a unique solution which satisfies*

$$u_\eta \in W^{2,2}(0,T;V), \quad \ddot{u}_\eta \in L^2(0,T;V') \quad \textit{and} \quad \varphi_\eta \in W^{1,2}(0,T;W). \tag{79}$$

Proof. For $\eta \in W^{1,2}(0,T;V')$ and from Riesz's representation theorem we define the function $f_\eta : [0,T] \to V'$ by

$$(f_\eta(t), v)_{V'\times V} = (f(t), v)_{V'\times V} - (\eta(t), v)_{V'\times V} \quad \forall v \in V,\ t \in [0,T]. \tag{80}$$

The regularity of $f \in W^{1,2}(0,T;V')$ and $\eta \in W^{1,2}(0,T;V')$, it follows that $f_\eta \in W^{1,2}(0,T;V')$.

The problem $\mathcal{P}_{V_\eta}$ becomes

$$\begin{aligned}(\ddot{u}_\eta(t), v - \dot{u}_\eta(t))_{V'\times V} + a_\mu(u_\eta(t), v - \dot{u}_\eta(t)) + j((S\dot{u}_\eta(t), \varphi_\eta(t)), v)\\ - j((S\dot{u}_\eta(t), \varphi_\eta(t)), \dot{u}_\eta(t)) \geq (f_\eta(t), v - \dot{u}_\eta(t))_{V'\times V} \quad \forall v \in V,\ t \in [0,T],\\ a_\beta(\varphi_\eta(t), \psi) - a_e(u_\eta(t), \psi) + J(S\dot{u}_\eta(t), \varphi_\eta(t), \psi)\\ \geq (q(t), \psi)_{W'\times W},\ \forall \psi \in W,\ t \in [0,T],\\ (u_\eta(0), \dot{u}_\eta(0)) = (u_0, u_1).\end{aligned}$$

Therefore, it follows from the results of theorem 4.2, that there exists a unique solution $(u_\eta, \varphi_\eta) \in W^{2,2}(0,T;V) \times W^{1,2}(0,T;W)$ such that $\ddot{u}_\eta \in L^2(0,T;V')$. □

For all $\eta \in W^{1,2}(0,T;V')$. Let (u_η, φ_η) the solution of problem $\mathcal{P}_{V_\eta}$ and $\mathcal{B}\eta(t)$ an element of V' defined by

$$\Big(\mathcal{B}\eta(t), v\Big)_{V'\times V} = \Big(\int_0^t \theta(t-s) u_\eta(s) ds, v\Big)_V + a_e^*(\varphi_\eta(t), v), \tag{81}$$

for all $v \in V$ and $t \in [0,T]$. L'operator $\mathcal{B}$ is well defined; if $\eta \in W^{1,2}(0,T;V')$ implies that $\mathcal{B}\eta$ belongs to $W^{1,2}(0,T;V')$.

We also note that

$$\Big(\big(\frac{d}{dt}\mathcal{B}\eta\big)(t), v\Big)_{V'\times V} = \Big(\theta(0) u_\eta(t) + \int_0^t \dot{\theta}(t-s) u_\eta(s) ds, v\Big)_V + a_e^*(\dot{\varphi}_\eta(t), v), \tag{82}$$

for all $\eta \in W^{1,2}(0,T;V')$, $v \in V$ and $t \in [0,T]$.

We have the following result.

Lemma 5.2. *L'operator $\mathcal{B} : W^{1,2}(0,T;V') \to W^{1,2}(0,T;V')$ admits a unique fixed point $\eta^* \in W^{1,2}(0,T;V')$.*

Proof. Let $\eta_i \in W^{1,2}(0,T;V')$, $i = 1,2$. To simplify the notation, we denote by (u_i, φ_i) the unique solution of problem $\mathcal{P}_{V_\eta}$. We choose $v = \dot{u}_2(s)$ in the first inequality, and $v = \dot{u}_1(s)$ in the second inequality, then we add the results, we result

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{ds}((\dot{u}_1(s) - \dot{u}_2(s), \dot{u}_1(s) - \dot{u}_2(s)))_H + \tfrac{1}{2}\tfrac{d}{ds} a_\mu(u_1(s) - u_2(s), u_1(s) - u_2(s)) \\
&\qquad \leq -(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s))_{V' \times V} + C\Big\{\|S\dot{u}_1(s) - S\dot{u}_2(s)\|^2_{L^2(\Gamma_3)} \\
&\qquad\qquad + \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2 + \|\varphi_1(s) - \varphi_2(s)\|_W^2\Big\},
\end{aligned}$$

by integrating the previous inequality between 0 and t with $t \in [0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V^2 + \frac{\mu^*}{2}\|u_1(t) - u_2(t)\|_V^2 \leq -(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t))_{V' \times V} \\
&+ \int_0^t \big(\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s)\big)_{V' \times V} ds + C\Big\{\int_0^t \|S\dot{u}_1(s) - S\dot{u}_2(s)\|^2_{L^2(\Gamma_3)} ds \\
&\qquad + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2 ds + \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_W^2 ds\Big\},
\end{aligned} \tag{83}$$

From (40), we have

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V^2 + \frac{\mu^*}{2}\|u_1(t) - u_2(t)\|_V^2 \leq C\Big\{\|\eta_1(t) - \eta_2(t)\|_{V'}\|u_1(t) - u_2(t)\|_V \\
&+ \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_{V'}\|u_1(s) - u_2(s)\|_V ds + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2 ds \\
&\qquad + \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_W^2 ds\Big\},
\end{aligned}$$

we use inequality

$$ab \leq \frac{a^2}{2m} + 2mb^2,$$

for all $a, b > 0$ and $m > 0$. For a practical choice of m we obtain

$$\frac{1}{2}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V^2 + \frac{\mu^*}{4}\|u_1(t) - u_2(t)\|_V^2 \leq C\Big\{\|\eta_1(t) - \eta_2(t)\|_{V'}^2$$

$$+\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_{V'}^2 ds+\int_0^t \|u_1(s)-u_2(s)\|_V^2 ds+\int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V^2 ds$$
$$+\int_0^t \|\varphi_1(s)-\varphi_2(s)\|_W^2 ds\Big\}. \qquad (84)$$

On the other hand, from (61) we also have

$$\frac{\beta^*}{4}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|_W^2 ds \le C\Big\{\int_0^t \|u_1(s)-u_2(s)\|_V^2 ds+\int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V^2 ds\Big\}. \qquad (85)$$

We combine (84) et (85) and we use the Gronwall inequality, we find

$$\|\dot u_1(t)-\dot u_2(t)\|_V^2+\|u_1(t)-u_2(t)\|_V^2 \le C\Big\{\|\eta_1(t)-\eta_2(t)\|_{V'}^2+\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_{V'}^2 ds\Big\}. \qquad (86)$$

We integrate (86) between 0 et T, we have

$$\|\dot u_1-\dot u_2\|_{L^2(0,T;V)}^2+\|u_1-u_2\|_{L^2(0,T;V)}^2 \le C\|\eta_1-\eta_2\|_{W^{1,2}(0,T;V')}^2. \qquad (87)$$

Therefore, for all $v\in V$ and $t\in[0,T]$, we have

$$(\mathcal{B}\eta_1(t),v)_{V'\times V}=\Big(\int_0^t \theta(t-s)u_1(s)ds,v\Big)_V+a_e^*(\varphi_1(t),v).$$

$$(\mathcal{B}\eta_2(t),v)_{V'\times V}=\Big(\int_0^t \theta(t-s)u_2(s)ds,v\Big)_V+a_e^*(\varphi_2(t),v).$$

By definition

$$\|\mathcal{B}\eta_1(t)-\mathcal{B}\eta_2(t)\|_{V'}=\sup_{\|v\|_V\le 1}|(\mathcal{B}\eta_1(t)-\mathcal{B}\xi_2(t),v)_{V'\times V}|,$$

which shows that

$$\|\mathcal{B}\eta_1(t)-\mathcal{B}\eta_2(t)\|_{V'}^2 \le C\Big\{\int_0^t \|u_1(s)-u_2(s)\|_V^2 ds+\|\varphi_1(t)-\varphi_2(t)\|_W^2\Big\}.$$

and

$$\|\mathcal{B}\eta_1-\mathcal{B}\eta_2\|_{L^2(0,T;V')}^2 \le C\|u_1-u_2\|_{W^{1,2}(0,T;V)}^2.$$

From (87) we have

$$\|\mathcal{B}\eta_1-\mathcal{B}\eta_2\|_{L^2(0,T;V')}^2 \le C\|\eta_1-\eta_2\|_{W^{1,2}(0,T;V')}^2. \qquad (88)$$

Therefore, from (82) we have

$$\Big(\big(\frac{d}{dt}\mathcal{B}\eta_1\big)(t),v\Big)_{V'\times V}=\Big(\theta(0)u_1(t)+\int\limits_0^t\dot{\theta}(t-s)u_1(s)ds,v\Big)_V+a_e^*(\dot{\varphi}_1(t),v),$$

$$\Big(\big(\frac{d}{dt}\mathcal{B}\eta_2\big)(t),v\Big)_{V'\times V}=\Big(\theta(0)u_2(t)+\int\limits_0^t\dot{\theta}(t-s)u_2(s)ds,v\Big)_V+a_e^*(\dot{\varphi}_2(t),v).$$

Thus

$$\begin{aligned}\Big\|\big(\frac{d}{dt}\mathcal{B}\eta_1\big)(t)-\big(\frac{d}{dt}\mathcal{B}\eta_2\big)(t)\Big\|_{V'}\leq\ & |\theta(0)|\|u_1(t)-u_2(t)\|_V+C\|\dot{\varphi}_1(t)-\dot{\varphi}_2(t)\|_W.\\ &+\int\limits_0^t|\dot{\theta}(t-s)|\|u_1(s)-u_2(s)\|_V ds\end{aligned}\tag{89}$$

The hypothesis b of $\mathbf{H}$(c) results

$$\begin{aligned}\Big\|\big(\frac{d}{dt}\mathcal{B}\eta_1\big)(t)-\big(\frac{d}{dt}\mathcal{B}\eta_2\big)(t)\Big\|^2_{V'}\leq C\Big\{&\|u_1(t)-u_2(t)\|_V^2+\int\limits_0^t\|u_1(s)-u_2(s)\|_V^2ds\\ &+\|\dot{\varphi}_1(t)-\dot{\varphi}_2(t)\|_W^2\Big\},\end{aligned}$$

by integrating the previous inequality between 0 and T, then using the estimates (85) and (86), we obtain

$$\Big\|\frac{d}{dt}\mathcal{B}\eta_1-\frac{d}{dt}\mathcal{B}\eta_2\Big\|_{L^2(0,T;V')}\leq C\|\eta_1-\eta_2\|_{W^{1,2}(0,T;V')}.\tag{90}$$

We combine (87) and (90), we obtain

$$\Big\|\mathcal{B}\eta_1-\mathcal{B}\eta_2\Big\|_{W^{1,2}(0,T;V')}\leq C\|\eta_1-\eta_2\|_{W^{1,2}(0,T;V')}.$$

This inequality shows that the operator $\mathcal{B}$ is a contraction on the Banach space $W^{1,2}(0,T;V')$ and, therefore, there exists a unique point fixed $\eta^*\in W^{1,2}(0,T;V')$ such that $\mathcal{B}\eta^*=\eta^*$. □

We have now all the ingredients to provide the proof of theorem 3.1.

Proof. **Existence.** Let $\eta^*\in W^{1,2}(0,T;V')$ be the fixed point of the operator $\mathcal{B}$. We denote by $(u_{\eta^*},\varphi_{\eta^*})$ the solution of problem $\mathcal{P}_{V_\eta}$ obtained for $\eta=\eta^*$, such that $\eta^*=\mathcal{B}\eta^*$, we result by (76)–(78) that $(u_{\eta^*},\varphi_{\eta^*})$ is a unique solution of problem $\mathcal{P}_V$, with regularity $(u_{\eta^*},\varphi_{\eta^*})\in W^{2,2}(0,T;V)\times W^{1,2}(0,T;W)$ such that $\ddot{u}_{\eta^*}\in L^2(0,T;V')$, which concludes the part of existence.

Uniqueness. Let (u_1,φ_1) and (u_2,φ_2) are solutions of problem $\mathcal{P}_V$ and let $t\in[0,T]$. Using an argument similar to that in the proof of (83), we obtain

$$\|\dot{u}_1(t)-\dot{u}_2(t)\|_V^2+\|u_1(t)-u_2(t)\|_V^2\leq C\Big\{\int\limits_0^t\|u_1(s)-u_2(s)\|_V^2ds$$

$$+\int_0^t \|\dot{u}_1(s)-\dot{u}_2(s)\|_V^2\,ds\Big\}.$$

The Gronwall inequality shows that

$$\|\dot{u}_1(t)-\dot{u}_2(t)\|_V^2+\|u_1(t)-u_2(t)\|_V^2\leq 0.$$

Which implies

$$\|u_1-u_2\|^2_{W^{1,2}(0,T;V)}=0. \tag{91}$$

Moreover, we have

$$\|\ddot{u}_1-\ddot{u}_2\|_{L^2(0,T;V')}\leq C\operatorname*{ess\,sup}_{t\in[0,T]}\int_0^T\|\ddot{u}_1(t)-\ddot{u}_2(t)\|_{V'}dt\leq 0. \tag{92}$$

From (91) and (92), we find that

$$\|u_1-u_2\|_{W^{2,2}(0,T;V)}=0. \tag{93}$$

So $u_1=u_2$.

On the other hand, we use (85) to find that

$$\|\varphi_1-\varphi_2\|^2_{L^2(0,T;W)}\leq C\Big\{\int_0^T\|u_1(t)-u_2(t)\|_V^2dt+\int_0^T\|\dot{u}_1(t)-\dot{u}_2(t)\|_V^2dt\Big\}.$$

From (91), we have

$$\|\varphi_1-\varphi_2\|_{L^2(0,T;W)}\leq C\|u_1-u_2\|_{W^{2,2}(0,T;V)}=0. \tag{94}$$

For $h\in\mathbb{R}_+^*$ such that $[t,t+h]\subset[0,T]$, we have

$$\begin{aligned}\varphi_1(t+h)-\varphi_1(t)&=\int_t^{t+h}\dot{\varphi}_1(s)ds\quad\text{a.e. } t\in(0,T),\\ \varphi_2(t+h)-\varphi_2(t)&=\int_t^{t+h}\dot{\varphi}_2(s)ds\quad\text{a.e. } t\in(0,T).\end{aligned} \tag{95}$$

Thus

$$(\varphi_1(t+h)-\varphi_1(t))-(\varphi_2(t+h)-\varphi_2(t))=\int_t^{t+h}(\dot{\varphi}_1(s)-\dot{\varphi}_2(s))ds, \tag{96}$$

therefore

$$\|(\varphi_1(t+h)-\varphi_1(t))-(\varphi_2(t+h)-\varphi_2(t))\|_W\leq\int_t^{t+h}\|\dot{\varphi}_1(s)-\dot{\varphi}_2(s)\|_W ds, \tag{97}$$

Since $\dot{\varphi} \in L^2(0,T;W)$, therefore, $\dot{\varphi} : [0,T] \to W$ is an absolutely continuous function and the inequality (97) shows that $\varphi : [0,T] \to W$ is absolutely continuous function, and

$$\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_W \leq \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h}\int_t^{t+h} \|\dot{\varphi}_1(s) - \dot{\varphi}_2(s)\|_W ds = 0. \tag{98}$$

By combining (94) and (98), finding $\varphi_1 = \varphi_2$, which ends the uniqueness part. □

Conflicts of interest : The authors declare no conflict of interest.

Data availability : In this section, please i would like to provide details regarding where data supporting reported results can be found : see please the links: https://www.researchgate.net/profile/Adel-Aissaoui
https://scholar.google.co.in/citations?user=9ePD5jYAAAAJ&hl=en

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