J. Appl. Math. & Informatics Vol. **41***(2023), No. 3, pp. 469 - 485*
https://doi.org/10.14317/jami.2023.469

APPROXIMATION OF DRYGAS FUNCTIONAL EQUATION IN QUASI-BANACH SPACE†

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Abstract. In this paper, we investigate the Hyers-Ulam-Rassias stability for a Drygas functional equation

$$g(u+v)+g(u-v)=2g(u)+g(v)+g(-v)$$

in the setting of quasi-Banach space using fixed point approach. Also, we give general results on hyperstability of a Drygas functional equation. The results obtain in this paper extend various previously known results in the setting of quasi-Banach space. Some examples are also illustrated.

AMS Mathematics Subject Classification : 39B32, 39B82, 46BXX.
Key words and phrases : Drygas functional equation, Hyers-Ulam-Rassias stability, hyperstability, Quasi-Banach spaces.

1. Introduction and Preliminaries

The theory of functional equations is a vast area of non-linear analysis which is rather hard to explore. Functional equations find many applications in the study of statistics, geometry, game theory, measure theory, dynamics, economics, and many other allied fields. The study of solutions and stability results of functional equations is a hot topic in the research field of analysis. The stability results of functional equations are employed in the non-linear analysis, especially in fixed point theory. The stability results are used to study the asymptotic properties of additive mappings.

In the theory of Ulam's stability, one can find efficient tools to evaluate the errors, that is, to study the existence of an exact solution of the perturbed functional equation which is not far from the given function. In 1940, the stability problem for the functional equations was first raised by Ulam [34]. Hyers [18]

Received March 14, 2021. Revised September 16, 2022. Accepted December 26, 2022.
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† The authors are thankful to AISTDF, DST India for the research grant vide project No. CRD/2018/000017 and DST-FIST (Govt. of India) for the grant SR/FST/MS-1/2017/13 to support this research work.

gave an affirmative partial answer to Ulam in Banach space. After that, Aoki [5] and Rassias [29] generalized Hyers theorem for additive and linear mapping by considering an unbounded Cauchy difference. In 1994, Găvruta[16] generalized Rassias' theorem and discussed the stability of linear functional equations.

A functional equation is hyperstable if a function satisfying this functional equation approximately is a true solution of it. In 1949, Bourgin [8] gave the first hyperstability result and concerned the ring homomorphisms. The hyperstability results of the several functional equations in the literature have been studied by many authors in recent years, (see [1] [2], [3], [6], [10], [11], [17], [22], and references cited therein).

The quasi-normed space is one of the interesting generalizations of the normed space (see [20], [21]). The difference between a norm and quasi-norm is that the modulus of concavity of a quasi-norm is greater than equal to 1, while that of a norm is equal to 1. The quasi-norm is not continuous in general, while a norm is always continuous. However, every p-norm is a continuous quasi-norm. By the Aoki-Rolewicz theorem [24] (see also [7]), each quasi-norm is equivalent to some p-norm. It is important to emphasize that the standard basic results of Banach space theory, such as the Uniform Boundedness Principle, Open Mapping Theorem, and Closed Graph Theorem, which depend only on completeness, apply to quasi-normed spaces. However, applications of convexity, such as the Hahn-Banach Theorem, are not applicable (see [20], page 1102).

Various authors study the stability problems of many different functional equations in the setting of quasi-normed space (see, for example, [13, 15, 25, 26]). Motivated by these, we investigate the stability of Drygas functional equation in quasi-normed space.

Now we present the basic notions and properties which are useful in the next section.

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{Z}$ stands for the set of integers, and $\mathbb{R}$ stands for the set of reals. Let $\mathbb{R}^+ := [0, \infty)$ be the set of nonnegative real numbers and $\mathscr{Y}^{\mathscr{X}}$ denotes the family of all mappings from a nonempty set $\mathscr{X}$ into a nonempty set $\mathscr{Y}$.

Definition 1.1. (see [7], [24]) A quasi-norm $\|.\|$ is a real-valued function on a linear space $\mathscr{X}$ satisfying the following axioms:

(1) $\| u \| \geq 0$ for all $u \in \mathscr{X}$ and $\| u \| = 0$ iff $u = 0$;
(2) $\| \lambda u \| = | \lambda | . \| u \|$ for all $\lambda \in \mathbb{R}$ and all $u \in \mathscr{X}$;
(3) there is a constant $K \geq 1$ such that $\| u + v \| \leq K(\| u \| + \| v \|)$ for all $u, v \in \mathscr{X}$.

The pair $(\mathscr{X}, \| . \|)$ is called a *quasi-normed* space if $\| . \|$ is a quasi-norm on $\mathscr{X}$.
A *quasi-Banach* space is a complete quasi-normed space.
A quasi-norm $\| . \|$ is called a p-norm $(0 < p \leq 1)$ if

$$\| u + v \|^p \leq \| u \|^p + \| v \|^p,$$

for all $u, v \in \mathscr{X}$. In this case, a quasi-Banach space is called a *p-Banach space*. Given a p-norm, the formula $d(u,v) := \| u-v \|^p$ gives us a translation invariant metric on $\mathscr{X}$. By the Aoki-Rolewicz theorem [24] (see also [7]), each quasi-norm is equivalent to some p-norm.

Remark 1.1. [4] The sequence space $\mathscr{X} = \ell^p$, $0 < p < 1$, with the function

$$\|u\| = \left(\sum_{i=1}^{\infty} | u_i |^p \right)^{\frac{1}{p}}$$

is not a normed space, because the condition triangle inequality of the norm is not satisfied. To explain this remark, we consider the following example.

Example 1.2. Suppose that sequence space $\mathscr{X} = \ell^{\frac{1}{2}}$, and $u = \{u_i\} = \{0, 1, 0, 0, 0, ...\} \in \ell^{\frac{1}{2}}$ and $v = \{v_i\} = \{0, 0, 2, 0, 0, ...\} \in \ell^{\frac{1}{2}}$. Then we have

$$\|u + v\| = \left(\sum_{i=1}^{\infty} | u_i + v_i |^{1/2} \right)^2 = (1 + \sqrt{2})^2$$

and

$$\|u\| + \|v\| = \left(\sum_{i=1}^{\infty} | u_i |^{1/2} \right)^2 + \left(\sum_{i=1}^{\infty} | v_i |^{1/2} \right)^2 = 3.$$

It is clear that

$$\|u + v\| > \|u\| + \|v\|.$$

Thus, the space $\ell^{\frac{1}{2}}$ is not a normed space.

Example 1.3. The sequence space $\mathscr{X} = \ell^p$, $0 < p < 1$, with the function

$$\|u\| = \left(\sum_{i=1}^{\infty} | u_i |^p \right)^{\frac{1}{p}}$$

is a quasi-Banach space (see [4]).

Theorem 1.4. *([23], Theorem 1). Let $(\mathscr{Y}, \|.\|)$ be a quasi-normed space, $p = \log_{2K} 2$ with $K \geq 1$ and*

$$|||u||| = \inf\{ \left(\sum_{j=1}^{n} \|u_j\|^p \right)^{\frac{1}{p}} | u = \sum_{j=1}^{n} u_j, u_j \in Y, n \geq 1\},$$

for all $u \in \mathscr{Y}$. Then $|||.|||$ is a p-norm on $\mathscr{Y}$, that is for all $u, v \in \mathscr{Y}$,

$$|||u + v|||^p \leq |||u|||^p + |||v|||^p. \tag{1}$$

Moreover, for all $u \in \mathscr{Y}$,

$$\frac{1}{2K} \|u\| \leq |||u||| \leq \|u\|. \tag{2}$$

If $\|.\|$ is a norm, then $p = 1$ and $|||.||| = \|.\|$.

Definition 1.5. (see [32]) Let $\mathscr{X}$ be a nonempty set, $\mathscr{Y}$ be a normed space, $\varepsilon \in \mathbb{R}_+^{\mathscr{X}^n}$ and $\mathcal{V}_1, \mathcal{V}_2$ be operators mapping from a non empty set $D \subset \mathscr{Y}^{\mathscr{X}}$ into $\mathscr{Y}^{\mathscr{X}^n}$. We say that the operator's equation

$$\mathcal{V}_1\varphi(u_1, u_2, ..., u_n) = \mathcal{V}_2\varphi(u_1, u_2, ..., u_n), \tag{3}$$

for $u_1, u_2, ..., u_n \in \mathscr{X}$ is $\varepsilon-$hyperstable provided that every $\varphi_0 \in D$ which satisfies

$$\| \mathcal{V}_1\varphi_0(u_1, u_2, ..., u_n) - \mathcal{V}_2\varphi_0(u_1, u_2, ..., u_n) \| \leq \varepsilon(u_1, u_2, ..., u_n)$$

fulfils the equation (3).

Theorem 1.6. *([9], Theorem 1)*

(1) *Let $\mathscr{X}$ be a nonempty set, $(\mathscr{Y}, d)$ be a complete metric space,*
(2) *$g_1, g_2, ..., g_k : \mathscr{X} \to \mathscr{X}$ and $l_1, l_2, ..., l_k : \mathscr{X} \to \mathbb{R}_+$ be given mappings.*
(3) *Let $\Lambda : \mathbb{R}_+^{\mathscr{X}} \to \mathbb{R}_+^{\mathscr{X}}$ be a linear operator defined by*

$$\Lambda\delta(u) := \sum_{i=1}^{k} \delta(g_i(u)), \tag{4}$$

for $\delta \in \mathbb{R}_+^{\mathscr{X}}$ and $u \in \mathscr{X}$.

(4) *If $\mathcal{T} : \mathscr{Y}^{\mathscr{X}} \to \mathscr{Y}^{\mathscr{X}}$ is an operator satisfying the inequality*

$$d(\mathcal{T}\xi(u), \mathcal{T}\mu(u)) \leq \sum_{i=1}^{k} l_i(u) d(\xi(g_i(u)), \mu(g_i(u))), \tag{5}$$

for all $\xi, \mu \in \mathscr{Y}^{\mathscr{X}}$, $u \in \mathscr{X}$.

(5) *There exist $\varepsilon : \mathscr{X} \to \mathbb{R}_+$ and a mapping $\varphi : \mathscr{X} \to \mathscr{Y}$ satisfy*

$$d(\mathcal{T}\varphi(u), \varphi(u)) \leq \varepsilon(u)$$

and for every $u \in \mathscr{X}$,

$$\varepsilon^*(u) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(u) < \infty.$$

Then for every $u \in \mathscr{X}$, the limit

$$\psi(u) := \lim_{n\to\infty} \mathcal{T}^n \varphi(u)$$

exists and the function $\psi \in \mathscr{Y}^{\mathscr{X}}$ so defined is the unique fixed point of $\mathcal{T}$ with

$$d(\varphi(u), \psi(u)) \leq \varepsilon^*(u),$$

for all $u \in \mathscr{X}$.

Using the concept of the papers [7, 9], Dung et al. [13] proved the following result.

Theorem 1.7. [13] *Suppose that*

(1) *$\mathscr{X}$ is a nonempty set, $\mathscr{Y}$ is a quasi-Banach space and $\mathcal{T} : \mathscr{Y}^{\mathscr{X}} \to \mathscr{Y}^{\mathscr{X}}$ is a given function .*

(2) *There exist $g_1, g_2, ..., g_k : \mathscr{X} \to \mathscr{X}$ and $l_1, l_2, ..., l_k : \mathscr{X} \to \mathbb{R}_+$ such that for all $\xi, \mu \in \mathscr{Y}^{\mathscr{X}}$, and for all $u \in \mathscr{X}$*

$$\| \mathcal{T}\xi(u) - \mathcal{T}\mu(u) \| \leq \sum_{i=1}^{k} l_i(u) \| (\xi - \mu)g_i(u) \|, \tag{6}$$

(3) *There exist function $\varepsilon : \mathscr{X} \to \mathbb{R}_+$ and $\varphi : \mathscr{X} \to \mathscr{Y}$ satisfy conditions*

$$\| \mathcal{T}\varphi(u) - \varphi(u) \| \leq \varepsilon(u), \tag{7}$$

for every $u \in \mathscr{X}$.

(4) *For every $u \in \mathscr{X}$, and $\theta = \log_{2K} 2$, with $K \geq 1$,*

$$\varepsilon^*(u) = \sum_{n=0}^{\infty} \left(\Lambda^n \varepsilon\right)^{\theta} (u) < \infty, \tag{8}$$

where $\Lambda : \mathbb{R}_+^{\mathscr{X}} \to \mathbb{R}_+^{\mathscr{X}}$ be a linear operator defined by

$$\Lambda\delta(u) := \sum_{i=1}^{k} l_i(u)\delta(g_i(u)), \tag{9}$$

for $\delta \in \mathbb{R}_+^{\mathscr{X}}$ and $u \in \mathscr{X}$.

Then we have

(5) *For every $u \in \mathscr{X}$, the limit*

$$\lim_{n\to\infty} \mathcal{T}^n \varphi(u) = \psi(u), \tag{10}$$

exists and the function $\psi : \mathscr{X} \to \mathscr{Y}$, so defined a fixed point $\mathcal{T}$ satisfying

$$\| \varphi(u) - \psi(u) \|^{\theta} \leq 4\varepsilon^*(u), \tag{11}$$

for all $u \in \mathscr{X}$.

(6) *For every $u \in \mathscr{X}$, if*

$$\varepsilon^*(u) \leq \left(M \sum_{n=1}^{\infty} \left(\Lambda^n \varepsilon\right)(u) \right)^{\theta} < \infty, \tag{12}$$

for some positive real number M, then the fixed point of $\mathcal{T}$ is unique.

To obtain a Jordan and Von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [12] consider the following functional equation

$$g(u) + g(v) = g(u - v) + \{g\left(\frac{u+v}{2}\right) - g\left(\frac{u-v}{2}\right)\}, \tag{13}$$

for all $u, v \in \mathbb{R}$, which can be reduced to the following equation (see, [30], Remark 9.2, pp. 131)

$$g(u + v) + g(u - v) = 2g(u) + g(v) + g(-v), \tag{14}$$

for all $u, v \in \mathbb{R}$. This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation

$$g(u+v)+g(u-v)=2g(u)+2g(v), \tag{15}$$

for all $u, v \in \mathbb{R}$.

The general solution of Drygas equation was given by Ebanks et al. [14]. It has the form $g(u) = H(u) + Q(v)$ for all $u \in \mathbb{R}$, where $H : \mathbb{R} \to \mathbb{R}$ is an additive function and $Q : \mathbb{R} \to \mathbb{R}$ is a quadratic function (see also [19]). A set-valued version of Drygas equation was considered by Smajdor [33]. Recently, hyperstability of a Drygas functional equation studied by various authors see [27], [32], [28] and [31].

In this paper, we discuss the generalized Hyers-Ulam-Rassias stability problem for a Drygas functional equation (14) in the setting of quasi-Banach spaces by using Theorem 1.7. Also, we obtain some hyperstability results for this equation. Our results extend the corresponding results of Sirouni et al. [32].

2. Main Result

Throughout in this section $\mathscr{X}$ is a nonempty set, we write $\mathscr{X}_0 := \mathscr{X} - \{0\}$, and we denoted by $Aut(\mathscr{X})$ for the family of all automorphisms of $\mathscr{X}$. The identity function on $\mathscr{X}$ will be denoted by $Id_{\mathscr{X}}$, and for each $m \in \mathscr{X}^{\mathscr{X}}$ we write $mu = m(u)$ for $u \in \mathscr{X}$ and we defined $-m$ by $-mu := -m(u), 2mu = mu + mu$ and $m' = m'u := (Id_{\mathscr{X}} - m)u = u - mu$ for $u \in \mathscr{X}$.

Theorem 2.1. *Let $\mathscr{X}$ be a quasi-normed space and $\mathscr{Y}$ be a quasi-Banach space. Assume that $g : \mathscr{X} \to \mathscr{Y}$ is a mapping such that*

$$||g(u+v)+g(u-v)-2\ g(u)-\ g(v)-g(-v)|| \leq \varepsilon(u,v), \tag{16}$$

where $\varepsilon : \mathscr{X}_0 \times \mathscr{X}_0 \to [0, \infty]$, $u, v \in \mathscr{X}_0$ such that $u + v \neq 0$ and $u - v \neq 0$.

Assume that

$$l(\mathscr{X}) := \{m \in Aut(\mathscr{X}) : m, -m, m', (Id_X - 2m) \in Aut(\mathscr{X}), \alpha_m < 1\} \tag{17}$$

is an nonempty set, where

$$\alpha_m := 2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(Id_X - 2m),$$

$$\lambda(m) := \inf\{t \in \mathbb{R}_+ : \varepsilon(mu, mv) \leq t\varepsilon(u, v)\ \forall u, v \in \mathscr{X}_0\},$$

for $m \in Aut(\mathscr{X}), K \geq 1$.

Then, for each non empty subset $\mathcal{A} \subset l(\mathscr{X})$ such that

$$a \circ b = b \circ a, (a, b \in \mathcal{A}), \tag{18}$$

there exists a unique function $D : \mathscr{X} \to \mathscr{Y}$ satisfying (14) and

$$\| D(u) - g(u) \|^{\theta} \leq 4\varepsilon^*(u), \tag{19}$$

for all $u \in \mathscr{X}_0$, where $\theta = \log_{2K} 2$ and $\varepsilon^*(u) := \inf\left\{\frac{\varepsilon^{\theta}(m'u, mu)}{1-\alpha_m^{\theta}} : m \in \mathcal{A}\right\}$.

Proof. Fix $m \in \mathcal{A}$. Replacing u by $m'u$ and v by mu in (16), we have

$$\|g(u) + g((Id_{\mathscr{X}} - 2m)u) - 2g(m'u) - g(mu) - g(-mu)\| \leq \varepsilon(m'u, mu) := \varepsilon_m(u), \quad (20)$$

for all $u \in \mathscr{X}_0$. We define the operators $\mathcal{T}_m : \mathscr{Y}^{\mathscr{X}_0} \to \mathscr{Y}^{\mathscr{X}_0}$ and $\Lambda_m : \mathbb{R}_+^{\mathscr{X}_0} \to \mathbb{R}_+^{\mathscr{X}_0}$ by

$$\mathcal{T}_m\xi(u) := 2\xi(m'u) + \xi(mu) + \xi(-mu) - \xi((Id_{\mathscr{X}} - 2m)u) \quad (21)$$

and

$$\Lambda_m\delta(u) := 2K\delta(m'u) + K^2\delta(mu) + K^3\delta(-mu) + K^3\delta((Id_{\mathscr{X}} - 2m)u), \quad (22)$$

for all $u \in \mathscr{X}_0$, $\xi \in \mathscr{Y}^{\mathscr{X}_0}$ and $\delta \in \mathbb{R}_+^{\mathscr{X}_0}$.
Then (20) becomes $\|g(u) - \mathcal{T}_m g(u)\| \leq \varepsilon_m(u)$, for all $u \in \mathscr{X}_0$. The operator Λ_m has the form given by (9) with $s = 4$ and $g_1(u) = m'u, g_2(u) = mu, g_3(u) = -mu, g_4(u) = (Id_{\mathscr{X}} - 2m)u$, $l_1(u) = 2K$, $l_2(u) = K^2$ and $l_3(u) = l_4(u) = K^3$ for all $u \in \mathscr{X}_0$. Further, we have

$$\begin{aligned}
&||\mathcal{T}_m\xi(u) - \mathcal{T}_m\mu(u)|| \\
&= ||2\xi(m'u) + \xi(mu) + \xi(-mu) - \xi((Id_{\mathscr{X}} - 2m)u) - 2\mu(m'u) \\
&\quad - \mu(mu) - \mu(-mu) + \mu((Id_{\mathscr{X}} - 2m)u)|| \\
&\leq 2K||\xi(m'u) - \mu(m'u)|| + K^2||\xi(mu) - \mu(mu)|| \\
&\quad + K^3||\xi(-mu) - \mu(-mu)|| + K^3||\xi((Id_{\mathscr{X}} - 2m)u) \\
&\quad - \mu((Id_{\mathscr{X}} - 2m)u)||, \\
&= \sum_{i=0}^{4} l_i(u) \parallel \xi(g_i(u)) - \mu(g_i(u)) \parallel,
\end{aligned}$$

for all $u \in \mathscr{X}_0$ and $\xi, \mu \in \mathscr{Y}^{\mathscr{X}_0}$. Using the definition of $\lambda(m)$, $\varepsilon(mu, mv) \leq \lambda(m)\varepsilon(m'u, mu)$ for all $u, v \in \mathscr{X}_0$ we have to show that $\Lambda_m^n\varepsilon_m(u) \leq \alpha_m^n\varepsilon(m'u, mu)$ for all $u \in \mathscr{X}_0$, where $\alpha_m = 2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(id_{\mathscr{X}} - 2m)$.

If $n = 0$, then $\varepsilon_m(u) = \varepsilon(m'u, mu)$. If $n = 1$, we have

$$\begin{aligned}
&\Lambda_m\varepsilon(u) \\
&= 2K\varepsilon_m(m'u) + K^2\varepsilon_m(mu) + K^3\varepsilon_m(-mu) \\
&\quad + K^3\varepsilon_m((id_{\mathscr{X}} - 2m)u) \\
&= 2K\varepsilon\,(m'(m'u), m(m'u)) + K^2\varepsilon\,(m'(mu), m(mu)) \\
&\quad + K^3\varepsilon\,(m'(-mu, m(-mu)) \\
&\quad + K^3\varepsilon\,(m'((Id_{\mathscr{X}} - 2m)u), m((Id_{\mathscr{X}} - 2m)u)) \\
&= 2K\varepsilon\,(m'(m'u), m'(mu)) + K^2\varepsilon\,(m(m'u), m(mu)) \\
&\quad + K^3\varepsilon\,(-m(m'u), -m(mu))
\end{aligned}$$

$$\begin{aligned}
&+K^3\varepsilon\left((Id_{\mathscr{X}}-2m)(m'(u),(Id_{\mathscr{X}}-2m)(mu)\right)\\
&\leq 2K\lambda(m')\varepsilon(m'u,mu)+K^2\lambda(m)\varepsilon(m'u,mu)\\
&+K^3\lambda(-m)\varepsilon(m'u,mu)+K^3\lambda(Id_{\mathscr{X}}-2m)\varepsilon(m'u,mu)\\
&=2K\lambda(m')+K^2\lambda(m)+K^3\lambda(-m)\\
&+K^3\lambda(Id_{\mathscr{X}}-2m)\varepsilon(m'u,mu)\\
&=\alpha_m\varepsilon(m'u,mu).
\end{aligned}$$

Now, further, if $r=2$, we have

$$\begin{aligned}
&\Lambda^2\varepsilon_m(u)\\
&=\Lambda\left[\Lambda\varepsilon_m(u)\right]\\
&=2K\Lambda_m\varepsilon_m(m'u)+K^2\Lambda_m\varepsilon_m(mu)+K^3\Lambda_m\varepsilon_m(-mx)\\
&+K^3\Lambda_m\varepsilon_m((Id_{\mathscr{X}}-2m)u)\\
&=2K\alpha_m\varepsilon\left(m'(m'u),m(m'u)\right)+K^2\alpha_m\varepsilon\left(m'(mu),m(mu)\right)\\
&+K^3\alpha_m\varepsilon\left(m'(-mu,m(-mu))\right)\\
&+K^3\varepsilon\left(m'((Id_{\mathscr{X}}-2m)u),m((Id_{\mathscr{X}}-2m)u)\right)\\
&=2K\alpha_m\varepsilon\left(m'(m'u),m'(mu)\right)+K^2\alpha_m\varepsilon\left(m(m'u),m(mu)\right)\\
&+K^3\alpha_m\varepsilon\left(-m(m'u),-m(mu)\right)\\
&+K^3\alpha_m\varepsilon\left((Id_{\mathscr{X}}-2m)(m'u),(Id_{\mathscr{X}}-2m)(mu)\right)\\
&\leq 2K\alpha_m\lambda(m')\varepsilon(m'u,mu)+K^2\alpha_m\lambda(m)\varepsilon(m'u,mu)\\
&+K^3\alpha_m\lambda(-m)\varepsilon(m'u,mu)+K^3\lambda(Id_{\mathscr{X}}-2m)\varepsilon(m'u,mu)\\
&=\alpha_m(2K\lambda(m')+K^2\lambda(m)+K^3\lambda(-m)\\
&+K^3\lambda(Id_{\mathscr{X}}-2m))\ \varepsilon(m'u,mu)\\
&=\alpha_m^2\varepsilon(m'u,mu).
\end{aligned}$$

Proceeding on the similar lines, we get

$$\Lambda_m^n\varepsilon_m(x)\leq\alpha_m^n\varepsilon(m'u,mu), \tag{23}$$

for all $u\in\mathscr{X}_0$ and $n\in\mathbb{N}_0$. Hence

$$\varepsilon^*(u)=\sum_{n=0}^{\infty}\left(\Lambda_m^n\varepsilon_m\right)^\theta(u)\leq\varepsilon^\theta(m'u,mu)\sum_{r=0}^{\infty}\alpha_m^{n\theta}=\frac{\varepsilon^\theta(m'u,mu)}{1-\alpha_m^\theta}<\infty,$$

for all $u\in\mathscr{X}_0$. Therefore by the Theorem 1.7, there exists a solution $D_u:\mathscr{X}\to\mathscr{Y}$ of the equation

$$D_m(u)=2D_m(m'u)+D_m(mu)+D_m(-mu)-D_m((Id_{\mathscr{X}}-2m)u), \tag{24}$$

for all $u\in\mathscr{X}_0$, which is a fixed point of $\mathcal{T}_m$ such that

$$\|D_m(u)-g(u)\|^\theta\leq 4\varepsilon^*(u), \tag{25}$$

for all $u \in \mathscr{X}_0$. Moreover, $D_m(u) = \lim_{r\to\infty} \mathcal{T}_m^r g(u)$ for all $u \in \mathscr{X}_0$.

Now, to prove that D_m satisfies the functional equation (14) on $\mathscr{X}_0$, we have to prove the following inequality

$$\begin{aligned}
&\| \mathcal{T}_m^r g(u+v) + \mathcal{T}_m^r g(u-v) - 2\mathcal{T}_m^r g(u) - \mathcal{T}_m^r g(v) - \mathcal{T}_m^r g(-v) \| \\
&\leq \ \alpha_m^r \varepsilon(u,v),
\end{aligned} \tag{26}$$

for all $u, v \in \mathscr{X}_0$ such that $u+v \neq 0$, $u-v \neq 0$, and $r \in \mathbb{N}_0$. Indeed if $r = 0$ then (26) is simply (16). So we suppose that (26) holds for $r \in \mathbb{N}$ and $u, v \in \mathscr{X}_0$ such that $u+v \neq 0$, $u-v \neq 0$. Then from (21) and the triangle inequality, we get

$$\begin{aligned}
&\| \mathcal{T}_m^{r+1} g(u+v) + \mathcal{T}_m^{r+1} g(u-v) - 2\mathcal{T}_m^{r+1} g(u) - \mathcal{T}_m^{r+1} g(v) \\
&- \mathcal{T}_m^{r+1} g(-v) \| \\
&= \| 2\mathcal{T}_m^r g(m'(u+v)) + \mathcal{T}_m^r g(m(u+v)) + \mathcal{T}_m^r g(-m(u+v)) \\
&- \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u+v)) + 2\mathcal{T}_m^r g(m'(u-v)) + \mathcal{T}_m^r g(m(u-v)) \\
&+ \mathcal{T}_m^r g(-m(u-v)) - \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u-v)) - 4\mathcal{T}_m^r g(m'(u)) \\
&- 2\mathcal{T}_m^r g(m(u)) - 2\mathcal{T}_m^r g(-m(u)) + 2\mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u)) \\
&- 2\mathcal{T}_m^r g(m'(v)) - \mathcal{T}_m^r g(m(v)) - \mathcal{T}_m^r g(-m(v)) \\
&+ \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(v)) - 2\mathcal{T}_m^r g(m'(-v)) - \mathcal{T}_m^r g(m(-v)) \\
&- \mathcal{T}_m^r g(-m(-v)) + \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(-v)) \| \\
&\leq 2K \| \mathcal{T}_m^r g(m'(u+v)) + \mathcal{T}_m^r g(m'(u-v)) - 2\mathcal{T}_m^r g(m'(u)) \\
&- \mathcal{T}_m^r g(m'(v)) - \mathcal{T}_m^r g(m'(-v)) \| \\
&+ K^2 \| \mathcal{T}_m^r g(m(u+v)) + \mathcal{T}_m^r g(m(u-v)) - 2\mathcal{T}_m^r g(m(u)) \\
&- \mathcal{T}_m^r g(m(v)) - \mathcal{T}_m^r g(m(-v) \| \\
&+ K^3 \| \mathcal{T}_m^r g(-m(u+v)) + \mathcal{T}_m^r g(-m(u-v)) - 2\mathcal{T}_m^r g(-m(u)) \\
&- \mathcal{T}_m^r g(-m(v)) - \mathcal{T}_m^r g(-m(-v) \| \\
&+ K^3 \| \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u+v)) + \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u-v)) \\
&- 2\mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(u)) - \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(v)) \\
&- \mathcal{T}_m^r g((Id_{\mathscr{X}} - 2m)(-v) \| \\
&\leq \alpha_m^r [2K\varepsilon(m'u, m'v) + K^2\varepsilon(mu, mv) + K^3\varepsilon(-mu, -mv) \\
&+ K^3\varepsilon((Id_{\mathscr{X}} - 2m)u, (Id_{\mathscr{X}} - 2m)v)] \\
&\leq \alpha_m^r [2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(Id_x - 2m)]\varepsilon(u,v) \\
&= \alpha_m^{r+1} \varepsilon(u,v).
\end{aligned}$$

By induction, we have shown that (26) holds for all $r \in \mathbb{N}$. Therefore from (2) and (26), we have

$$\begin{aligned} & |||\mathcal{T}_m^r g(u+v) + \mathcal{T}_m^r g(u-v) - 2\mathcal{T}_m^r g(u) - \mathcal{T}_m^r g(v) - \mathcal{T}_u^r g(-v)|||^\theta \\ & \leq \; K^{r\theta} \alpha_m^{r\theta} \varepsilon^\theta(u,v). \end{aligned} \tag{27}$$

Letting $r \to \infty$ in (27) and using the definition of $l(\mathscr{X})$, we have

$$D_m(u+v) + D_m(u-v) = 2D_m(u) + D_m(v) + D_m(-v), \tag{28}$$

for all $u, v \in \mathscr{X}_0$. Thus we have prove that for every for $m \in \mathcal{A}$ there exists a function $D_m : \mathscr{X}_0 \to \mathscr{Y}$ which is the solution of functional equation (14) on $\mathscr{X}_0$ and satisfies

$$||g(u) - D_m(u)||^\theta \leq 4 \left(\frac{\varepsilon^\theta(m'u, mu)}{1 - \alpha_m^\theta} \right) = 4\varepsilon^*(u),$$

for all $u \in \mathscr{X}_0$.

Now, we prove that $D_m = D_q$ for all $m, q \in \mathcal{A}$. Fix m, q and note that D_q satisfies (25) with m replaced by q. Hence by replacing (u, v) with $(m'u, mu)$ in (28) and using (1) and (2), we get $TD_j = D_j$, for $j = m, q$ and

$$\begin{aligned} |||D_m(u) - D_q(u)|||^\theta &\leq |||D_m(u) - g(u)|||^\theta + |||D_q(u) - g(u)|||^\theta \\ &\leq \left(\frac{4\varepsilon_m^\theta(u)}{1 - \alpha_m^\theta} \right) + \left(\frac{4\varepsilon_q^\theta(u)}{1 - \alpha_q^\theta} \right), \end{aligned}$$

for all $u \in \mathscr{X}_0$. It follows from the linearity of Λ and (23) that

$$\begin{aligned} |||D_m(u) - D_q(u)|||^\theta &= |||\mathcal{T}^n D_m(u) - \mathcal{T}^n D_q(u)|||^\theta \\ &\leq 4 \left(\frac{\Lambda^n \varepsilon_m^\theta(u)}{1 - \alpha_m^\theta} \right) + 4 \left(\frac{\Lambda^n \varepsilon_q^\theta(u)}{1 - \alpha_q^\theta} \right) \\ &\leq (\alpha_m)^n U_m(u) + (\alpha_q)^n U_q(u), \end{aligned}$$

where $U_m(u) = 4\frac{\varepsilon_m^\theta(u)}{1-\alpha_m^\theta}$ for all $u \in \mathscr{X}_0$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we get $D_m = D_q = D$. Thus, we have

$$\|g(u) - D(u)\|^\theta \leq U_m(u),$$

for all $u \in \mathscr{X}_0, m \in \mathcal{A}$. Thus, we derive (19). Due to (28), it is easy to notice that D is a solution of (14). Now to prove the uniqueness of the mapping D, let us assume that there exists a mapping $D' : \mathscr{X} \to \mathscr{Y}$ which satisfies (14) and inequality

$$\|g(u) - D'(u)\|^\theta \leq 4\varepsilon^*(u).$$

Using(2), we have

$$|||D(u) - D'(u)|||^\theta \leq 8\varepsilon^*(u).$$

Further $\mathcal{T}D'(u) = D'(u)$ for all $u \in \mathscr{X}_0$. Consequently, with a fixed $m \in \mathcal{A}$

$$\begin{aligned} |||D(u) - D'(u)|||^\theta &= |||\mathcal{T}^n D(u) - \mathcal{T}^n D'(u)|||^\theta \\ &\leq 8\Lambda^n \varepsilon^*(u) \\ &\leq \frac{8\Lambda^n \varepsilon_m^\theta(u)}{1-\alpha_m^\theta} \\ &\leq \frac{8\alpha_m^n \varepsilon_m^\theta(u)}{1-\alpha_m^\theta}, \end{aligned}$$

for all $u \in \mathscr{X}_0, m \in \mathcal{A}$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we get $D = D'$. The proof of the theorem is complete.

□

In the following theorem, we prove the hyperstability of the equation (14) in the Banach spaces.

Theorem 2.2. *Let $\mathscr{X}$ be a quasi-normed space and $\mathscr{Y}$ be a quasi-Banach space, and ε be as in the above Theorem 2.1. Suppose that there exists a non empty set $\mathcal{A} \in l(\mathscr{X})$ such that $a \ o \ b = b \ o \ a$ for all $a, b \in \mathcal{A}$ and*

$$\begin{cases} \inf\limits_{m\in\mathcal{A}} \varepsilon^\theta(m'u, mu) & = 0 \\ \sup\limits_{m\in\mathcal{A}} \alpha_m < 1. \end{cases} \tag{29}$$

$u \in \mathscr{X}_0$, then every $g : \mathscr{X} \to \mathscr{Y}$ satisfying (16) is a solution of (14) on $\mathscr{X}_0$.

Proof. Suppose that $g : \mathscr{X} \to \mathscr{Y}$ is a mapping which is satisfies (16). Then, by the Theorem 2.1, there exists a mapping $D : \mathscr{X} \to \mathscr{Y}$, which satisfies (14) and $||g(u) - D(u)||^\theta \leq \varepsilon^*(u)$ for all $u \in \mathscr{X}_0$. Since, from (29), $\varepsilon^*(u) = 0$ for all $u \in \mathscr{X}_0$. This implies that $g(u) = D(u)$ for all $u \in \mathscr{X}_0$, where

$$g(u+v) + g(u-v) = 2g(u) + g(v) + g(-v),$$

for all $u, v \in \mathscr{X}_0$. Which satisfies the functional equation (14) on $\mathscr{X}_0$. □

From Theorems 2.1 and 2.2, we can obtain the following corollaries as natural results.

Corollary 2.3. *Let $\mathscr{X}$ be a quasi-normed space, and $\mathscr{Y}$ be a quasi-Banach space. Assume that $p < 0$, $q < 0$ and φ is a positive number. If $g : \mathscr{X} \to \mathscr{Y}$ satisfies*

$$\begin{aligned} &\| \ g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v) \ \|^\theta \\ &\leq \varphi^\theta \left(\| \ u \ \|^p + \| \ u \ \|^q\right)^\theta, \end{aligned} \tag{30}$$

for all $u, v \in \mathscr{X}_0$, then g is a solution of the functional equation (14) on $\mathscr{X}_0$.

Proof. The proof follows from the above Theorem 2.2 by taking $\varepsilon^\theta(u, v) = \varphi^\theta$ $(\| \ u \ \|^p + \| \ v \ \|^q)^\theta$ for all $u, v \in \mathscr{X}_0$ with some real numbers $\varphi \geqslant 0$, $p < 0$ and

$q < 0$. For each $j \in \mathbb{N}$, define m_j: $\mathscr{X}_0 \to \mathscr{X}_0$ by $m_j x := m_j(x) = -ju$ and m'_j:$\mathscr{X}_0 \to \mathscr{X}_0$ by $m'_j u := m'_j(u) = (1+j)u$. Then

$$\begin{aligned}\varepsilon^\theta(m_j u, m_k v) &= \varepsilon^\theta(-ju, -kv)\\ &= [\varphi\,(\| -ju \|^p + \| -kv \|^q)]^\theta\\ &= [\varphi j^p \| u \|^p + \varphi k^q \| v \|^q]^\theta\\ &\leq [(j^p + k^q)\varphi(\| u \|^p + \| v \|^q)]^\theta\\ &= (j^p + k^q)^\theta \varepsilon^\theta(u, v),\end{aligned}$$

for all $u, v \in \mathscr{X}_0$ and $k, j \in \mathbb{N}$. Hence

$$\lim_{j\to\infty} \varepsilon^\theta(m'_j u, m_j v) \leq \lim_{j\to\infty} \left((1+j)^p + j^q)\right)^\theta \varepsilon^\theta(u, v) = 0,$$

for all $u, v \in \mathscr{X}_0$ and $k, j \in \mathbb{N}$. Then (29) is valid with $\lambda(m_j) = j^p + j^q$ for $j \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $j \geq n_0$ and

$$\alpha_{m_j} = 2\left((1+j)^p + (1+j)^q\right) + 2(j^p + j^q) + (1+2j)^p + (1+2j)^q < 1.$$

Therefore we can say that (17) is satisfied with $\mathcal{A} := \{m_j \in Aut(\mathscr{X}) : j \in \mathbb{N}_{n0}\}$. Hence, by the Theorem 2.2, every $g : \mathscr{X} \to \mathscr{Y}$ satisfying (30) is a solution of the functional equation (14) on $\mathscr{X}_0$. □

Now, we extend the main result of Piszczek et al. [27] (Theorem 2) in the framework of quasi-Banach space.

Corollary 2.4. *Let $\mathscr{X}$ be a quasi-normed space and $\mathscr{Y}$ be a quasi-Banach space. Assume that $p < 0$ and φ is a positive number. If $g : \mathscr{X} \to \mathscr{Y}$ satisfies*

$$\begin{aligned}&\| g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v) \|^\theta\\ &\leq \varphi^\theta \left(\| u \|^p + \| v \|^p\right)^\theta,\end{aligned} \tag{31}$$

for all $u, v \in \mathscr{X}_0$, then g is a solution of the functional equation (14) on $\mathscr{X}_0$.

Proof. It is easily seen that the function ε given by

$$\varepsilon^\theta(u, v) = \left[\varphi\left(\| u \|^p + \| v \|^p\right)\right]^\theta,$$

for all $u, v \in \mathscr{X}_0$ satisfies (29), since

$$\begin{aligned}\varepsilon^\theta(ju, kv) = [\varphi \| ju \|^p + \varphi \| kv \|^p]^\theta &\leq [\varphi(j^p + k^p)\left(\| u \|^p + \| v \|^p\right)]^\theta\\ &= (j^p + k^p)^\theta \varepsilon^\theta(u, v),\end{aligned}$$

for all $u, v \in \mathscr{X}_0, k, j \in \mathbb{N}$ and $kj \neq 0$. The remaining part of the proof is similar to the Corollary 2.3. □

Remark 2.1. Piszczek et al. [27] obtained Corollary 2.4 in the setting of a Banach space.

If $\mathscr{X}$ is a quasi-normed space and $g : \mathscr{X} \to \mathscr{Y}$ satisfies (31) for $u, v \in \mathscr{X}_0$, with $p < 0$, then by Theorem 2.2, we know that g satisfies the Drygas functional equation on $\mathscr{X}_0$. It is easy to see that if $g(0) = 0$, then g satisfies the Drygas functional equation on $\mathscr{X}$. So we have the following corollary.

Corollary 2.5. *Let $\mathscr{X}$ be a quasi-normed space, $\mathscr{Y}$ be a quasi-Banach space. Assume that $p < 0$ and φ is a positive number. If $g : \mathscr{X} \to \mathscr{Y}$ satisfies $g(0) = 0$ and inequality*

$$\begin{aligned} &\| g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v) \|^\theta \\ &\le \varphi^\theta \left(\| u \|^p + \| v \|^p\right)^\theta, \end{aligned} \tag{32}$$

for all $u, v \in \mathscr{X}_0$, then g is a solution of the functional equation (14) on $\mathscr{X}$.

Corollary 2.6. *Let $\mathscr{X}$ be a quasi-normed space, $\mathscr{Y}$ be a quasi-Banach space. Assume that $p + q < 0$ and φ is a positive number. If $g : \mathscr{X} \to \mathscr{Y}$ satisfies*

$$\begin{aligned} &\| g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v) \|^\theta \\ &\le \varphi^\theta \left(\| u \|^p \| v \|^q\right)^\theta, \end{aligned} \tag{33}$$

for all $u, v \in \mathscr{X}_0$, then g is a solution of the functional equation (14) on $\mathscr{X}_0$.

Proof. It is easily seen that the function ε given by

$$\varepsilon^\theta(u, v) = \left(\varphi\left(\| u \|^p \| v \|^q\right)\right)^\theta,$$

for all $u, v \in \mathscr{X}_0$ satisfies (29), since

$$\begin{aligned} \varepsilon^\theta(ju, kv) = \varphi^\theta\left(\| ju \|^p \| kv \|^q\right)^\theta &\le \varphi^\theta (j^p k^q)^\theta \left(\| u \|^p \| v \|^q\right)^\theta \\ &= (j^p k^q)^\theta \varepsilon^\theta(u, v), \end{aligned}$$

for all $u, v \in \mathscr{X}_0$, $k, j \in \mathbb{N}$ and $kj \neq 0$. On the similar lines of the Corollary 2.3, we get the results. □

By an analogous conclusion, the function ε given by

$$\varepsilon^\theta(u, v) = \varphi^\theta\left(\| u \|^p + \| v \|^q + \| u \|^p \| v \|^q\right)^\theta,$$

for all $u, v \in \mathscr{X}_0$ satisfies (29), since

$$\begin{aligned} \varepsilon^\theta(ju, kv) &= \varphi^\theta\left(\| ju \|^p + \| kv \|^q + \| ju \|^p \| kv \|^q\right)^\theta \\ &= \varphi^\theta\left(j^p \| u \|^p + k^q \| v \|^q + j^p k^q \| u \|^p \| v \|^q\right)^\theta \\ &\le \left(j^p + k^q + j^p k^q\right)^\theta \varepsilon^\theta(u, v), \end{aligned}$$

for all $u, v \in \mathscr{X}_0, k, j \in \mathbb{N}$ and $kj \neq 0$. So we have the following corollary.

Corollary 2.7. *Let $\mathscr{X}$ be a complex quasi-normed space, $\mathscr{Y}$ be a quasi-Banach space. Assume that $p < 0$, $q < 0$, $p + q < 0$ and φ is a positive number. If $g : \mathscr{X} \to \mathscr{Y}$ satisfies*

$$\begin{aligned} &\| g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v) \|^\theta \\ &\le \varphi^\theta \left(\| u \|^p + \| v \|^q + \| u \|^p \| v \|^q\right)^\theta, \end{aligned} \tag{34}$$

for all $u, v \in \mathscr{X}_0$, *then* g *is a solution of the functional equation (14) on* $\mathscr{X}_0$.

The following result corresponds to the results on the non-homogeneous Drygas functional equation (35).

Corollary 2.8. *Let* $\mathscr{X}$ *be a quasi-normed space,* $\mathscr{Y}$ *be a quasi-Banach space,* ε *as in Theorem 2.1 and* $H : \mathscr{X}^2 \to \mathscr{Y}$. *Suppose that* $\| H(u,v) \|^\theta \leq \varepsilon^\theta(u,v)$ *for all* $u, v \in \mathscr{X}_0$, *where* $H(u_0, v_0) \neq 0$ *for some* $u_0, v_0 \in \mathscr{X}_0$ *and there exists a nonempty* $\mathcal{A} \in l(\mathscr{X})$ *such that (18) and (29) satisfies. Then the non-homogeneous equation*

$$g(u+v) + g(u-v) = 2g(u) + 2g(v) + g(-v) + H(u,v), \tag{35}$$

for all $u, v \in \mathscr{X}_0$, *has no solution in the class of functions* $g : \mathscr{X} \to \mathscr{Y}$.

Proof. Let us assume the $g : \mathscr{X} \to \mathscr{Y}$ is a solution to (35). Then

$$\begin{aligned}
&\| g(u+v) + g(u-v) - 2g(u) - 2g(v) - g(-v) \|^\theta \\
&= \| 2g(u) + g(v) + g(-v) + H(u,v) - 2g(u) - g(v) - g(-v) \|^\theta \\
&= \| H(u,v) \|^\theta \\
&\leq \varepsilon^\theta(u,v),
\end{aligned}$$

for all $u, v \in \mathscr{X}_0$. Consequently, by Theorem 2.2, g is a solution of (14). Therefore, we have

$$H(u_0, v_0) = g(u_0 + v_0) + g(u_0 - v_0) - 2g(u_0) + g(v_0) - g(-v_0) = 0,$$

which is contradiction. Hence non-homogeneous equation (35) has no solution in the class of functions $g : \mathscr{X} \to \mathscr{Y}$. □

The following example shows that the assumption in the above Corollary is essential.

Example 2.9. Let $\mathscr{X} = \mathscr{Y} = L^{\frac{1}{2}}[0,1]$ and $\| u \|_{\mathscr{X}} = \| u \|_{\mathscr{Y}} = \left(\int_0^1 | u(t) |^{\frac{1}{2}} \, dt \right)^2$ for all $u \in \mathscr{X}$, where
$L^{\frac{1}{2}}[0,1] = \{ g : [0,1] \to \mathbb{R} : | g |^{\frac{1}{2}}$ is Lebesgue integrable$\}$ and $g(x) = u^4 + u^2$ for all $u \in \mathscr{X}$, $\varepsilon(u,v) = \left(\int_0^1 \left(2\sqrt{3} \, | u(t)v(t) | \right) dt \right)^2$ for all $u, v \in \mathscr{X}$ such that $u+v, u-v \neq 0$.

$$||g(u+v) + g(u-v) - 2g(u) - g(v) - g(-v)]|| \leq \varepsilon(u,v) \tag{36}$$

but g does not satisfy the functional equation (14).

Proof. Note that $\mathscr{X}$ and $\mathscr{Y}$ are quasi-Banach spaces

$$\begin{aligned}
&||g(u+v)+g(u-v)-2g(u)-g(v)-g(-v)]||| \\
&= \bigg(\int_0^1 \Big((u(t)+v(t))^4 + (u(t)+v(t))^2 + (u(t)-v(t))^4 \\
&+ (u(t)-v(t))^2 - 2u^4(t) - 2u^2(t) - v^4(t) - v^2(t) \\
&- v^4(t) - v^2(t) \Big)^{\frac{1}{2}} dt \bigg)^2 \\
&= \bigg(\int_0^1 \big(u^4(t) + 4v^3(t)u(t) + 6u^2(t)v^2(t) + 4v^3(t)u(t) + v^4(t) \\
&+ u^2(t) + v^2(t) + 2u(t)v(t) + u^4(t) - 4u^3(t)v(t) + 6u^2(t)v^2(t) \\
&- 4u(t)v^3(t) + v^4(t) + u^2(t) + v^2(t) - 2u(t)v(t) - 2u^4(t) \\
&- 2u^2(t) - v^4(t) - v^2(t) - v^4(t) - v^2(t) \big)^{\frac{1}{2}} dt \bigg)^2 \\
&= \bigg(\int_0^1 \big(12u^2(t)v^2(t) \big)^{\frac{1}{2}} dt \bigg)^2 \\
&= \bigg(\int_0^1 \Big(2\sqrt{3} \mid u(t)v(t) \mid \Big) dt \bigg)^2 \\
&= \varepsilon(u,v).
\end{aligned}$$

This proves that (36) holds, but g does not satisfy the functional equation (14). □

Example 2.10. Let $\mathscr{X} = [-1,1]\backslash\{0\}$ and let $g : \mathscr{X} \to \mathbb{R}$ be defined as $g(u) = \mid u \mid$, $u \in \mathscr{X}$. Then for all $u, v \in \mathscr{X}$ such that $u+v, u = v \neq 0$, ϕ be a positive number and

$$\|g(u+v)+g(u-v)-2g(u)-g(v)-g(-v)\|^\theta \leq (\phi(\|u\|^p + \|v\|^p)^\theta$$

with $p < 0$, but g does not satisfy the functional equation (14) on $\mathscr{X}$.

Example 2.11. Let $\mathscr{X} = \mathbb{R}\backslash[-1,1]$ and let $g : \mathscr{X} \to \mathbb{R}$ be defined as $g(u) = C$, for all $u \in \mathscr{X}$, for some $C > 0$. Then for all $u, v \in \mathscr{X}$ such that $u+v, u-v \neq 0$, ϕ be a positive number and

$$\|g(u+v)+g(u-v)-2g(u))-g(v)-g(-v)\|^\theta \leq (\phi(\|u\|^p + \|v\|^p))^\theta$$

with $p \geq 0$, but g does not satisfy the functional equation (14) on $\mathscr{X}$.

Remark 2.2. If $\mathscr{X}$ is a normed space and $\mathscr{Y}$ is a Banach space and $K = 1$ in Theorem 2.1, we obtain the corresponding results of Sirouni et al. [32].

Question. Prove or disprove the conclusion of Theorem 2.1 in the case $\mathscr{Y}$ is a normed space.

Conflicts of interest : The authors declare that they have not any conflict of interest.

Acknowledgments : The authors are thankful to the learned referees for their valuable suggestions.

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