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DISSIPATIVE RANDOM DYNAMICAL SYSTEMS AND LEVINSON CENTER

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Abstract. In this work, some various types of Dissipativity in random dynamical systems are introduced and studied: point, compact, local, bounded and weak. Moreover, the notion of random Levinson center for compactly dissipative random dynamical systems presented and prove some essential results related with this notion.

1. Introduction

Random Dynamical System(RDS) considered as branch of the dynamical system that involve two essential components:

- the noise model,
- the system model which is disconcerted by the noise.

The noise, through this paper, will be exhibited by a metric dynamical system in the view of the ergodic theory. The RDS is an importance in the modeling of several phenomena in biology, physics, ets. In 1945 [14] Ulam and Neumann the first lesson the RDS. Arnold (1998) [2] are introduced the concept of RDS.

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The aim of this paper is studying different types of dissipativity for RDSs and the notion of Levinson center. Dissipative systems have certain importance in many science such as physics and engineering. The hypothesis of dissipation yield results in essential constraint on behavior of its dynamic. Many papers devoted to the study the dissipativety in RDSs, see, for example, Gu [7], Huang [8], Kloeden [9], Kuehn [10], Kuksin [11], Wang [15], Xiaoying [17], Yuhong [18] for more details.

There are several applications of the theory of dissipativity: systems with a finite number of degrees of freedom, stability of feedback systems, electrical networks, thermodynamic. Like a deterministic case, the theory of stochastic dissipativity play an essential role in lecturing stochastic robustness [16], risk-sensitive disturbance rejection, stability in probability of feedback interconnections, and optimality with averaged performance measures for stochastic dynamical systems [12], the stochastic dissipativity theory can be used to design feedback controllers that add dissipation and guarantee stability robustness in probability allowing stochastic stabilization to be understood in physical terms [13].

Furthermore, in 2021, Cheban [5] give a numerous applications these results to different classes of evolution equations (ordinary differential equations, difference equations, functional differential equations and some class of partial differential equations of parabolic type).

2. Random dynamical systems, some general concepts

General facts about RDSs are stated in this section. For more details, see for example [1] and [6].

Definition 2.1. ([6]) Let $(\Omega, \mathcal{F},\ \mathbb{P})$ be a probability space and $\theta : \mathbb{T}\times\Omega \longrightarrow \Omega$ be a measurable function satisfy the following

$$\theta_0 = \text{ id},\ \theta_{\mathrm{t}} \circ \theta_{\mathrm{s}} = \theta_{\mathrm{t+s}} \text{ for all t, s} \in \mathbb{T};\text{ and } \theta_{\mathrm{t}}\ \mathbb{P}\ = \mathbb{P} \text{ for all t} \in \mathbb{T}.$$

A set $\mathrm{B} \in \mathcal{F}$ is called θ-invariant if θ_tB=B for all t$\in \mathbb{T}$. An MDS θ called ergodic under $\mathbb{P}$ if for any θ-invariant set $\mathrm{B} \in \mathcal{F}$ we have either $\mathbb{P}$(B)=0 or $\mathbb{P}$(B)=1.

Definition 2.2. ([6]) Let X be a topological space and $\mathbb{T}$ be a locally compact group. The random dynamical system (RDS) is a pair $(\theta,\ \varphi)$ involving an MDS θ and a cocycle φ over θ of continuous mappings of X, that is, a measurable mapping

$$\varphi\ \mathbb{T}\times\Omega\times X \longrightarrow X,\ (t,\ \omega,\ x) \longmapsto \varphi(t,\ \omega,\ x) \text{ such that}$$

(1) for every $\mathrm{t} \in \mathbb{T}$ and $\omega \in \Omega$, the function $x \longmapsto \varphi(\mathrm{t}, \omega, x) \equiv \varphi(t, \omega)x$ is continuous,
(2) for all t, $\mathrm{s} \in \mathbb{T}$ and $\omega \in \Omega$, the function $\varphi(t,\omega) := \varphi(t,\omega,\cdot)$ fulfill:

$$\varphi(0, \omega) = \text{ id}, \ \varphi(t+s, \omega) = \varphi(t, \theta_{\mathrm{s}}\omega) \circ \varphi(s, \omega).$$

The property (2) called cocycle property of φ.

Definition 2.3. ([6]) Consider the metric space (X, d).

(1) A random set is the set-valued function $\omega \longmapsto \mathrm{M}(\omega) \neq \emptyset$ such that for any $x \in X$ the function

$$\omega \longmapsto \ \mathrm{dist}_{\mathrm{X}}(x, \mathrm{M}(\omega))$$

is measurable. The random set M is called a random closed set if $\mathrm{M}(\omega)$ is closed for each $\omega \in \Omega$ and any adjective applying on $\mathrm{M}(\omega)$ is closed for each $\omega \in \Omega$ applied similarly on M .
(2) A random set $\{\mathrm{M}(\omega)\}$ is said to be bounded if for some $x_0 \in X$ and some positive random variable $r(\omega)$ the following fulfill:

$$M(\omega) \subset \{x \in X : d(x, x_0) \leq r(\omega)\} \ \text{ for all } \omega \in \ \Omega.$$

(3) A tempered random variable (t.r.v) is a measurable function $\varepsilon : \Omega \longrightarrow \mathbb{R}$ with

$$\lim_{t \longrightarrow +\infty} \frac{1}{|t|} \log |\varepsilon(\theta_t \omega)| = 0.$$

Definition 2.4. ([6]) Consider the RDS (θ, φ). A set-valued mapping $\omega \longmapsto S(\omega)$ is called forward invariant (backward invariant) if for all $t > 0$ and $\omega \in \Omega$ we have $\varphi(t, \omega)S(\omega) \subseteq S(\theta_t\omega)$ (resp. $S(\theta_t\omega) \subseteq \varphi(t, \omega) S(\omega)$).

Definition 2.5. ([6]) A collection $\mathcal{U}$ of random sets is called a universe of sets if

(1) every members of $\mathcal{U}$ is closed, and
(2) $\mathcal{U}$ is closed with respect to inclusions.

Definition 2.6. ([6]) An absorbing random set for RDS (θ, φ) in the universe $\mathcal{U}$ is a random set A have the property that if for every M$\in \mathcal{U}$ and for all ω there exists $\mathrm{t}_0(\omega)$ with

$$\varphi(\mathrm{t}, \theta_{-\mathrm{t}}\omega)M(\theta_{-\mathrm{t}}\omega) \subset A(\omega) \ \text{ for all } \mathrm{t} \geq \mathrm{t}_0(\omega), \ \ \omega \in \Omega.$$

Definition 2.7. Consider the RDS (θ, φ). A random set M is said to be orbitally stable if for any t.r.v ε and any non-negative number t, there exists t.r.v δ such that

$$d(x, M(\omega)) < \delta(\omega) \ \text{ implies } \ d(\varphi(t, \theta_{-t}\omega)x, M(\omega)) < \varepsilon(\omega).$$

Definition 2.8. ([6]) A random closed set $\{M(\omega)\}$ from a universe $\mathcal{M}$ is called a random attractor of RDS $(\theta,\ \phi)$ in $\mathcal{M}$ if $B(\omega)$ is proper subset of X for every $\omega \in \Omega$ and:

(1) B is an invariant set, that is, $\varphi(t,\omega) B(\omega) = B(\theta_t\omega)$ for $t \geq 0,\ \omega \in \Omega$;
(2) B is an attracting in $\mathcal{U}$, that is, for all $M \in \mathcal{U}$

$$\lim_{n\to+\infty} d_X \{\varphi(t,\theta_{-t}\omega) M(\theta_{-t}\omega), B(\omega)\} = 0,\ \ \omega \in \Omega,$$

where $d_X\{A \setminus B\} = \sup_{x\in A} dis_X(x, B)$.

Definition 2.9. ([3]) Let (X, d) be a metric space. $K \subset X$ is said to be precompact or totally bounded if every sequence in K admits a subsequence converges to a point of X.

3. Compactly dissipative random dynamical systems

Some limiting properties of RDSs are established in this section, and essential results related with concept of compactly dissipative RDSs are proved.

Definition 3.1. ([6]) Let $M : \omega \to M(\omega)$ be a random set. The set-valued function

$$\omega \longmapsto \Gamma_M(\omega) := \bigcap \overline{\gamma_M^t(\omega)} = \bigcap \overline{\cup \varphi(\tau,\theta_{-\tau}\omega) M(\theta_{-\tau}\omega)},\ t > 0,\ \tau \geq t$$

is called the omega-limit set of the trajectories starting from M.

Proposition 3.2. ([6]) *An element x belong to the omega-limit set $\Gamma_M(\omega)$ if and only if there exist sequences $t_n \to +\infty$ and $y_n \in M(\theta_{-t_n}\omega)$ such that*

$$x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n.$$

Proposition 3.3. *Consider RDS (θ, φ), the following fulfill for all random sets A and B in X.*

(1) If $A \subseteq B$, *then* $\Gamma_A(\omega) \subseteq \Gamma_B(\omega)$.
(2) $\Gamma_{A\cup B(\omega)} \subseteq \Gamma_A(\omega) \cup \Gamma_B(\omega)$.
(3) *If the random set A is forward invariant (backward invariant, invariant), then $\Gamma_A(\omega) \subseteq \bar{A}$ (resp. $\bar{A} \subseteq \Gamma_A(\omega)$, $\Gamma_A(\omega) = \bar{A}$).*
(4) $\overline{\cup\{\Gamma_x | x \in M\}} \subseteq \Gamma_M(\omega)$.

Proof. (1) Let $x \in \Gamma_A(\omega)$. Then there exist sequences $t_n \to +\infty$, $y_n \in A(\theta_{-t_n}\omega)$ such that $x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n$. Since $A(\omega) \subset B(\omega)$, for all $\omega \in \Omega$, $A(\theta_{-t}\omega) \subset B(\theta_{-t}\omega)$. So $y_n \in B(\theta_{-t}\omega)$ and $x \in \Gamma_B(\omega)$. Hence $\Gamma_A(\omega) \subset \Gamma_B(\omega)$.

(2) Let $x \in \Gamma_{A\cup B(\omega)}$. Then there exist sequences $t_n \to +\infty, y_n \in (A \cup B)(\theta_{-t_n}\omega)$ such that

$$x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n.$$

Since $(A \cup B)(\omega) \subseteq A(\omega) \cup B(\omega)$, for all $\omega \in \Omega$, $(A \cup B)(\theta_{-t_n}\omega) \subseteq A(\theta_{-t_n}\omega) \cup B(\theta_{-t_n}\omega)$. So, $y_n \in A(\theta_{-t_n}\omega) \cup B(\theta_{-t_n}\omega)$, therefor $y_n \in A(\theta_{-t_n}\omega)$ or $y_n \in B(\theta_{-t_n}\omega)$. Hence $x \in \Gamma_A(\omega)$ or $x \in \Gamma_B(\omega)$, then we get $x \in \Gamma_A(\omega) \cup \Gamma_B(\omega)$.

(3) Suppose that A is forward invariant. Let $x \in \Gamma_A(\omega)$. Then there exist $t_n \to +\infty$, $y_n \in A(\theta_{-t_n}\omega)$ such that

$$x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n.$$

Since $y_n \in A(\theta_{-t_n}\omega)$ for all n,

$$\varphi(t_n, \theta_{-t_n}\omega) y_n \in \varphi(t_n, \theta_{-t_n}\omega) A(\theta_{-t_n}\omega) \subseteq A(\omega), \text{ for all } n.$$

Put $z_n = \varphi(t_n, \theta_{-t_n}\omega) y_n$, for all n, then for every n and for all $\omega \in \Omega$, we get $z_n \in A(\omega)$, and $x = \lim_{n\to+\infty} z_n$. Hence x belong to $\bar{A}(\omega)$ for all $\omega \in \Omega$.

(4) Let $x \in M \Rrightarrow x \subset M \Rrightarrow \Gamma_x(\omega) \subset \Gamma_M(\omega)$ (by(1)), where $x \in M$ $\Rrightarrow \bigcup_{x\in M} \Gamma_x(\omega) \subset \Gamma_M(\omega) \Rrightarrow \overline{\bigcup_{x\in M} \Gamma_x(\omega)} \subset \Gamma_M(\omega) = \Gamma_M(\omega)$ (by definition of $\Gamma_M(\omega)$). □

Proposition 3.4. *The necessary condition for the omega limit set $\Gamma_M(\omega)$ being invariant is the trajectory $\gamma_M^t(\omega)$ is precompact random set.*

Proof. To prove that $\varphi(t,\omega)\Gamma_M(\omega) = \Gamma_M(\theta_t\omega)$. Let $x \in \Gamma_M(\omega)$. Then by Proposition 3.2 there exist sequences $t_n \to +\infty$ and $y_n \in M(\theta_{-t_n}\omega)$ with $x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n$. Therefore,

$$\begin{aligned}\varphi(t,\omega) x &= \lim_{n\to+\infty} \varphi(t,\omega) \circ \varphi(t_n, \theta_{-t_n}\omega) y_n \\ &= \lim_{n\to+\infty} \varphi(t + t_n, \theta_{-t-t_n} \circ \theta_{t}\omega) y_n.\end{aligned}$$

According to Proposition 3.2, we have $\varphi(t,\omega) x \in \Gamma_M(\theta_t\omega)$. Thus

$$\varphi(t,\omega)\Gamma_M(\omega) = \Gamma_M(\theta_t\omega).$$

Suppose that $x \in \Gamma_M(\theta_t\omega)$ for some $t > 0$ and $\omega \in \Omega$. By Proposition 3.2

$$x = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n} \circ \theta_t\, \omega) y_n,$$

where $y_n \in M(\theta_{-t_n} \circ \theta_t\omega)$ and $t_n \longrightarrow \infty$. From the cocycle property,

$$x = \lim_{n\to+\infty} \varphi(t,\omega) z_n \quad \text{with } z_n = \varphi(t_n - t, \theta_{-t_n+t}\omega) y_n.$$

Then $\{z_n\}$ is a sequence in $\gamma_M^t(\omega)$. Since $\gamma_M^t(\omega)$is precompact, $\{z_n\}$ admits subsequence $\{z_{n_k}\}$ such that $z_{n_k} \longrightarrow a \in X$.

Furthermore, from Proposition 3.2 we have $a \in \Gamma_M(\omega)$. Also we have $x = \varphi(t,\omega)a$. Therefore, $\Gamma_M(\theta_t\omega) \subseteq \varphi(t,\omega)\Gamma_M(\omega)$ for all positive real number t and $\omega \in \Omega$. So $\Gamma_M(\omega)$ is invariant. □

Theorem 3.5. *Consider the RDS (θ,φ). The next statements are equivalent for every random set $B \subseteq X$:*

(1) *The sequence $\{\varphi(t_k,\theta_{-t}\omega)x_k\}$ is random precompact for all sequence $t_k \to +\infty$ and $\{x_k\} \subseteq B(\theta_{-t}\omega)$.*

(2) (a) $\Gamma_B(\omega) \neq \emptyset$ *and compact,*

(b) $\Gamma_B(\omega)$ *is invariant, and*

$$\lim_{t\to+\infty} \sup_{x\in B(\theta_{-t}\omega)} d\left(\varphi(t,\theta_{-t}\omega)x, \Gamma_B(\omega)\right) = 0. \tag{3.1}$$

(3) *For some compact random set $\emptyset \neq K \subseteq X$ we have*

$$\lim_{t\to+\infty} \sup_{x\in B(\theta_{-t}\omega)} d\left(\varphi(t,\theta_{-t}\omega)x, K(\omega)\right) = 0.$$

Proof. To prove (1) ⇒ (2). Let $\{x_k\} \subseteq B(\theta_{-t}\omega)$, where $t_k \to +\infty$. Then according to (1), the sequence $\{\varphi(t,\theta_{-t}\omega)x_k\}$ convergent. Assume

$$\bar{x} = \lim_{t\to+\infty} \varphi\left(t_k,\theta_{-t_k}\omega\right)x_k.$$

Then $\bar{x} \in \Gamma_B(\omega)$, so, $\Gamma_B(\omega)$ is nonempty. Let us show that $\Gamma_B(\omega)$ is compact. Let $\varepsilon_k \downarrow 0$ and $y_k \subseteq \Gamma_B(\omega)$. Then there is $x_k \in B(\theta_{-t}\omega)$ and $t_k \geq k$ with

$$d(\varphi(t_k,\theta_{-t_k}\omega)x_k, y_k) < \varepsilon_k.$$

According to condition (1), the sequence $\{\varphi(t_k,\theta_{-t_k}\omega)x_k\}$ is precompact, and since $\varepsilon_k \downarrow 0$, so $\{y_k\}$ is precompact. Easy to get the positive invariance of $\Gamma_B(\omega)$ from the definition. To show $\Gamma_B(\omega)$ to be invariant it is enough to prove it is negatively invariant. Let $y \in \Gamma_B(\omega)$ and $t \in T$. Hence there is $\{x_k\} \subseteq B(\theta_{-t}\omega)$ and $t_k \longrightarrow +\infty$ such that

$$\begin{aligned} y &= \lim_{k\to+\infty} \varphi(t_k,\theta_{-t_k}\omega)x_k \\ &= \lim_{k\to+\infty} \varphi(t_k - t + t,\theta_{-t_k}\omega)x_k \\ &= \lim_{k\to+\infty} \varphi(t_k - t,\theta_{-t_k}\omega)x_k. \end{aligned}$$

As $t_k - t \longrightarrow +\infty$, according to condition (1), the sequence $\{\varphi(t_k - t,\theta_{-t_k}\omega)x_k\}$ can be considered convergent.

Assume $y_k = \lim_{k\to+\infty} \varphi(t_k - t,\theta_{-t_k}\omega)x_k$. Then $y = \varphi(t,\theta_{-t}\omega)y_k$ and $y_t \in \Gamma_B(\omega)$, that is, $y \in \varphi(t,\theta_{-t}\omega)\Gamma_B(\omega)$. The invariance of $\Gamma_B(\omega)$ is proved at the same way.

Now to prove (3.1) fulfill. Assume that (3.1) is invalid, then for some $\varepsilon_0 > 0$, $t_k \rightarrow +\infty$, and $x_k \in B$ such that

$$d\left(\varphi\left(t_k, \theta_{-t_k}\omega\right) x_k, \Gamma_B\left(\omega\right)\right) \geq \varepsilon_0. \tag{3.2}$$

According to condition (2), the sequence $\{\ \varphi(t_k, \theta_{-t_k}\omega)x_k\}$ convergent. Let $y = \lim_{t\to+\infty}\ \varphi(t_k, \theta_{-t_k}\omega)x_k$. Then $y \in \Gamma_B\left(\omega\right)$. Take $k \longrightarrow +\infty$ in (3.2), we get $y \notin \Gamma_B\left(\omega\right)$. This is a contradiction and this end the proof of (1) $\Rightarrow$(2).

It is evident that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). □

Corollary 3.6. *Let $M \subseteq X$ be nonempty random set and $\gamma_M^t(\omega)$ be relatively compact. Then, we have $\Gamma_M\left(\omega\right) \neq \emptyset$ is invariant and compact such that*

$$\lim_{t\to+\infty}\ \sup_{x\in M(\theta_{-t}\omega)}\ d(\varphi(t, \theta_{-t}\omega)x,\ \Gamma_M\left(\omega\right)) = 0. \tag{3.3}$$

The proof follows directly from above theorem.

The converse of Corollary 3.6 holds, nonetheless we will state the concept of measure of non-compactness before formulating it.

Definition 3.7. ([4]) Let $B(X)$ be the collection of bounded sets in X, the mapping $\mu : B(X) \longrightarrow \mathbb{R}^+$ filling the following axioms is said to be a measure of non-compactness on X:

(1) $\mu(U) = 0$ if and only if $\bar{U}$ is compact, $U \in B(X)$.
(2) $\mu\left(U \cup V\right) = \max\left\{\mu(U),\ \mu(V)\right\}, U,\ V \in B(X)$.

Definition 3.8. ([4]) The measure of non-compactness of Kuratowski μ : $B(X) \longrightarrow \mathbb{R}^+$ is defined by

$$\mu(U) := \inf\{\ \varepsilon > 0 \mid U \text{ has a finite } \varepsilon - \text{covering}\}.$$

Lemma 3.9. *For every nonempty precompact random M, the trajectory $\gamma_M^t(\omega)$ is precompact if $\Gamma_M\left(\omega\right)$ nonempty, compact and* (3.3) *holds.*

Proof. Suppose that ε is a tempered random variable. Then from (3.3) there exists a number $L\left(\varepsilon\right) > 0$ so that

$$M_\varepsilon := \cup\varphi\left(t, \theta_{-t}\omega\right) M\left(\theta_{-t}\omega\right) \subseteq B\left(\Gamma_M\left(\omega\right), \varepsilon\right). \tag{3.4}$$

Assume that the measure of non-compactness of Kuratowski of K is denoted by $\mu(K)$. Then from (3.4), we get

$$\begin{aligned}\mu(\gamma_M^t(\omega)) &= \mu((\varphi\left(t, \theta_{-t}\omega\right) M\left(\theta_{-t}\omega\right) : t \in [0,\ L(\varepsilon)] \cup M_\varepsilon(\omega)) \\ &= \max(\mu(\varphi(t, \theta_{-t}\omega),\ [0,\ L(\varepsilon)]),\ \mu(M_\varepsilon(\omega)) = \mu(M_\varepsilon) \\ &\leq 2\varepsilon.\end{aligned}$$

So we get $\mu(\gamma_M^t(\omega)) = 0$. □

Theorem 3.10. *Suppose that $M \subseteq X$ is a nonempty bounded random set and precompact. Then $\gamma_M^t(\omega)$ is a precompact random set if and only if the following statements hold:*

(1) $\Gamma_M(\omega) \neq \emptyset$;
(2) $\Gamma_M(\omega)$ *is compact.*
(3) *Equality* (3.3) *fulfill* .

Proof. Applying Corollary 3.6 and Lemma 3.9 we get the result. □

Definition 3.11. ([6]) Consider the RDS (θ, φ) and the universe $\mathcal{U}$. Then (θ, φ) is called dissipative in $\mathcal{U}$, if for some absorbing set A for the RDS (θ, φ) in $\mathcal{U}$ and some closed random ball $B_{r(\omega)}(x_0)$ with center $x_0 \in X$ and radius $r(\omega)$ we have

$$A(\omega) \subset B_{r(\omega)}(x_0), \text{ for all } \omega \in \Omega.$$

Definition 3.12. (Compact RDS) Consider RDS (θ, φ) and the universe $\mathcal{U}$. Then (θ, φ) is called compact in $\mathcal{U}$, if

(1) $(\theta,\ \varphi)$ is dissipative in $\mathcal{U}$ and
(2) the absorbing set A is a random compact set.

Definition 3.13. Consider the RDS (θ, φ) and the universe $\mathcal{U}$. Then (θ, φ) is said to be $\mathcal{U}$-strong dissipative if for every tempered random variable ε and $U \in \mathcal{U}$, there is positive number $t_0(\varepsilon, U) > 0$ so that

$$\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega) \subset B_{\varepsilon(\omega)}(K) \text{ for all } t \geq t_0(\omega) \text{ and } \omega \in \Omega,$$

where K is a certain fixed random set in X depending only on $\mathcal{U}$ and

$$B_{\varepsilon(\omega)}(K) \equiv \{x \in X : \text{dist}_X(x, K(\omega)) \leq r(\omega)\}.$$

Here, K will be called the attractor of $\mathcal{U}$.

Remark 3.14. From the fact that every singleton set is a random set, one can show that the dissipativity implies $\mathcal{U}$-strong dissipativity.

Definition 3.15. The RDS $(\theta,\ \varphi)$ is said to be:

(1) point dissipative if for every $x \in X^{\Omega}$, there is a random set $K \subset X$ so that,

$$\lim_{t \longrightarrow +\infty} d\left(\varphi\left(t, \theta_{-t}\omega\right) x\left(\theta_{-t}\omega\right), K\left(\omega\right)\right) = 0. \tag{3.5}$$

(2) compact dissipative if for any compact random set $D \subset X$ there is a random set $K \subset X$ so that

$$\lim_{t \longrightarrow +\infty} \sup\left\{d\left(\varphi\left(t, \theta_{-t}\omega\right) x, K\left(\omega\right)\right) : x \in D(\theta_{-t}\omega)\right\} = 0.$$

(3) locally dissipative if for every $y \in X$ there are $\delta_y(\omega) > 0$ and random set $K \subset X$ so that

$$\lim_{t \longrightarrow +\infty} \sup \left\{ d\left(\varphi\left(t, \theta_{-t}\omega\right) x, K\left(\omega\right)\right) : x \in B_{\delta_y(\omega)}(y) \right\} = 0.$$

(4) bounded dissipative if for any bounded random set B in X there is random set K in X so that

$$\lim_{t \longrightarrow +\infty} \sup \left\{ d\left(\varphi\left(t, \theta_{-t}\omega\right) x, K\left(\omega\right)\right) : x \in B(\theta_{-t}\omega) \right\} = 0.$$

According to this, we have the following:

Definition 3.16. The RDS $(\theta,\ \varphi)$ is said to be:

(1) point κ-dissipative if (θ, φ) is point dissipative and K in (3.5) is compact.
(2) point $\mathfrak{b}$-dissipative if (θ, φ) is point dissipative and K in (3.5) is bounded.
(3) compact κ-dissipative system if (θ, φ) is compact dissipative and K in (3.5) is compact.
(4) compact $\mathfrak{b}$-dissipative system if (θ, φ) is compact dissipative and K in (3.5) is bounded.
(5) locally κ-dissipative system if (θ, φ) is locally dissipative and K in (3.5) is compact.
(6) locally $\mathfrak{b}$-dissipative system if (θ, φ) is locally dissipative and$K$ in (3.5) is bounded.

From the Definitions 3.15, 3.16 above it follows that:

Proposition 3.17. *In any RDS (θ, φ) the following statement hold:*

(1) *bounded κ-dissipativity implies local k dissipativity.*
(2) *bounded $\mathfrak{b}$-dissipativity implies local b-dissipativity.*
(3) *local κ-dissipativity implies compact k -dissipativity.*
(4) *local $\mathfrak{b}$-dissipativity implies compact b-dissipativity.*
(5) *compact κ-dissipativity implies point k -dissipativity.*
(6) *compact $\mathfrak{b}$-dissipativity implies point b-dissipativity.*

Proof. (3) and (4): Let $(\theta,\ \varphi)$ be local $\kappa(\mathfrak{b})$-dissipative. Then there is a nonempty compact (bounded) random set $K \subseteq X$ so that for every tempered random variable $\varepsilon > 0$ and $x \in X$, there exist $\delta_p(\omega) >\ 0$ and $l = l(\varepsilon, x) > 0$ for which

$$d\left(\varphi\left(t, \theta_{-t}\omega\right) y, K\right) < \varepsilon\left(\omega\right) \tag{3.6}$$

for every $t\ \geq l$ and $y \in B(p,\ \delta_p(\omega))$.

Consider a nonempty compact random$M(\omega)$ in X. Then for every $x \in M(\omega)$, there is $\delta = \delta_x(\omega) > 0$ and $l = l(\varepsilon,\ x) > 0$ such that (3.6) valid. Consider the open covering

$$\{B_{\delta_x(\omega)}(x)\ |x \in M\} \text{ of } M.$$

By the compactness of M and completeness of the space X, there a finite subcovering

$$\{B_{\delta_{x_i}(\omega)}(x_i)\,|i = 1, 2, \dots, m\}.$$

Assume $L(\varepsilon, M) := \max\{\{l(\varepsilon, x_i)|\, i = 1,\ 2, \dots,\ m\}$. Then from (3.6) it follows that

$$d\big(\varphi(t, \theta_{-t}\omega)\, x, (\omega)\big) < \varepsilon(\omega)$$

for every $x \in M$ and $t \geq L(\varepsilon,\ M)$. □

Lemma 3.18. *Let* $M \subseteq X$ *be a random set. If* $\overline{\gamma_M^t}(\omega)$ *is compact and* $\Gamma_M(\omega) \subseteq M$*, then*

$$\Gamma_M(\omega) = \bigcap \{\varphi(t, \theta_{-t}\omega)\, M(\theta_{-t}\omega)|\, t \in T, \omega \in \Omega\}. \tag{3.7}$$

Proof. Set

$$I(M) := \bigcap \{\varphi(t, \theta_{-t}\omega)\, M(\theta_{-t}\omega)|\, t \in T, \omega \in \Omega\}.$$

Then $I(M) \subseteq \Gamma_M(\omega)$. To show the reverse inclusion is holds, if $\Gamma_M(\omega) \subseteq M$. Indeed, by Theorem 3.5 and Corollary 3.6, $\Gamma_M(\omega)$ is an invariant random set. So,

$$\Gamma_M(\omega) = \varphi(t, \theta_{-t}\omega)\Gamma_M(\theta_{-t}\omega) \subseteq \varphi(t, \theta_{-t}\omega)M(\theta_{-t}\omega) \text{ for every } t \in T.$$

Therefore, $\Gamma_M(\omega) \subseteq I(M)$ and consequently $\Gamma_M(\omega) = I(M)$. □

4. Random Levinson center(RLS)

We will study the k-dissipative RDSs, so we will neglect the prefix k without lead to misapprehension. Consider a compact dissipative RDS (θ, φ) and a nonempty compact random set K in X, that is an attractor for compact subsets of X. Then the following equality hold for every compact subset $M \subseteq X$,

$$\lim_{t\to+\infty} \sup_{x\in M(\theta_{-t}\omega)} d\,(\varphi(t, \theta_{-t}\omega)\, x, K(\omega)) = 0.$$

Hence $\Gamma_K(\omega) \subseteq K$, and consequently,

$$J_X(\omega) := \Gamma_K(\omega) = \cap\{\varphi(t, \theta_{-t}\omega)\, K(\theta_{-t}\omega)|\, t \in T,\ \omega \in \Omega\}. \tag{4.1}$$

To prove that the $J_X(\omega)$ does not be governed by the select of K attracting compact random sets in X. Indeed, if we indicate by $J_K(\omega) := \Gamma_K(\omega)$ and K_1 every other compact random set attracting any compact random sets in X, so there is $L = L(K,\ K_1,\ \varepsilon) > 0$ such that $\varphi(t, \theta_{-t}\omega)K \subseteq K_1$ and $\varphi(t, \theta_{-t}\omega)K_1 \subseteq K_1$ for all $t \geq L$. Since $J_K(\omega) = \Gamma_K(\omega) \subseteq K$ and $J_{K_1}(\omega) = \Gamma_{K_1}(\omega) \subseteq K_1$, it follows from the invariance of $J_{K_1}(\omega)$ and $J_K(\omega)$ that $J_K(\omega) \subseteq K_1$,
$J_{K_1}(\omega) \subseteq K$, $J_K(\omega) \subseteq \varphi(t, \theta_{-t}\omega)K_1 \subseteq K_1$, and $J_{K_1}(\omega) \subseteq \varphi(t, \theta_{-t}\omega)$K for all $t \in T$, and consequently,

$J_K(\omega) = J_{K_1}(\omega)$. Hence $J_X(\omega)$ defined by (4.1) does not be governed by the select of attractor K.

Definition 4.1. we will call the set $J_X(\omega)$ defined by equality (4.1) the random Levinson Center (RLC) of the compact dissipative RDS (θ, φ).

Next, we will prove some essential properties of RLC.

Theorem 4.2. *Let $J_X(\omega)$ be a RLC of a compact dissipative RDS (θ, φ). Then:*

1. *$J_X(\omega)$ is a compact invariant random set.*
2. *$J_X(\omega)$ is an orbitally stable.*
3. *$J_X(\omega)$ is an attractor of the collection of compact random subsets of X.*
4. *$J_X(\omega)$ is the maximal compact invariant random (MCI) set of (θ, φ).*

Proof. (1) The first statement yield by the definition of $J_X(\omega)$ and Theorem 3.5.

(2) To prove that $J_X(\omega)$ is an orbitally stable. Assume contrary that $J_X(\omega)$ is not orbitally stable. Then there is a tempered random variable (t.r.v) $\varepsilon_0(\omega) > 0$, $\delta_n \longrightarrow 0 (\delta_n > 0)$, $x_n \in B(J_X(\omega), \delta_n)$, and $t_n \longrightarrow +\infty$ such that

$$d(\varphi(t_n, \theta_{-t_n}\omega) x_n, J_X(\omega)) \geq \varepsilon_0(\omega). \tag{4.2}$$

Since $x_n \in B(J_X(\omega), \delta_n)$ and $\delta_n \longrightarrow 0$, by the compactness of $J_X(\omega)$, we can consider $\{x_n\}$ to be convergent. Since the RDS (θ, φ) is compact dissipative, the set $\gamma^t_{\{x_n\}}(\omega)$ is precompact.

Now the set $\widetilde{K} = K \cup \overline{\gamma^t_{\{x_n\}}(\omega)}$ is a random set and it is an attractor for the collection of compact random sets in X, and so, $\Gamma_{\widetilde{K}}(\omega) = \Gamma_K(\omega) = J_X(\omega)$. Especially, $\Gamma_{\gamma^{t}_{\{x_n\}}}{}^{-}(\omega) \subseteq \Gamma_K(\omega) = J_X(\omega)$. By the compactness of $\overline{\Upsilon^t_{\{x_n\}}(\omega)}$, the sequence $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}$ is convergent. Let $p = \lim_{n \longrightarrow +\infty} \varphi(t_n, \theta_{-t_n}\omega)x_n$. Then $p \in \Gamma_{\gamma^{t}_{\{x_n\}}}{}^{-}(\omega)$. Also from it follows (4.2) that $p \notin J_X(\omega)$. This is a contradiction. Thus the second statement is proved.

(3) Let M be a compact random subset of X. The trajectory $\gamma^t_M(\omega)$ is precompact by the compact dissipativity of (θ, φ), and by Theorem 3.10, conditions (1) and (2) are hold. Especially, for every $\varepsilon(\omega) > 0$ there is $L(\varepsilon) > 0$ such that $\varphi(t, \theta_{-t}\omega)M(\theta_{-t}\omega) \subseteq B(\Gamma_M(\omega), \varepsilon)$ for all $t \geq L(\varepsilon)$. The set $\widetilde{K} = K \cup \Gamma_M(\omega)$ is also an attractor of compact subsets of X, so, $\Gamma_{\widetilde{K}}(\omega) = \Gamma_K(\omega) = J_X(\omega)$. So $\Gamma_M(\omega) \subseteq \Gamma_{\widetilde{K}}(\omega) = J_X(\omega)$, and hence

$$d(\varphi(t, \theta_{-t}\omega)M(\theta_{-t}\omega), J_X(\omega)) \leq d(\varphi(t, \theta_{-t}\omega)M(\theta_{-t}\omega), \Gamma_M(\omega)) < \varepsilon(\omega),$$

that is,

$$\lim_{t\longrightarrow+\infty} d(\varphi(t,\theta_{-t}\omega)M(\theta_{-t}\omega),\ J_X(\omega)) = 0$$

for all M compact random subset of X, where $d(A,B) := \sup_{a\in A} d(a,B)$ and $d(a,\ B) = \sup_{a\in A(\theta_{-t}\omega)} \inf\{d(a,\ b)|b \in B(\omega))\}$.

(4) If $J_1(\omega)$ is a compact invariant random set in X. Then by (3) above we have

$$\lim_{t\to+\infty} d(\varphi(t,\theta_{-t}\omega)J_1(\theta_{-t}\omega),\ J_X(\omega)) = 0. \tag{4.3}$$

Since $J_1(\omega)$ is invariant, $\varphi(t,\theta_{-t}\omega)J_1(\theta_{-t}\omega) = J_1(\omega)$ for all $t \in T$. From this and (4.3), we get $J_1(\omega) \subseteq J_X(\omega)$. □

Let $\{K_\lambda(\theta_{-t}\omega)|\lambda \in \Lambda\}$ denote the collection of nonempty compact forward invariant random sets that attract every compact random set in X.

Theorem 4.3. *Let $J_X(\omega)$ be the RLC of compact dissipative RDS $(\theta,\ \varphi)$. Then*

$$J_X(\omega) = \cap K_\lambda(\theta_{-t}\omega)|\lambda\ \in \Lambda,\ \omega \in \Omega.$$

Proof. Suppose that $K(\omega) := \cap\{K_\lambda(\theta_{-t}\omega)|\lambda\ \in \Lambda, \omega \in \Omega\}$. Note first that $J_X(\omega) \subseteq K(\omega)$, so, $K(\omega) \neq \emptyset$. Indeed $J_X(\omega) = \Gamma_{K_\lambda}(\omega) \subseteq K_\lambda(\omega)$, for every $\lambda \in \Lambda$ i.e., $J_X(\omega) \subseteq K(\omega)$. To show that $J_X(\omega) \supseteq K(\omega)$ holds. Since $J_X(\omega)$ it is nonempty and positively invariant and attracts every random compact set in X, then $J_X(\omega) \in \{K_\lambda(\theta_{-t}\omega)|\ \lambda \in \Lambda,\ \omega \in \Omega\}$, and consequently, $K(\theta_{-t}\omega) \subseteq J_X(\theta_{-t}\omega)$. □

Lemma 4.4. *Let $J_X(\omega)$ be the LRC of compact dissipative RDS $(\theta,\ \varphi)$, and $K(\omega)$ a nonempty compact random set attracting every compact set in X. Then*

$$J_X(\theta_{-t}\omega) = \cap\{\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega):\ t \in T\}.$$

Proof. Since $K(\omega)$ is an attractor of compact sets in X, $\Gamma_K(\omega) \subseteq K(\theta_{-t}\omega)$, and by Lemma 3.18,

$$J_X(\theta_{-t}\omega) := \Gamma_K(\omega) = \bigcap\{\varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega):\ t \in T\}.$$

□

Definition 4.5. The stable manifold of a random set M in X is a set

$$W^s(M) = \{x \in X : \lim_{t\longrightarrow+\infty} d(\varphi(t,\theta_{-t}\omega)x,\ M(\omega)) = 0\}.$$

Lemma 4.6. *Let M be a forward invariant compact attractor set and for every t the function $\varphi(t,\theta_{-t}\omega) : X \longrightarrow X$ is continuous. Then we have the following:*

(1) *$W^s(M)$ is an open set.*
(2) *If K is a compact set in $W^s(M)$, then*

$$\lim_{t\longrightarrow+\infty} d(\varphi\,(t,\theta_{-t}\omega)\,K\,(\theta_{-t}\omega)\,,M(\omega))=0. \tag{4.4}$$

Proof. (1) If M is a random attracting, then there exists a tempered random variable $\delta\,(\omega)>\ 0$ so that $B(M,\ \delta(\omega))\subset W^s(M)$. To end the proof we need to show that if $p\in\ W^s(M)\ B(M,\ \delta(\omega))$, there exists $\eta\ >\ 0$ with $B(p,\ \eta)\subset W^s(M)$. Since $p\ \in\ W^s(M)$, there is $t_p\ >\ 0$ with $\hat{p}\ :=\ \varphi\left(t_p,\theta_{-t_p}\omega\right)p\ \in B(M,\ \delta\ (\omega))$. Since $B(M,\ \delta\ (\omega))$ is open, there exists $\alpha>0$ with $B(\hat{p},\ \alpha)\subset B(M,\ \delta\ (\omega))$. But $\varphi\left(t_p,\theta_{-t_p}\omega\right):X\longrightarrow X$ is continuous, thus there is $\eta>0$ such that

$$\varphi\left(t_p,\theta_{-t_p}\omega\right)B(p,\ \eta)\subset B(M,\ \delta\ (\omega))\subset W^s(M).$$

Hence $W^s(M)$ is an open set.

(2) Let $\varepsilon(\omega)\ >\ 0$ be a tempered random variable (t.r.v) and let K be a compact random set in $W^s(M)$. For the (t.r.v) $\varepsilon(\omega)>0$, choose the (t.r.v) $\delta(\varepsilon,\omega)>0$, taking into version the stability of M. Since every point of $W^s(M)$ attracting by M, for every element x in $K(\omega)$ there is a (t.r.v) $\rho_x(\ \varepsilon,\omega)>0$ and $l(x,\ \varepsilon)>0$ such that

$$\varphi\,(t,\theta_{-t}\omega)\,B\,(x,\ \rho_x\,(\ \varepsilon,\omega))\subseteq B\,(M,\ \varepsilon) \tag{4.5}$$

for all $t\geq l(x,\varepsilon)$. Since K is compact and $\{B(x,\ \rho_x(\ \varepsilon,\omega))|\ x\in K\}$ is an open cover of K, this open cover admits a finite open subcover

$$\{B\,(x_i,\rho_x\,(\ \varepsilon,\omega))|\,i=1,\ldots,\ n\}.$$

Set $L\,(M,\varepsilon)\,:=\max\{\{l\,(x_i,\varepsilon)|\,i\,=\,1,\ldots,\ n\}$. Then $\varphi(t,\theta_{-t}\omega)M(\theta_{-t}\omega)\subseteq B(K,\ \varepsilon)$ for all $t\geq L(M,\ \varepsilon)$. This means that M attracts K. □

Definition 4.7. Set

$$J''\,(\omega):=\bigcap\{M_\lambda\,(\omega)\,|\lambda\in\Lambda,\omega\in\Omega\},$$

where $\mathcal{M}:=\{M_\lambda\,(\omega)\,|\lambda\in\Lambda,\omega\in\Omega\}$ is a collection of nonempty random sets in X such that every member of $\mathcal{M}$ is
(1) forward invariant,
(2) compact, and
(3) globally asymptotically stable (GAS).

Theorem 4.8. *Let $J_X\,(\omega)$ be the RLC of compact dissipative RDS (θ,φ). Then $J_X(\omega)=\ J''\,(\omega)$.*

Proof. First, by Lemma 4.6 we have $J'\,(\omega)\subseteq\ J''\,(\omega)$, and according to Theorem 4.3, $J_X\,(\omega)=\ J'\,(\omega)\subseteq\ J''\,(\omega)$. Now, for certain $\lambda_0\in\Lambda\ M_{\lambda_0}=J_X\,(\omega)$, and so, $J''\,(\omega)\subseteq J_X\,(\omega)$. Hence $J_X(\omega)=\ J''\,(\omega)$. □

Theorem 4.9. *Consider a compact dissipative RDS (θ, φ) and K a nonempty compact invariant random set in X. Then the following statements are equivalent:*

(1) *K is RLC of (θ, φ).*
(2) *K is globally asymptotically stable (GAS).*
(3) *K is a maximal compact invariant set (MCI) in X.*

Proof. (1) implies (2) by Theorem 4.2. To show that (2) implies (1). Let $J_X(\omega)$ be the *RLC* of (θ, φ). By Theorem 4.8, $J_X(\omega) \subseteq K$. But by Theorem 4.2, the *RLC* attracts all compact sets in X, and since K is an invariant, we have $K \subseteq J_X(\omega)$. So $J_X(\omega) = K$.

Finally, (1) implies (3) by Theorem 4.2. To proof (3) implies (1). Let $J_X(\omega)$ be the *RLC* of (θ, φ). But by Theorem 4.2, the set $J_X(\omega)$ is compact and invariant, and since K is the maximal compact invariant set we have $J_X(\omega) \subseteq K$. By Theorem 4.2, the *LC* is attracts the collection of compact subsets of X and since K is invariant, then $K \subseteq J_X(\omega)$. So, $K = J_X(\omega)$. □

5. Conclusion

In this work, some essential facts related to the RDSs are stated, and introduce the concept of orbitally stable RDS without the study of its properties. Then some new properties of the omega limits set in RDSs are proved to study the compactly dissipative RDSs. Some types of dissipative RDSs were introduced and studied. We proved some relations among different dissipative types. The RLC of a compact dissipative RDS (θ, φ) is a compact invariant random set, an orbitally stable attractor of the collection of compact random subsets in phase space and it is the maximal compact invariant random set (MCI) of RDS. Moreover, any nonempty compact invariant random set in a compact dissipative RDS is RLC, GAS and MCI.

References

[1] L. Arnold, *Random dynamical systems*, Dynamical systems, Springer, Berlin, Corrected 2nd printing, 2003.
[2] L. Arnold and I. Chueshov, *Order-Preserving Random Dynamical Systems: Equilibria, attractors, applications*, Dyn. Stab. of Sys., **13** (1998), 265–280.
[3] T. Bühler and D.A. Salamon, *Functional analysis*m Amer, Math. Soc., 2018.
[4] D.N. Cheban, *Nonautonomous Dynamics*, Springer Int. Publishing, 2020.
[5] D. Cheban, *On The Structure of The Levinson Center for Monotone Dissipative Non-autonomous Dynamical Systems*, Adv. Math. Res. Nova Scie. Pub., **29** (2021), 173–218.
[6] I. Chueshov, *Monotone random systems theory and applications*, Springer, 2004.
[7] A. Gu, S. Zhou and Q. Jin, *Random Attractors for Partly Dissipative Stochastic Lattice Dynamical Systems with Multiplicative White Noises*, Acta Math. Appl. Sin. English Ser., **31** (2015), 567–576.

[8] J. Huang, *Random attractor of stochastic partly dissipative systems perturbed by Lévy noise*, J. Ineq. Appl., **2012** (2012), 1-13.
[9] P. Kloeden and R. Pavani, *Dissipative synchronization of nonautonomous and random systems*, GAMM-Mitt., **32** (2009), 80–92.
[10] C. Kuehn, A. Neamţu and A. Pein, *Random attractors for stochastic partly dissipative systems*, Nonlinear Diff. Equ. Appl., NoDEA, **27** (2020), 1–37.
[11] S. Kuksin and A. Shirikyan, *On dissipative systems perturbed by bounded random kick-forces*, Ergod. Th. & Dynam. Sys., **22** (2002), 1487–1495.
[12] A. Mazurov and P. Pakshin, *Stochastic dissipativity with risk-sensitive storage function and related control problems*, ICIC Expr. Lett., **3** (2009), 53–60.
[13] T. Rajpurohit and W. Haddad, *Dissipativity Theory for Nonlinear Stochastic Dynamical Systems*, IEEE Trans. Auto. Control., **62** (2017), 1684-1699.
[14] S.M. Ulam and J. Von Neumann, *Random ergodic theorems*, Bull. Amer. Math. Soc., **51** (1945), 660.
[15] Y. Wang, Y. Liu and Z. Wang, *Random attractors for partly dissipative stochastic lattice dynamical systems*, J. Diff. Equ. Appl., **14** (2008), 799–817.
[16] Z. Wu, M. Cui, X. Xie and P. Shi, *Theory of stochastic dissipative systems*, IEEE Trans. Autom. Control, **56** (2011), 1650–1655.
[17] S. Xiaoying and M. Qiaozhen, *Existence of random attractors for weakly dissipative plate equations with memory and additive white noise*, Comput. Math. Appl., **73** (2017), 2258–2271.
[18] L. Yuhong, Z. Brzeźniak and Z. Jianzhon, *Conceptual Analysis and Random Attractor for Dissipative Random Dynamical Systems*, Acta Math. Scienttia, **28** (2008), 253–268.