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# IMPROVED BOUNDS OF POLYNOMIAL INEQUALITIES WITH RESTRICTED ZERO

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**Abstract.** Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$. Then Malik [12] obtained the following inequality:

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.$$

In this paper, we shall first improve as well as generalize the above inequality. Further, we also improve the bounds of two known inequalities obtained by Govil et al. [8].

## 1. Introduction

Let $p(z)$ be a polynomial of degree $n$. Then, according to a famous well-known classical result due to Bernstein [3],

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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Inequality (1.1) is sharp and equality holds if $p(z)$ has all its zeros at the origin. If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then Erdös conjectured and later Lax [11] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Inequality (1.2) is best possible and equality holds for $p(z) = a + bz^n$, where $|a| = |b|$.

For the class of polynomials $p(z)$ of degree $n$ not vanishing in $|z| < k$, $k \geq 1$, Malik [12] proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is best possible and equality holds for $p(z) = (z + k)^n$.

Chan and Malik [6] considered a polynomial of the type $p(z) = a_o + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, and obtained the following extention of inequality (1.3).

**Theorem 1.1.** ([6]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{1.4}$$

*The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where $n$ is a multiple of $\mu$.*

Next, Bidkham and Dewan [4] generalized inequality (1.3) and obtained

**Theorem 1.2.** ([4]) *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$,*

$$\max_{|z|=R} |p'(z)| \leq \frac{n(R+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|. \tag{1.5}$$

*The result is best possible and equality in (1.5) holds for $p(z) = (z + k)^n$.*

Aziz and Zargar [2] considered the class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, not vanishing in $|z| < k$, $k \geq 1$ and proved the following extension of Theorem 1.1 and generalization of Theorem 1.2.

**Theorem 1.3.** ([2]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$,*

$$\max_{|z|=R} |p'(z)| \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)|. \tag{1.6}$$

*The result is best possible and equality in* (1.6) *holds for* $p(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$, *where* $n$ *is a multiple of* $\mu$.

For a given polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree $n$ having no zero in $|z| < k$, $k \geq 1$, it is indeed desirable to know the dependence of

$$\max_{|z|=1} |p'(z)| / \max_{|z|=1} |p(z)|, \tag{1.7}$$

on the coefficients $a_0, a_1, . . ., a_m$, $1 \leq m \leq n$. It is clear that these coefficients are not quite arbitrary. Govil et al. [8] obtained the following which gives the dependence of (1.7) on $a_0$, $a_1$ and $a_2$.

**Theorem 1.4.** ([8]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n$ *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\begin{aligned} \max_{|z|=1} |p'(z)| \ &\leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \\ &\quad \times \max_{|z|=1} |p(z)|, \end{aligned} \tag{1.8}$$

*where* $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

It is really of interest to investigate inequalities in the reversed direction of Bernstein type discussed above and Turán [15] was the first who obtained such inequalities that if $p(z)$ has all its zeros in $z \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.9}$$

The result is sharp and equality holds in (1.9) for the polynomial having all its zeros on $|z| = 1$.

Malik [12] generalized inequality (1.9) by proving that if $p(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.10}$$

The result is sharp and extremal polynomial being $p(z) = (z + k)^n$.

For the same class of polynomials, by involving certain co-efficients of the polynomial, Govil et al. [8] obtained the following result.

**Theorem 1.5.** ([8]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n$ *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n\,|a_n| + |a_{n-1}|}{(1+k^2)\,n\,|a_n| + 2\,|a_{n-1}|} \max_{|z|=1} |p(z)|. \tag{1.11}$$

## 2. Lemmas

The following lemmas are needed for the proofs of the theorems and the corollaries in next section.

**Lemma 2.1.** ([13]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, then*

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1. \tag{2.1}$$

**Lemma 2.2.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \le R \le k$,*

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu + k^{2\mu}R)} \le \frac{R^{\mu-1}}{R^\mu + k^\mu}. \tag{2.2}$$

*Proof.* Since $p(z) \ne 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(Rz) \ne 0$ in $|z| < \frac{k}{R}$, $\frac{k}{R} \ge 1$, where $0 < R \le k$. Hence applying Lemma 2.1 to $P(z)$, we get

$$\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0|}\left(\frac{k}{R}\right)^\mu \le 1. \tag{2.3}$$

Now, (2.3) becomes

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1,$$

which is equivalent to

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu + k^{2\mu}R)} \le \frac{R^{\mu-1}}{R^\mu + k^\mu}.$$

□

**Lemma 2.3.** ([5]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \le R \le k$,*

$$\exp\left\{ n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt \right\} \le \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}. \tag{2.4}$$

**Lemma 2.4.** ([13]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \le n\frac{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}|\frac{a_\mu}{a_0}|(k^{\mu+1} + k^{2\mu})}\max_{|z|=1}|p(z)|. \tag{2.5}$$

*Inequality* (2.5) *is sharp and equality holds for the polynomial* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* $n$ *is a multiple of* $\mu$.

**Lemma 2.5.** ([9]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* $n$ *having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$,

$$\begin{aligned}\max_{|z|=R} |p(z)| \quad &\leq \quad \exp\left\{ n \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu + k^{2\mu} t)} dt \right\} \\ &\qquad \times \max_{|z|=r} |p(z)|. \end{aligned} \tag{2.6}$$

**Lemma 2.6.** ([14]) *If* $p(z)$ *is the polynomial of degree* $n$ *having no zero in* $|z| < k$, $k \geq 1$, *then for* $|z| \leq k$, $|\xi| \leq k$, *where* $\xi$ *is a real or complex number, we have*

$$(\xi - z)p'(z) + np(z) \neq 0. \tag{2.7}$$

**Lemma 2.7.** ([8]) *If* $f(z)$ *is analytic and* $|f(z)| \leq 1$ *in* $|z| \leq 1$, *then for* $|z| \leq 1$,

$$|f(z)| \leq \frac{(1-|a|)|z|^2 + |bz| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |bz| + (1-|a|)}, \tag{2.8}$$

*where* $a = f(0)$, $b = f'(0)$. *The example*

$$f(z) = \frac{a + \frac{b}{1+a}z - z^2}{1 - \frac{b}{1+a}z - az^2},$$

*shows that the estimate is sharp.*

**Lemma 2.8.** ([7]) *If* $p(z)$ *is a polynomial of degree* $n$, *then on* $|z| = 1$,

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{2.9}$$

*where*

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

**Lemma 2.9.** *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n$ *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$1 - \frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu-\lambda^2|} \geq 0, \tag{2.10}$$

*where* $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

*Proof.* From Lemma 2.1, we have

$$|\lambda| = \frac{k}{n}\frac{|a_1|}{|a_0|} \leq 1.$$

Now,

$$(1-k+k^2+k|\lambda|) - (1+k^2|\lambda|) = k(k-1)(1-|\lambda|) \geq 0$$

or

$$(1-k+k^2+k|\lambda|) \geq (1+k^2|\lambda|),$$

which implies that

$$(1+k)\left[(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|\right] \geq (1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu-\lambda^2|,$$

which is equivalent to

$$1 - \frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \geq 0.$$

□

**Lemma 2.10.** ([8]) *If $p(z)$ is a polynomial of degree $n$, then*

$$\max_{|z|=1}|p'(z)| \geq n\max_{|z|=1}|p(z)| - \max_{|z|=1}|q'(z)|, \tag{2.11}$$

*where*

$$q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}.$$

**Lemma 2.11.** ([10]) *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then*

$$\frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \leq n\frac{1+k|\lambda|}{1+k^2+2k|\lambda|}, \tag{2.12}$$

*where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.*

Lemma 2.11 was conjectured by Govil et al. [8] and later precisely proved by Krishnadas et al [10].

**Lemma 2.12.** *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$\begin{aligned}&\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\\ &\geq \frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|},\end{aligned} \tag{2.13}$$

*where* $\omega = \frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega = \frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.

*Proof.* If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ is a polynomial of degree at most $n$ having no zero in $|z| < 1/k$, $1/k \geq 1$. Applying Lemma 2.12 to $q(z)$, we have

$$\frac{n}{1+\frac{1}{k}}\frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \leq n\frac{1+\frac{1}{k}|\omega|}{1+\frac{1}{k^2}+2\frac{1}{k}|\omega|},$$

which is equivalent to

$$\begin{aligned} n - &\frac{n}{1+\frac{1}{k}}\frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \\ &\geq n - n\frac{1+\frac{1}{k}|\omega|}{1+\frac{1}{k^2}+2\frac{1}{k}|\omega|}. \end{aligned}$$

This simplifies to

$$\begin{aligned} &\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|} \\ &\geq \frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}. \end{aligned}$$

□

## 3. Main results

In this paper, under the same set of hypotheses, we first obtain an improvement of Theorem 1.3 by involving some of the coefficients of the polynomial $p(z)$. In fact, we obtain

**Theorem 3.1.** *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* $n$ *having no zero in* $|z| < k$, $k \geq 1$, *then for* $0 < r \leq R \leq k$,

$$\begin{aligned} \max_{|z|=R}|p'(z)| \quad \leq \quad & n\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu+k^{2\mu}R\right)} \\ & \times \ \exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\} \\ & \times \ \max_{|z|=r}|p(z)|. \end{aligned} \tag{3.1}$$

*Proof.* Since $p(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(Rz) \neq 0$ in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$, where $0 < R \leq k$. Hence applying Lemma 2.4 to $P(z)$, we have

$$\max_{|z|=1} |P'(z)| \leq n \frac{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu \left(\frac{k}{R}\right)^{\mu+1}}{1 + \left(\frac{k}{R}\right)^{\mu+1} + \frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu \left(\frac{k^{\mu+1}}{R^{\mu+1}} + \frac{k^{2\mu}}{R^{2\mu}}\right)} \max_{|z|=1} |P(z)|.$$

This gives

$$R \max_{|z|=R} |p'(z)| \leq n \frac{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu \left(\frac{k}{R}\right)^{\mu+1}}{1 + \left(\frac{k}{R}\right)^{\mu+1} + \frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu \left(\frac{k^{\mu+1}}{R^{\mu+1}} + \frac{k^{2\mu}}{R^{2\mu}}\right)} \max_{|z|=R} |p(z)|,$$

which is equivalent to

$$\begin{aligned} \max_{|z|=R} |p'(z)| \quad &\leq \quad n \frac{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|\frac{k^{\mu+1}}{R}}{R + \frac{k^{\mu+1}}{R^\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(\frac{Rk^{\mu+1}}{R} + \frac{Rk^{2\mu}}{R^\mu}\right)} \max_{|z|=R} |p(z)| \\ &= \quad n \frac{R^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu + k^{2\mu}R\right)} \max_{|z|=R} |p(z)|. \end{aligned}$$

Using Lemma 2.5 for $\max_{|z|=R} |p(z)|$, we obtain

$$\begin{aligned} \max_{|z|=R} |p'(z)| = n &\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu + k^{2\mu}R\right)} \\ &\times \exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right)} dt\right\} \max_{|z|=r} |p(z)|. \end{aligned}$$

This completes the proof of Theorem 3.1. □

**Remark 3.2.** To show that Theorem 3.1 is in general, an improvement of Theorem 1.3, it is sufficient to show that

$$\begin{aligned} &\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu + k^{2\mu}R)} \\ &\quad \times \exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu + k^{2\mu}t)} dt\right\} \qquad (3.2) \\ &\leq \frac{R^{\mu-1}}{R^\mu + k^\mu}\left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}. \end{aligned}$$

By Lemma 2.2, we have

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu+k^{2\mu}R)} \leq \frac{R^{\mu-1}}{R^\mu+k^\mu}. \tag{3.3}$$

Also by Lemma 2.3, we have

$$\exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\} \leq \left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}. \tag{3.4}$$

Multiplying inequalities (3.3) and (3.4), we have inequality (3.2).

**Remark 3.3.** Using inequality (3.2) of Remark 3.2, Theorem 3.1 reduces to Theorem 1.3 which further generalizes Theorem 1.2.

**Remark 3.4.** Putting $r = 1$, in Theorem 3.1, we have the following generalization of Lemma 2.4 proved by Qazi [13].

**Corollary 3.5.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$,*

$$\begin{aligned}\max_{|z|=R}|p'(z)| \leq\ & n\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu+k^{2\mu}R\right)} \\ & \times\exp\left\{n\int_1^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\} \\ & \times\max_{|z|=1}|p(z)|. \end{aligned}\tag{3.5}$$

**Remark 3.6.** If we assign $r = R = 1$ and $\mu = 1$ in Theorem 3.1, we obtain the following inequality proved by Govil et al. [8], which further improves the bound given by inequality (1.3) due to Malik [12].

**Corollary 3.7.** *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq n\frac{1+\frac{1}{n}\left|\frac{a_1}{a_0}\right|k^2}{1+k^2+2\frac{1}{n}\left|\frac{a_1}{a_0}\right|k^2}\max_{|z|=1}|p(z)|. \tag{3.6}$$

**Remark 3.8.** Using the fact

$$\frac{1}{n}\left|\frac{a_1}{a_0}\right|k \leq 1$$

from Lemma 2.1, inequality (3.6) of Corollary 3.7 reduces to inequality (1.3) proved by Malik [12].

**Remark 3.9.** Putting $r = R = \mu = k = 1$, inequality (3.1) of Theorem 3.1 reduces to Erdös-Lax inequality (1.2).

Next, we consider polynomials of degree $n \geq 3$ and prove the following theorem which is an improvement of Theorem 1.4 by involving $\min\limits_{|z|=k}|p(z)|$. In fact, we prove

**Theorem 3.10.** *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 3$ having no zero in $|z| < k$, $k \geq 1$, then*

$$\begin{aligned}\max_{|z|=1}|p'(z)| \leq{}& \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\max_{|z|=1}|p(z)|\\&-\frac{n}{k^n}\left(1-\frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right)\\&\times\min_{|z|=k}|p(z)|, \qquad (3.7)\end{aligned}$$

*where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$. The result is best possible and equality in (3.7) holds for*

$$p(z) = a_0\frac{1}{k^n}(z+k)^{n_1}\left(z^2 + 2kz\frac{na-n_1}{n-n_1} + k^2\right)^{\frac{n-n_1}{3}},$$

*where $a$ is an arbitrary real number and $n_1$ is an integer such that $\frac{n}{3} \leq n_1 \leq n$, $n - n_1$ is even.*

*Proof.* Consider a new polynomial $Q(z) = p(z) + m\alpha z^n$, where $\alpha$ is a real or complex number such that $|\alpha| < \left(\frac{1}{k}\right)^n$, $m = \min\limits_{|z|=k}|p(z)|$.

Now, on $|z| = k$

$$\begin{aligned}|m\alpha z^n| &< m\frac{1}{k^n}k^n\\&= m\\&\leq |p(z)|.\end{aligned}$$

Then by Rouche's theorem, $p(z)$ and $Q(z)$ must have same number of zeros in $|z| < k$ and hence $Q(z)$ has no zero in $|z| < k$. And for $|z| < k$, $|\xi| < k$, where

$\xi$ is a real or complex number, by Lemma 2.6, we have

$$nQ(z) + (\xi - z)Q'(z) \neq 0,$$

that is,

$$nQ(z) - zQ'(z) \neq -\xi Q'(z). \tag{3.8}$$

Consequently, for $|z| \leq k$

$$\left|\frac{Q'(z)}{nQ(z) - zQ'(z)}\right| \leq \frac{1}{k}. \tag{3.9}$$

Hence if

$$f(z) = \frac{kQ'(kz)}{nQ(kz) - kzQ'(kz)}, \tag{3.10}$$

then $|f(z)| \leq 1$ for $|z| \leq 1$.

Also

$$f(0) = \frac{ka_1}{na_0} = \lambda \tag{3.11}$$

and

$$f'(0) = (n-1)\left\{\frac{2k^2a_2}{n(n-1)a_0} - \left(\frac{ka_1}{na_0}\right)^2\right\} = (n-1)(\mu - \lambda^2). \tag{3.12}$$

Then for $|z| \leq 1$, we use Lemma 2.7 to conclude that

$$|f(z)| \leq \frac{(1-|\lambda|)|z|^2 + (n-1)|\mu - \lambda^2||z| + |\lambda|(1-|\lambda|)}{|\lambda|(1-|\lambda|)|z|^2 + (n-1)|\mu - \lambda^2||z| + (1-|\lambda|)}.$$

Thus in particular for $|z| = 1$, we have

$$|Q'(z)| \leq \frac{1}{k}\frac{(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + |\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + (1-|\lambda|)k^2}|nQ(z) - zQ'(z)|. \tag{3.13}$$

If $q(z) = z^n\overline{Q(\frac{1}{\bar{z}})}$, then on $|z| = 1$, $|nQ(z) - zQ'(z)| = |q'(z)|$. Therefore inequality (3.13) becomes

$$|Q'(z)| \leq \frac{1}{k}\frac{(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + |\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + (1-|\lambda|)k^2}|q'(z)|. \tag{3.14}$$

From Lemma 2.8, we have

$$\max_{|z|=1}\big(|Q'(z)| + |q'(z)|\big) \leq n\max_{|z|=1}|Q(z)|. \tag{3.15}$$

Combining inequalities (3.14) and (3.15), we get

$$\max_{|z|=1}|Q'(z)| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu - \lambda^2|}\max_{|z|=1}|Q(z)|,$$

which is equivalent to

$$\max_{|z|=1}|p'(z)+\alpha mnz^{n-1}| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \max_{|z|=1}|p(z)+\alpha mz^n|. \tag{3.16}$$

Suppose $z_0$ on $|z|=1$ is such that

$$\max_{|z|=1}|p'(z)| = |p'(z_0)|. \tag{3.17}$$

Now,

$$|p'(z_0)+n\alpha mz_0^{n-1}| \leq \max_{|z|=1}|p'(z)+n\alpha mz^{n-1}|. \tag{3.18}$$

In the left hand side of inequality (3.18) for suitable choice of the argument of $\alpha$, we have

$$|p'(z_0)+n\alpha mz_0^{n-1}| = |p'(z_0)|+n|\alpha|m. \tag{3.19}$$

Using (3.19) and (3.17) in inequality (3.18), we have

$$\max_{|z|=1}|p'(z)|+n|\alpha|m \leq \max_{|z|=1}|p'(z)+n\alpha mz^{n-1}|. \tag{3.20}$$

Combining inequalities (3.20) and (3.16), we have

$$\max_{|z|=1}|p'(z)|+n|\alpha|m \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \max_{|z|=1}|p(z)+\alpha mz^n|. \tag{3.21}$$

Again suppose $z_1$ on $|z|=1$ is such that

$$\begin{aligned}\max_{|z|=1}|p(z)+\alpha mz^n| &= |p(z_1)+\alpha mz_1^n| \\ &\leq |p(z_1)|+|\alpha|m \\ &\leq \max_{|z|=1}|p(z)|+|\alpha|m. \end{aligned} \tag{3.22}$$

Using inequality (3.22) in inequality (3.21), we have

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \left\{\max_{|z|=1}|p(z)|+|\alpha|m\right\}-n|\alpha|m,$$

which on taking limit as $|\alpha| \to \frac{1}{k^n}$ becomes

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \left\{ \max_{|z|=1} |p(z)| + \frac{1}{k^n} m \right\} - n\frac{1}{k^n} m,$$

which on simplification gives

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \max_{|z|=1} |p(z)| - \frac{n}{k^n} \left\{ 1 - \frac{1}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \right\} m.$$

This completes the proof of Theorem 3.10. □

**Remark 3.11.** To show that Theorem 3.10 is indeed an improvement of Theorem 1.4, it is sufficient to show

$$\left( 1 - \frac{1}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \right) \ge 0. \tag{3.23}$$

From Lemma 2.9, we have inequality (3.23).

**Remark 3.12.** Using inequality (2.12) of Lemma 2.11, Theorem 3.10 reduces to the following result which improves the bound given by Govil et al. [8, Theorem 1, (10)].

**Corollary 3.13.** *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \ge 3$ having no zero in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1} |p'(z)| \le n \frac{1+k|\lambda|}{1+k^2+2k|\lambda|} \max_{|z|=1} |p(z)| - \frac{n}{k^n} \left( 1 - \frac{1+k|\lambda|}{1+k^2+2k|\lambda|} \right) \min_{|z|=k} |p(z)|, \tag{3.24}$$

*where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$.*

**Remark 3.14.** Using the fact

$$|\lambda| = \frac{1}{n} \left| \frac{a_1}{a_0} \right| k \le 1$$

from Lemma 2.1, inequality (3.24) of Corollary 3.13 reduces to the following result which is an improvement of inequality (1.3) proved by Malik [12].

**Corollary 3.15.** *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n\geq 3$ having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|-\frac{n}{k^{n-1}+k^n}\min_{|z|=k}|p(z)|. \tag{3.25}$$

**Remark 3.16.** Putting k=1 in Theorem 3.10, we obtain the following inequality proved by Aziz and Dawood [1], which further improves the bound given by Erdös-Lax inequality (1.2).

**Corollary 3.17.** *If $p(z)$ is a polynomial of degree $n\geq 3$ having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\left\{\max_{|z|=1}|p(z)|-\min_{|z|=1}|p(z)|\right\}. \tag{3.26}$$

As an application of Theorem 3.10, we obtain the following result which is an improvement of the result proved by Govil et al. [8, Corollary 2, (17)].

**Theorem 3.18.** *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, $a_0\neq 0$, is a polynomial of degree $n\geq 3$ having all its zeros in $|z|\leq k$, $k\leq 1$, then*

$$\begin{aligned}\max_{|z|=1}|p'(z)|\geq{}& \frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\\ &\times\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\},\end{aligned} \tag{3.27}$$

*where $\omega=\frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega=\frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.*

*Proof.* If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, $a_0\neq 0$, is a polynomial of degree $n\geq 3$, then $q(z)=z^n\overline{p(1/\overline{z})}$ is also a polynomial of degree $n\geq 3$, then on $|z|=1$,

$$|p(z)|=|q(z)|. \tag{3.28}$$

Also by Lemma 2.10, we have

$$\max_{|z|=1}|p'(z)|\geq n\max_{|z|=1}|p(z)|-\max_{|z|=1}|q'(z)|,$$

that is

$$n\max_{|z|=1}|p(z)|-\max_{|z|=1}|p'(z)|\leq \max_{|z|=1}|q'(z)|. \tag{3.29}$$

If $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then $q(z)$ has no zero in $|z| < 1/k$, $1/k \geq 1$. Hence applying Theorem 3.10 to $q(z)$, we have

$$\begin{aligned}\max_{|z|=1}|q'(z)| \leq &\frac{n}{1+\frac{1}{k}}\frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}\max_{|z|=1}|q(z)|\\ &-nk^n\left\{1-\frac{1}{1+\frac{1}{k}}\frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}\right\}\\ &\times\min_{|z|=k}|q(z)|, \end{aligned} \tag{3.30}$$

where $\omega = \frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega = \frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.
Since $q(z) = z^n\overline{p(1/\overline{z})}$,

$$\begin{aligned}\min_{|z|=\frac{1}{k}}|q(z)| &= \min_{|z|=\frac{1}{k}}|z^n\overline{p(1/\overline{z})}|\\ &= \frac{1}{k^n}\min_{|z|=\frac{1}{k}}|p(1/\overline{z})|\\ &= \frac{1}{k^n}\min_{|z|=k}|p(z)|. \end{aligned} \tag{3.31}$$

Combining inequalities (3.29) and (3.30) and then using (3.28) and (3.31), we obtain

$$\begin{aligned}&n\max_{|z|=1}|p(z)|-\max_{|z|=1}|p'(z)|\\ &\leq \frac{n}{1+\frac{1}{k}}\frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}\max_{|z|=1}|p(z)|\\ &\quad-\frac{nk^n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\frac{1}{k^n}\min_{|z|=k}|p(z)|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geq &\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\\ &\times\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\}.\end{aligned} \tag{3.32}$$

Hence, the proof of Theorem 3.18 is complete. □

**Remark 3.19.** Theorem 3.18 improves upon the result of Govil et al. [8, Corollary 2,(17)] by involving $\min_{|z|=k}|p(z)|$.

So, to show that Theorem 3.18 is an improvement of the result due to Govil et al. [8, Corollary 2, (17)], it is sufficient to show

$$\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\geq 0,$$

which is equivalent to show

$$1-|\omega|\geq 0.$$

Applying Lemma 2.1 to $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, where $p(z)$ is as defined in Theorem 3.18, we have

$$|\omega|=\frac{1}{nk}\frac{|a_{n-1}|}{|a_n|}\leq 1.$$

**Remark 3.20.** Using inequality (2.13) of Lemma 2.12, Theorem 3.18 reduces to the following result which is an improvement of Theorem 1.5 due to Govil et al. [8].

**Corollary 3.21.** *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$, $a_0\neq 0$, *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $|z|\leq k$, $k\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\geq\frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\}. \tag{3.33}$$

**Remark 3.22.** Putting $k=1$ in Theorem 3.18, we obtain the following refinement of inequality (1.9) which was proved by Aziz and Dawood [1].

**Corollary 3.23.** *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$, $a_0\neq 0$, *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $z\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\geq\frac{n}{2}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=1}|p(z)|\right\}. \tag{3.34}$$

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## References


[1] A. Aziz and Q.M. Dawood, *Inequalities for a polynomial and its derivative*,J. Approx. Theory, **54** (1988), 306-313.

[2] A. Aziz and B.A. Zargar, *Inequalities for a polynomial and its derivative*, Math. Inequal. Appl., **1**(4) (1998), 543-550.

[3] S. Bernstein, *Lecons Sur Les Propriétés extrémales et la meilleure approximation desfunctions analytiques d'une fonctions reele*, Gauthier-Villars, Paris, 1926.

[4] M. Bidkham and K.K. Dewan, *Inequalities for polynomial and its derivative*, J. Math. Anal. Appl., **116** (1992), 319-324.

[5] B. Chanam and K.K. Dewan, *Inequalities for polynomial and its derivative*, J. Math. Anal. Appl., **336** (2007), 171-179.
[6] T.N. Chand and M.A. Malik, *On Erdös-Lax theorem*, Proc. Indian Acad. Sci., **92**(3) (1983), 191-193.
[7] N.K. Govil and Q.I. Rahaman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math.Soc., **137** (1969), 501-517.
[8] N.K. Govil, Q.I. Rahaman and G. Schmeisser, *On the derivative of a polynomial*, Illinoise J. Math., **23** (1979), 319-329.
[9] V.K. Jain, *On maximum modulus of polynomials with zeros outside a circle*, Glasnik Mathematicki, **29** (1994), 267-274.
[10] Kshetrimayum Krishnadas, Reingachand Ngamchui and Barchand Chanam, *Generalized and Extended Versions of Ankeny-Rivlin and Improved,Geneneralized, and Extended Versions of Rivlin Type Inequalities for the $s^{th}$ Derivative of a Polynomial*, MDPI mathematics, **9**(8), (2021), 887.
[11] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509-513.
[12] M.A. Malik, *On the derivative of a polynomial*, J. London Math. Soc., **1** (1969), 57-60.
[13] M.A. Qazi, *On the maximum modulus of polynomials*, Proc. Amer. Math. Soc., **115** (1992), 337-343.
[14] G. Szegö, *Bemerkungen zu einem satz von J.H. Grace*,etc., Math. Z., **13** (1992), 28-55.
[15] P. Turán, *Über die ableitung von polynomen*, Compositio Math., **7** (1939), 89-95.