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ON CONTROLLABILITY FOR FRACTIONAL VOLTERRA-FREDHOLM SYSTEM

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Abstract. In this manuscript, we study the sufficient conditions for controllability of Volterra-Fredholm type fractional integro-differential systems in a Banach space. Fractional calculus and the fixed point theorem are used to derive the findings. Some examples are provided to illustrate the obtained results.

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1. Introduction

Fractional integro-differential equation has recently given a natural foundation for mathematical modeling of many real-world events, particularly in the control, biological, and medical domains [13, 24, 28, 29, 30, 35]. Theoretical and practical foundations have been built for the study of such situations. Many disciplines of physics and technological sciences use fractional integro-differential equations and control issues. Fractional integro-differential equations [3, 12, 14, 15, 16, 17, 18, 19, 20, 38, 39, 40] are measured as an alternative model to nonlinear differential equations. The authors [26, 27, 31, 32, 33, 37] have thoroughly explored the theory of fractional integrals and derivatives.

Agarwal et al. [1] investigated the existence and uniqueness of solutions for various kinds of starting and BVP of fractional differential equations, as well as inclusions related the fractional Caputo derivative in finite dimensional spaces.

In both finite and infinite dimensional spaces, the concept of controllability is crucial. Many writers have looked at the topic of controllability [2, 4, 5, 6, 8, 9, 22, 25, 38]. In Banach spaces, Balachandran et al. [10, 11] developed certain fractional integro-differential systems with controllers. The purpose of this study is to use Gronwall type inequalities to examine the controllability of fractional integro-differential systems with boundary conditions.

This paper is motivated by the recent works [10, 11, 22, 34, 38] and its main purpose is to establish sufficient conditions for the controllability of the mixed-type Volterra-Fredholm integro-differential system with control of the form:

$$\begin{aligned}&{}^{c}\mathbb{D}^{\delta}\mathfrak{y}(\rho) = \Delta(\rho, \mathfrak{y}(\rho), (S\mathfrak{y})(\rho), (H\mathfrak{y})(\rho)) + Bu(\rho), \quad \rho \in \Upsilon := [0, \vartheta], \quad (1.1)\\ &\Xi\mathfrak{y}(0) + \mathit{\Xi}\mathfrak{y}(\vartheta) = \mathfrak{E},\end{aligned}$$

where $0 < \delta < 1$, the control function $u(\cdot)$ is given in $L^2(\Upsilon, \mathfrak{U})$, a Banach space of admissible control functions with $\mathfrak{U}$, B is a bounded operator, $\Delta : \Upsilon \times \Psi \times \Psi \times \Psi \to \Psi$ is continuous, Ψ is a Banach space and $\Xi, \mathit{\Xi}, \mathfrak{E} \in \mathbb{R}$ with $\Xi + \mathit{\Xi} \neq 0$, and S, H are nonlinear operators given by

$$(S\mathfrak{y})(\rho) = \int_0^{\rho} k(\rho, \varsigma)\mathfrak{y}(\varsigma)d\varsigma, \ (H\mathfrak{y})(\rho) = \int_0^{\vartheta} h(\rho, \varsigma)\mathfrak{y}(\varsigma)d\varsigma,$$

with $\gamma_0^s = \max\left\{\int_0^{\rho} k(\rho, \varsigma)d\varsigma : (\rho, \varsigma) \in \Upsilon \times \Upsilon\right\}$ and
$\gamma_0^h = \max\left\{\int_0^{\vartheta} h(\rho, \varsigma)d\varsigma : (\rho, \varsigma) \in \Upsilon \times \Upsilon\right\}$, where $k, h \in C(\Upsilon \times \Upsilon, \mathbb{R}^+)$.

2. Preliminaries

Let $C(\Upsilon, \Psi)$ denote the Banach space with $\|\mathfrak{y}\|_\infty := \sup\{\|\mathfrak{y}(\rho)\| : \rho \in \Upsilon\}$. For measurable functions $\xi : \Upsilon \to \mathbb{R}$ define the norm

$$\|\xi\|_{L^p(\Upsilon,\mathbb{R})} = \left(\int_\Upsilon |\xi(\rho)|^p d\rho\right)^{\frac{1}{p}}, \ \ 1 \le p < \infty.$$

Let $L^p(\Upsilon, \mathbb{R})$ denote the Banach space of all Lebesgue measurable functions with ξ and $\|\xi\|_{L^p(\Upsilon,\mathbb{R})} < \infty$.

Definition 2.1. The Riemann-Liouville fractional order integral of order $\delta > 0$ of a function Δ is defined as

$$I^\delta_{\Xi+}\Delta(\rho) = \int_\Xi^\rho \frac{(\rho - \varsigma)^{\delta-1}}{\Gamma(\delta)}\Delta(\varsigma)d\varsigma.$$

Definition 2.2. The Riemann-Liouville fractional order derivative of order δ of Δ is defined by

$$\left(D^\delta_{\Xi+}\Delta\right)(\rho) = \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{d\rho}\right)^n \int_\Xi^\rho (\rho-\varsigma)^{n-\delta-1}\Delta(\varsigma)d\varsigma, \ \ n = -[-\delta].$$

Definition 2.3. For a function Δ given on the interval $[\Xi, \mathit{\Xi}]$, the Caputo fractional order derivative of order δ of Δ is defined by

$$\left({}^cD^\delta_{\Xi+}\Delta\right)(\rho) = \frac{1}{\Gamma(n-\delta)}\int_\Xi^\rho (\rho-\varsigma)^{n-\delta-1}\Delta^{(n)}(\varsigma)d\varsigma.$$

Now, let us initiate the definition of a solution of the problem (1.1).

Definition 2.4. A function $\mathfrak{y} \in C^1(\Upsilon, \Psi)$ is said to be a solution of the problem (1.1), if $\mathfrak{y}$ satisfies the equation

$${}^c\mathbb{D}^\delta\mathfrak{y}(\rho) = \Delta(\rho, \mathfrak{y}(\rho), (S\mathfrak{y})(\rho), (H\mathfrak{y})(\rho)) + Bu(\rho) \ \text{ a.e. on } \ \Upsilon$$

and the condition $\Xi\mathfrak{y}(0) + \mathit{\Xi}\mathfrak{y}(\vartheta) = \mathfrak{E}$.

Lemma 2.5. (Lemma 3.2, [1]) *Let $\mathfrak{y} \in C(\Upsilon, \Psi)$ be a function such that*

$$\mathfrak{y}(\rho) = \frac{1}{\Gamma(\delta)}\int_0^\rho (\rho-\varsigma)^{\delta-1}\bar{\Delta}(\varsigma)d\varsigma - \frac{1}{\Xi + \mathit{\Xi}}\left[\frac{\mathit{\Xi}}{\Gamma(\delta)}\int_0^\vartheta (\vartheta-\varsigma)^{\delta-1}\bar{\Delta}(\varsigma)d\varsigma - \mathfrak{E}\right]$$

if and only if $\mathfrak{y}$ is a solution of

$$\begin{cases} {}^c\mathbb{D}^\delta\mathfrak{y}(\rho) = \bar{\Delta}(\rho), \ \ 0 < \delta < 1, \ \ \rho \in \Upsilon \\ \Xi\mathfrak{y}(0) + \mathit{\Xi}\mathfrak{y}(\vartheta) = \mathfrak{E}. \end{cases} \tag{2.1}$$

We get the following result as a result of Lemma 2.5 which is essential in what follows.

Lemma 2.6. *If* $\mathfrak{y} \in C(\Upsilon, \Psi)$ *is a function such that*

$$\mathfrak{y}(\rho) = \frac{1}{\Gamma(\delta)} \int_0^\rho (\rho - \varsigma)^{\delta-1} [\Delta(\varsigma, \mathfrak{y}(\varsigma), (S\mathfrak{y})(\varsigma), (H\mathfrak{y})(\varsigma)) + Bu(\varsigma)] d\varsigma - \frac{1}{\Xi + \mathit{\Xi}} \times \left(\frac{\mathit{\Xi}}{\Gamma(\delta)} \int_0^\vartheta (\vartheta - \varsigma)^{\delta-1} [\Delta(\varsigma, \mathfrak{y}(\varsigma), (S\mathfrak{y})(\varsigma), (H\mathfrak{y})(\varsigma)) + Bu(\varsigma)] d\varsigma - \mathfrak{E} \right),$$

for every $\rho \in \Upsilon$ *if and only if* $\mathfrak{y}$ *is a solution of the problem* (1.1).

Definition 2.7. The system (1.1) is said to be controllable on Υ if for all $\mathfrak{y}_0, \mathfrak{y}_1 \in \Psi$, there exists a control $u \in L^2(\Upsilon, \mathfrak{U})$ such that the solution $\mathfrak{y}(\rho)$ of the problem (1.1) satisfies $\mathfrak{y}(\vartheta) = y_1$.

First, we'll provide the assumptions that we'll be using.

(H1) $\Delta : \Upsilon \times \Psi \times \Psi \times \Psi \to \Psi$ is measurable on Υ.

(H2) $W : L^2(\Upsilon, \mathfrak{U}) \to \Psi$, defined by

$$Wu = \frac{1}{\Gamma(\delta)} \int_0^\vartheta (\vartheta - \varsigma)^{\delta-1} Bu(\varsigma) d\varsigma,$$

induces an invertible operator W defined on $L^2(\Upsilon, \mathfrak{U})/kerW$ and there exist $K > 0$ such that $\left\|BW^{-1}\right\| \le K$.

(H3) $\Delta : \Upsilon \times \Psi \times \Psi \times \Psi \to \Psi$ is continuous and there exists a constant $\beta_1 \in (0, \delta)$ and real-valued functions $\xi_1(\rho), \xi_2(\rho), \xi_3(\rho) \in L^{\frac{1}{\beta_1}}(\Upsilon, \Psi)$ such that

$$\|\Delta(\rho, \mathfrak{x}(\rho), (S\mathfrak{x})(\rho), (H\mathfrak{x})(\rho)) - \Delta(\rho, \mathfrak{y}(\rho), (S\mathfrak{y})(\rho), (H\mathfrak{y})(\rho))\| \le \xi_1(\rho)\|\mathfrak{x} - \mathfrak{y}\| + \xi_2(\rho)\|S\mathfrak{x} - S\mathfrak{y}\| + \xi_3(\rho)\|H\mathfrak{x} - H\mathfrak{y}\|,$$

for each $\rho \in \Upsilon$ and all $\mathfrak{x}, \mathfrak{y} \in \Psi$.

(H4) There exist a constant $\beta_2 \in (0, \delta)$ and real-valued function $\varrho(\rho) \in L^{\frac{1}{\beta_2}}(\Upsilon, \Psi)$ such that

$$\|\Delta(\rho, \mathfrak{y}, S\mathfrak{y}, H\mathfrak{y})\| \le \varrho(\rho),$$

for each $\rho \in \Upsilon$ and all $\mathfrak{y} \in \Psi$.

(H5) There exists a constant $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\delta}$ and $N > 0$ such that

$$\|\Delta(\rho, u, Su, Hu)\| \le N \left(1 + \gamma_0^s \|u\|^\lambda + \gamma_0^h \|u\|^\lambda\right)$$

for each $\rho \in \Upsilon$ and all $u \in \Psi$.

Lemma 2.8. ([23]) *Let* $\mathfrak{y} \in C(\Upsilon, \Psi)$ *satisfy the following inequality*

$$\|\mathfrak{y}(\rho)\| \le \Xi + \mathit{\Xi} \int_0^\rho (\rho - \varsigma)^{\delta-1} \|\mathfrak{y}(\varsigma)\|^\lambda d\varsigma + \mathfrak{E} \int_0^\rho (\vartheta - \varsigma)^{\delta-1} \|\mathfrak{y}(\varsigma)\|^\lambda d\varsigma, \quad (2.2)$$

where $\delta \in (0,1), \lambda \in \left[0, 1-\frac{1}{p}\right)$, *for some* $1 < p < \frac{1}{1-\delta}$, *and* $\Xi, \mathit{\Xi}, \mathfrak{E} \geq 0$. *Then there exists a constant* $M^* > 0$ *such that* $\|\mathfrak{y}(\rho)\| \leq M^*$.

Let us assume that $M = \left\|\xi_1 + \gamma_0^s \xi_2 + \gamma_0^h \xi_3\right\|_{L^{\frac{1}{\beta_1}}(\Upsilon,\Psi)}$, $H^p = \|\rho\|_{L^{\frac{1}{\beta_2}}(\Upsilon,\Psi)}$.

3. Controllability result

Theorem 3.1. *If the assumptions* (H1)-(H5) *are fulfilled and if*

$$\Lambda_{\delta,\beta_1,\vartheta}(\rho) = \frac{M\vartheta^{\delta-\beta_1}}{\Gamma(\delta)\left(\frac{\delta-\beta_1}{1-\beta_1}\right)^{1-\beta_1}}\left(1+\frac{K\vartheta^\delta}{\delta\Gamma(\delta)}\right)\left(1+\frac{\mathit{\Xi}}{\Xi+\mathit{\Xi}}\right) < 1, \tag{3.1}$$

then the problem (1.1) *is controllable on* Υ.

Proof. Using the assumption (H2), for an arbitrary function $u(\rho)$ define the control

$$\begin{aligned} u(\rho) = W^{-1}\Bigg\{&\frac{\mathfrak{E}}{\mathit{\Xi}} - \frac{\Xi+\mathit{\Xi}}{\mathit{\Xi}}\mathfrak{y}(0) \\ &-\frac{1}{\Gamma(\delta)}\int_0^\vartheta (\vartheta-\varsigma)^{\delta-1}\Delta(\varsigma,\mathfrak{y}(\varsigma),(S\mathfrak{y})(\varsigma),(H\mathfrak{y})(\varsigma))d\varsigma\Bigg\}(\rho). \end{aligned} \tag{3.2}$$

Let

$$\begin{aligned} r \geq & \frac{\vartheta^{\delta-\beta_2}H^p}{\Gamma(\delta)\left(\frac{\delta-\beta_2}{1-\beta_2}\right)^{1-\beta_2}}\left(1+\frac{\vartheta^\delta}{\Gamma(\delta+1)}\right)\left(1+\frac{\mathit{\Xi}}{\Xi+\mathit{\Xi}}\right) \\ &+\left(\frac{\mathfrak{E}}{\mathit{\Xi}}\left(1+\frac{\mathit{\Xi}}{\Xi+\mathit{\Xi}}\right)+\|\mathfrak{y}(0)\|\left(1+\frac{\Xi+\mathit{\Xi}}{\mathit{\Xi}}\right)\right)\frac{K\vartheta^\delta}{\Gamma(\delta+1)}+\frac{\mathfrak{E}}{\Xi+\mathit{\Xi}}. \end{aligned}$$

It suffices to display that when employing the above, the map G on $B_r := \{\mathfrak{y} \in C(\Upsilon,\Psi) : \|\mathfrak{y}\| \leq r\}$ is defined as follows

$$\begin{aligned} &(G\mathfrak{y})(\rho) \\ &= \frac{1}{\Gamma(\delta)}\int_0^t (\rho-\varsigma)^{\delta-1}[\Delta(\varsigma,\mathfrak{y}(\varsigma),(S\mathfrak{y})(\varsigma),(H\mathfrak{y})(\varsigma))+Bu(\varsigma)]d\varsigma - \frac{1}{\Xi+\mathit{\Xi}} \\ &\times\left(\frac{\mathit{\Xi}}{\Gamma(\delta)}\int_0^\vartheta (\vartheta-\varsigma)^{\delta-1}[\Delta(\varsigma,\mathfrak{y}(\varsigma),(S\mathfrak{y})(\varsigma),(H\mathfrak{y})(\varsigma))+Bu(\varsigma)]d\varsigma - \mathfrak{E}\right), \end{aligned} \tag{3.3}$$

for every $\rho \in \Upsilon$. As a result, the existence of a solution to the problem (1.1) equates to the mapping G having a fixed point on B_r. We shall employ the Banach contraction principle to prove that G has a fixed point. Obviously, $G\mathfrak{y}(\vartheta) = \mathfrak{y}_1$, this signifies that the control u directs the solution to the problem

(1.1) from the starting state $\mathfrak{y}_0$ to $\mathfrak{y}_1$ in time ϑ provided we are able to get a fixed point of the mapping G.

First we show that G maps $C(\Upsilon, B_r)$ into itself.

Step 1. For each $\mathfrak{y} \in B_r$, by (H3) and Holder inequality, we get

$$\begin{aligned}
\|(G\mathfrak{y})(\rho)\| \leq & \frac{1}{\Gamma(\delta)} \int_0^\rho (\rho-\varsigma)^{\delta-1} \rho(\varsigma) d\varsigma + \frac{K}{\Gamma(\delta)} \int_0^\rho (\rho-\varsigma)^{\delta-1} \\
& \times \left(\frac{\mathfrak{E}}{\Xi} + \frac{\Xi+\Xi}{\Xi} \|\mathfrak{y}(0)\| + \frac{1}{\Gamma(\delta)} \int_0^\vartheta (\vartheta-\tau)^{\delta-1} \rho(\tau) d\tau \right) d\varsigma \\
& + \frac{\Xi}{\Xi+\Xi\Gamma(\delta)} \int_0^\vartheta (\vartheta-\varsigma)^{\delta-1} \rho(\varsigma) d\varsigma \\
& + \frac{K\Xi}{(\Xi+\Xi)\Gamma(\delta)} \int_0^\vartheta (\vartheta-\varsigma)^{\delta-1} \left(\frac{\mathfrak{E}}{\Xi} + \frac{\Xi+\Xi}{\Xi} \|\mathfrak{y}(0)\| \right. \\
& \left. + \frac{1}{\Gamma(\delta)} \int_0^\vartheta (\vartheta-\tau)^{\delta-1} \rho(\tau) d\tau \right) d\varsigma + \frac{\mathfrak{E}}{\Xi+\Xi} \\
\leq & \frac{1}{\Gamma(\delta)} \left(\int_0^\rho (\rho-\varsigma)^{\frac{\delta-1}{1-\beta_2}} d\varsigma \right)^{1-\beta_2} \left(\int_0^\rho (\rho(\varsigma))^{\frac{1}{\beta_2}} d\varsigma \right)^{\beta_2} \\
& + \frac{K}{\Gamma(\delta)} \int_0^\rho (\rho-\varsigma)^{\delta-1} \left[\frac{\mathfrak{E}}{\Xi} + \frac{\Xi+\Xi}{\Xi} \|\mathfrak{y}(0)\| \right. \\
& \left. + \frac{1}{\Gamma(\delta)} \left(\int_0^\vartheta (\vartheta-\tau)^{\frac{\delta-1}{1-\beta_2}} d\tau \right)^{1-\beta_2} \left(\int_0^\vartheta (\rho(\tau))^{\frac{1}{\beta_2}} d\tau \right)^{\beta_2} \right] d\varsigma \\
& + \frac{\Xi}{\Xi+\Xi\Gamma(\delta)} \left(\int_0^\vartheta (\vartheta-\varsigma)^{\frac{\delta-1}{1-\beta_2}} d\varsigma \right)^{1-\beta_2} \left(\int_0^\vartheta (\rho(\varsigma))^{\frac{1}{\beta_2}} d\varsigma \right) \\
& + \frac{K\Xi}{(\Xi+\Xi)\Gamma(\delta)} \int_0^\vartheta (\vartheta-\varsigma)^{\delta-1} \left[\frac{\mathfrak{E}}{\Xi} + \frac{\Xi+\Xi}{\Xi} \|\mathfrak{y}(0)\| \right. \\
& \left. + \frac{1}{\Gamma(\delta)} \left(\int_0^\vartheta (\vartheta-\tau)^{\frac{a-1}{1-\beta_2}} d\tau \right)^{1-\beta_2} \left(\int_0^\vartheta (\rho(\tau))^{\frac{1}{\beta_2}} d\tau \right)^{\beta_2} \right] d\varsigma + \frac{\mathfrak{E}}{\Xi+\Xi} \\
\leq & \frac{\vartheta^{\delta-\beta_2} H^p}{\Gamma(\delta) \left(\frac{\delta-\beta_2}{1-\beta_2} \right)^{1-\beta_2}} \left(1 + \frac{\vartheta^\delta}{\Gamma(\delta+1)} \right) \left(1 + \frac{\Xi}{\Xi+\Xi} \right) \\
& + \left(\frac{\mathfrak{E}}{\Xi} \left(1 + \frac{\Xi}{\Xi+\Xi} \right) + \|\mathfrak{y}(0)\| \left(1 + \frac{\Xi+\Xi}{\Xi} \right) \right) \frac{K\vartheta^\delta}{\Gamma(\delta+1)} + \frac{\mathfrak{E}}{\Xi+\Xi} \\
\leq & r.
\end{aligned}$$

Then, G maps from $C(\Upsilon, B_r)$ to $C(\Upsilon, B_r)$.

Step 2. On B_r, G is a contraction mapping. By using (H2) and Holder inequality, for $\mathfrak{x}, \mathfrak{y} \in B_r$ and any $\rho \in \Upsilon$, we obtain

$$
\begin{aligned}
&\|(G\mathfrak{x})(\rho) - (G\mathfrak{y})(\rho)\| \\
&\leq \frac{1}{\Gamma(\delta)} \int_0^{\rho} \big[(\rho-\varsigma)^{\delta-1} \|\Delta(\varsigma, \mathfrak{x}(\varsigma), (S\mathfrak{x})(\varsigma), (H\mathfrak{x})(\varsigma)) \\
&\quad - \Delta(\varsigma, \mathfrak{y}(\varsigma), (S\mathfrak{y})(\varsigma), (H\mathfrak{y})(\varsigma))\|\big] d\varsigma \\
&\quad + \frac{\mathit{\Xi}}{(\Xi + \mathit{\Xi})\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\varsigma)^{\delta-1} \|\Delta(\varsigma, \mathfrak{x}(\varsigma), (S\mathfrak{x})(\varsigma), (H\mathfrak{x})(\varsigma)) \\
&\quad - \Delta(\varsigma, \mathfrak{y}(\varsigma), (S\mathfrak{y})(\varsigma), (H\mathfrak{y})(\varsigma))\| d\varsigma \\
&\quad + \frac{K}{\Gamma(\delta)} \int_0^{\rho} (\rho-\varsigma)^{\delta-1} \Bigg(\frac{\Xi + \mathit{\Xi}}{\mathit{\Xi}} \|\mathfrak{x}(0) - \mathfrak{y}(0)\| \\
&\quad + \frac{1}{\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\tau)^{\delta-1} \|\Delta(\tau, \mathfrak{x}(\tau), (S\mathfrak{x})(\tau), (H\mathfrak{x})(\tau)) \\
&\quad - \Delta(\tau, \mathfrak{y}(\tau), (S\mathfrak{y})(\tau), (H\mathfrak{y})(\tau))\| d\tau \Bigg) d\varsigma \\
&\quad + \frac{K\mathit{\Xi}}{(\Xi + \mathit{\Xi})\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\varsigma)^{\delta-1} \Bigg(\frac{\Xi + \mathit{\Xi}}{\mathit{\Xi}} \|\mathfrak{x}(0) - \mathfrak{y}(0)\| \\
&\quad + \frac{1}{\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\tau)^{\delta-1} \|\Delta(\tau, \mathfrak{x}(\tau), (S\mathfrak{x})(\tau), (H\mathfrak{x})(\tau)) \\
&\quad - \Delta(\tau, \mathfrak{y}(\tau), (S\mathfrak{y})(\tau), (H\mathfrak{y})(\tau))\| d\tau \Bigg) d\varsigma \\
&\leq \frac{\|\mathfrak{x} - \mathfrak{y}\|}{\Gamma(\delta)} \int_0^{\rho} (\rho-\varsigma)^{\delta-1} \left[\xi_1(\varsigma) + \gamma_0^s \xi_2(\varsigma) + \gamma_0^h \xi_3(\varsigma)\right] d\varsigma \\
&\quad + \frac{\mathit{\Xi}\|\mathfrak{x} - \mathfrak{y}\|}{(\Xi + \mathit{\Xi})\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\varsigma)^{\delta-1} \left[\xi_1(\varsigma) + \gamma_0^s \xi_2(\varsigma) + \gamma_0^h \xi_3(\varsigma)\right] d\varsigma \\
&\quad + \frac{K}{\Gamma(\delta)} \int_0^{\rho} (\rho-\varsigma)^{\delta-1} \Bigg(\frac{\Xi + \mathit{\Xi}}{\mathit{\Xi}} \|\mathfrak{x}(0) - \mathfrak{y}(0)\| \\
&\quad + \frac{\|\mathfrak{x} - \mathfrak{y}\|}{\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\tau)^{\delta-1} \left[\xi_1(\varsigma) + \gamma_0^s \xi_2(\varsigma) + \gamma_0^h \xi_3(\varsigma)\right] d\tau \Bigg) d\varsigma \\
&\quad + \frac{K\mathit{\Xi}}{(\Xi + \mathit{\Xi})\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\varsigma)^{\delta-1} \Bigg(\frac{\Xi + \mathit{\Xi}}{\mathit{\Xi}} \|\mathfrak{x}(0) - \mathfrak{y}(0)\| \\
&\quad + \frac{1}{\Gamma(\delta)} \int_0^{\vartheta} (\vartheta-\tau)^{\delta-1} \left[\xi_1(\varsigma) + \gamma_0^s \xi_2(\varsigma) + \gamma_0^h \xi_3(\varsigma)\right] d\tau \Bigg) d\varsigma
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\|\mathfrak{x}-\mathfrak{y}\|}{\Gamma(\delta)}\left(\int_0^\rho (\rho-\varsigma)^{\frac{\delta-1}{1-\beta_1}}d\varsigma\right)^{1-\beta_1} \\
&\quad \times\left(\int_0^\rho \left[\xi_1(\varsigma)+\gamma_0^s\xi_2(\varsigma)+\gamma_0^h\xi_3(\varsigma)\right]^{\frac{\rho}{\beta_1}}d\varsigma\right)^{\beta_1} \\
&\quad + \frac{\mathit{\Xi}\|x-\mathfrak{y}\|}{(\Xi+\mathit{\Xi})\Gamma(\delta)}\left(\int_0^\vartheta (\vartheta-\varsigma)^{\frac{\delta-1}{1-\beta_1}}d\varsigma\right)^{1-\beta_1} \\
&\quad \times\left(\int_0^\vartheta \left[\xi_1(\varsigma)+\gamma_0^s\xi_2(\varsigma)+\gamma_0^h\xi_3(\varsigma)\right]^{\frac{1}{\beta_1}}d\varsigma\right)^{\beta_1} \\
&\quad + \frac{K}{\Gamma(\delta)}\int_0^\rho (\rho-\varsigma)^{\delta-1}\left(\frac{\Xi+\mathit{\Xi}}{\mathit{\Xi}}\|\mathfrak{x}(0)-\mathfrak{y}(0)\|\right. \\
&\quad + \frac{\|\mathfrak{x}-\mathfrak{y}\|}{\Gamma(\delta)}\left(\int_0^\vartheta (\vartheta-\tau)^{\frac{\delta-1}{1-\beta_1}}d\tau\right)^{1-\beta_1} \\
&\quad \left.\times\left(\int_0^\vartheta \left[\xi_1(\tau)+\gamma_0^s\xi_2(\tau)+\gamma_0^h\xi_3(\tau)\right]^{\frac{1}{\beta_1}}d\tau\right)^{\beta_1}\right)d\varsigma \\
&\quad + \frac{K\mathit{\Xi}}{(\Xi+\mathit{\Xi})\Gamma(\delta)}\int_0^\vartheta (\vartheta-\varsigma)^{\delta-1}\left(\frac{\Xi+\mathit{\Xi}}{\mathit{\Xi}}\|\mathfrak{x}(0)-\mathfrak{y}(0)\|\right. \\
&\quad + \frac{\|\mathfrak{x}-\mathfrak{y}\|}{\Gamma(\delta)}\left(\int_0^\vartheta (\vartheta-\tau)^{\frac{\delta-1}{1-\beta_1}}d\tau\right)^{1-\beta_1} \\
&\quad \left.\times\left(\int_0^\vartheta \left[\xi_1(\tau)+\gamma_0^s\xi_2(\tau)+\gamma_0^h\xi_3(\tau)\right]^{\frac{1}{\beta_1}}d\tau\right)^{\beta_1}\right)d\varsigma \\
&\leq \frac{M\vartheta^{\delta-\beta_1}}{\Gamma(\delta)\left(\frac{\delta-\beta_1}{1-\beta_1}\right)^{1-\beta_1}}\left(1+\frac{\mathit{\Xi}}{\Xi+\mathit{\Xi}}\right)\left(1+\frac{K\vartheta^\delta}{\Gamma(\delta+1)}\right)\|\mathfrak{x}-\mathfrak{y}\| \\
&\quad + \frac{K\vartheta^\delta}{\Gamma(\delta+1)}\left(1+\frac{\Xi+\mathit{\Xi}}{\mathit{\Xi}}\right)\|\mathfrak{x}(0)-\mathfrak{y}(0)\| \\
&\leq \Lambda_{\delta,\beta_1,\vartheta}(\rho)\|\mathfrak{x}-\mathfrak{y}\|.
\end{aligned}$$

Because Δ and G are continuous and by (2.2), $0\leq\Lambda_{\delta,\beta_1,\vartheta}(\rho)<1$, thus G is a contraction map. Hence there exists a unique fixed point $\mathfrak{y}\in C(\Upsilon,B_r)$ such that $G\mathfrak{y}(\rho)=\mathfrak{y}(\rho)$. Any fixed point of G is a solution of (1.1) which fulfills $\mathfrak{y}(\vartheta)=\mathfrak{y}_1$. Then the system (1.1) is controllable on Υ. □

4. Applications

We provide some examples in this part to elucidate the use of our key findings.

Example 4.1. Consider the BVP as follows:

$$
\begin{aligned}
{}^{c}\mathbb{D}^{\delta}\mathfrak{y}(\rho) &= \frac{e^{-\ell\rho}|\mathfrak{y}(\rho)|}{(1+e^{\rho})(1+|\mathfrak{y}(\rho)|)} + \int_0^{\rho} \frac{e^{-\ell\rho-\varsigma}}{16}\frac{|\mathfrak{y}(\rho)|}{1+|\mathfrak{y}(\rho)|}d\varsigma \\
&\quad + \int_0^1 \frac{e^{-\ell\rho-\varsigma}}{(1+15e^{\varsigma})}\frac{|\mathfrak{y}(\rho)|}{1+|\mathfrak{y}(\rho)|}d\varsigma,\ \ \rho\in\Upsilon:=[0,1], \\
\mathfrak{y}(0)+\mathfrak{y}(1) &= 0, \qquad (4.1)
\end{aligned}
$$

where $\ell > 0$ is a constant, $\delta \in (0,1)$. Let $\mathfrak{y}_1, \mathfrak{y}_2 \in [0,\infty)$ and $\rho \in \Upsilon$. Then we have

$$
|\Delta(\rho,\mathfrak{x}(\rho),(S\mathfrak{x})(\rho),(H\mathfrak{x})(\rho)) - \Delta(\rho,\mathfrak{y}(\rho),(S\mathfrak{y})(\rho),(H\mathfrak{y})(\rho))| \le \frac{9e^{-\ell\rho}}{16}|\mathfrak{x}-\mathfrak{y}|
$$

and

$$
|\Delta(\rho,\mathfrak{y},S\mathfrak{y},H\mathfrak{y})| \le \frac{9e^{-\ell\rho}}{16}.
$$

For $\rho\in\Upsilon, \xi_1(\rho)=\xi_2(\rho)=\xi_3(\rho)=\frac{e^{-\ell\rho-\varsigma}}{32}$, let $M = \|\frac{9e^{-\ell\rho}}{16}\|_{L^{\frac{1}{\beta_1}}(\Upsilon,\Psi)}$.

$$
\Lambda_{\delta,\beta_1,\vartheta}(\rho) = \frac{M\vartheta^{\delta-\beta_1}}{\Gamma(\delta)\left(\frac{\delta-\beta_1}{1-\beta_1}\right)^{1-\beta_1}}\left(1+\frac{K\vartheta^{\delta}}{\delta\Gamma(\delta)}\right)\left(1+\frac{\Xi}{\Xi+\Xi}\right) < 1.
$$

Then all the hypothesises in Theorem 3.1 are fulfilled, our conclusions can be utilized to the system (4.1).

Example 4.2. Consider the fractional system proposed by Volterra and Fredholm:

$$
\begin{cases}
\frac{\partial^r}{\partial\tau^r}\chi(\tau,\eta) = \Delta\chi(\tau,\eta)+l_0(\eta)\sin\chi(\tau,\eta)+l_1\int_0^{\tau}e^{-\chi(\varrho,\eta)}d\varrho+l_2\int_0^1 e^{-\chi(\varrho,\eta)}d\varrho \\
\qquad\qquad\qquad +Bx(\tau), \quad \tau\in V=[0,1], \ \ \eta\in\mathfrak{G}, \\
\chi(\tau,\eta)=0, \quad \tau\in[0,1], \ \ \eta\in\partial\mathfrak{G}, \\
\chi(0,\eta)+\int_0^c j(\varrho)\,\mathrm{In}\left(1+|\chi(\varrho,\eta)|^{\frac{1}{2}}\right)d\varrho = 0, \quad \chi'(0,\eta)=\chi_1(\eta), \ \ \eta\in\mathfrak{G},
\end{cases}
$$

where $\frac{\partial^r}{\partial\tau^r}$ denotes Caputo fractional derivative of order r $(1.5 \le r < 2)$, $j \in L^1(V,\mathbb{R}^+)$, l_0 is continuous on $\mathfrak{G}$ and $l_1, l_2 > 0$, $\mathfrak{G}\subset\mathbb{R}^N$ is a bounded domain, $\mathfrak{U}=\mathfrak{Z}=L^2(\mathfrak{G})$.

Suppose that $\mathfrak{A}$ to be the Laplace map with Dirichlet conditions as $\mathfrak{A}=\Delta$ and

$$
\varpi(\mathfrak{A}) = \left\{g\in H_0^1(\mathfrak{G}), \mathfrak{A}g\in L^2(\mathfrak{G})\right\}.
$$

Then, we have $\varpi(\mathfrak{A}) = H_0^1(\mathfrak{G})\cap H^2(\mathfrak{G})\cdot\mathfrak{A}$ produces $C(\tau)$ for $\tau\ge 0$ in the view of [7]. Let $\hbar_{\mathfrak{s}}=\mathfrak{s}^2\pi^2$ and $\mu_{\mathfrak{s}}(\eta)=\sqrt{(2/\pi)}\sin(\mathfrak{s}\pi\eta)$, for all $\mathfrak{s}\in\mathbb{N}$.

Suppose that $\{-\hbar_{\mathfrak{s}}, \mu_{\mathfrak{s}}\}_{\mathfrak{s}=1}^{\infty}$ is an eigensystem of the factor $\mathfrak{A}$. Then $0 < \hbar_1 \leq \hbar_2 \leq \cdots\ \hbar_{\mathfrak{s}} \to \infty$ when $\mathfrak{s} \to \infty$, $\{\mu_{\mathfrak{s}}\}_{\mathfrak{s}=1}^{\infty}$ forms an orthonormal foundation of $\mathfrak{Z}$. Moreover

$$\mathfrak{A}\chi = -\sum_{\mathfrak{s}=1}^{\infty} \hbar_{\mathfrak{s}} \left(\chi, \mu_{\mathfrak{s}}\right) \mu_{\mathfrak{s}}, \quad \chi \in \varpi(\mathfrak{A}),$$

where $(\cdot, \cdot)$ indicate the inner product in $\mathfrak{Z}$. Accordingly, $C(\tau)$ is defined by

$$C(\tau)\chi = \sum_{\mathfrak{s}=1}^{\infty} \cos\left(\sqrt{\hbar_{\mathfrak{s}}}\tau\right) \left(\chi, \mu_{\mathfrak{s}}\right) \mu_{\mathfrak{s}}, \quad \chi \in \mathfrak{Z},$$

which is connected with the sine family $\{S(\tau), \tau \geq 0\}$ in $\mathfrak{Z}$ defined by

$$S(\tau)\chi = \sum_{\mathfrak{s}=1}^{\infty} \frac{1}{\sqrt{\hbar_{\mathfrak{s}}}} \sin\left(\sqrt{\hbar_{\mathfrak{s}}}\tau\right) \left(\chi, \mu_{\mathfrak{s}}\right) \mu_{\mathfrak{s}}, \quad \chi \in \mathfrak{Z}$$

and $\|C(\tau)\|_{L_c(\mathfrak{Z})} \leq 1$ for any $\tau \geq 0$. Since $r = \frac{3}{2}$, we know that $\tau = \frac{3}{4}$, and then $\|C_c(\tau)\|_{L_c(\mathfrak{Z})} \leq 1$ for any $\tau \geq 0$.

The control operator $B : \mathfrak{U} \to \mathfrak{Z}$ is defined by

$$Bx = \sum_{\mathfrak{s}=1}^{\infty} a\hbar_{\mathfrak{s}} \left(\bar{x}, \mu_{\mathfrak{s}}\right) \mu_{\mathfrak{s}}, \quad a > 0.$$

In the above

$$\bar{x} = \begin{cases} x_{\mathfrak{s}}, & \mathfrak{s} = 1, 2, \ldots, N, \\ 0, & \mathfrak{s} = N+1, N+2, \ldots \end{cases}$$

for N in $\mathbb{N}$. Indicate $W : L^2(V, \mathfrak{U}) \to \mathfrak{Z}$ as follows:

$$W_X = \int_0^{\varrho} (1-\varrho)^{-\frac{1}{4}} T_{\frac{3}{4}}(1-\varrho) Bx(\varrho) d\varrho.$$

Thus, $|x| = \left(\sum_{\mathfrak{s}=1}^{\infty} \left(x, \mu_{\mathfrak{s}}\right)^2\right)^{\frac{1}{2}}$ for $x \in \mathfrak{U}$, we have

$$|Bx| = \left(\sum_{\mathfrak{s}=1}^{\infty} a^2 \hbar_{\mathfrak{s}}^2 \left(\bar{x}, \mu_{\mathfrak{s}}\right)^2\right)^{\frac{1}{2}} \leq aN\hbar_N|x|,$$

which implies that there exists $P_1 > 0$ such that

$$\|B\|_{L_c(\mathfrak{U}, \mathfrak{Z})} \leq P_1.$$

Assume $x(\varrho,\eta)=\chi(\eta)\in\mathfrak{U}$, $\bar{\chi}$ indicate $\chi_{\mathfrak{s}}$ if $\mathfrak{s}=1,2,\ldots,N$ or 0 if $\mathfrak{s}=N+1,\cdots$. Thus, we have

$$\begin{aligned}
W_x &= \int_0^1 (1-\varrho)^{-\frac{1}{4}}\frac{3}{4}\int_0^\infty \xi S_{\frac{3}{4}}(\xi) S\left((1-\varrho)^{\frac{3}{4}}\xi\right) B\chi d\xi d\varrho \\
&= a\int_0^1 (1-\varrho)^{-\frac{1}{4}}\frac{3}{4}\int_0^\infty \xi S_{\frac{3}{4}}(\xi)\sum_{\mathfrak{s}=1}^{N}\sqrt{\hbar_{\mathfrak{s}}}\sin\left(\sqrt{\hbar_{\mathfrak{s}}}(1-\varrho)^{\frac{3}{4}}\xi\right)\Bigg)(\bar{\chi},\mu_{\mathfrak{s}})\,\mu_{\mathfrak{s}} d\xi d\varrho \\
&= a\sum_{\mathfrak{s}=1}^{N}\int_0^\infty S_{\frac{3}{4}}(\xi)\left(1-\cos\left(\sqrt{\hbar_{\mathfrak{s}}}\xi\right)\right)d\xi\,(\bar{\chi},\mu_{\mathfrak{s}})\,\mu_{\mathfrak{s}} \\
&= a\sum_{\mathfrak{s}=1}^{\infty}\left(1-E_{\frac{3}{2},1}(-\hbar_{\mathfrak{s}})\right)(\chi,\mu_{\mathfrak{s}})\mu_{\mathfrak{s}}.
\end{aligned}$$

In [21], let $v=E_{\frac{3}{2},1}\left(-\frac{1}{10}\right)$. Then, for all $\mathfrak{s}\in\mathbb{N}$, we possess $-1<E_{\frac{3}{2},1}(-\hbar_{\mathfrak{s}})\le v<1$ which implies

$$0<1-v\le 1-E_{\frac{3}{2},1}(-\hbar_{\mathfrak{s}})<2.$$

Thus, we assort W is surjective ever after, for each $\chi=\sum_{\mathfrak{s}=1}^{\infty}(\chi,\mu_{\mathfrak{s}})\,\mu_{\mathfrak{s}}\in\mathfrak{Z}$, we clarify $W^{-1}:\mathfrak{Z}\to L^2(V,\mathfrak{U})/\ker W$ by

$$\left(W^{-1}\chi\right)(\tau,\eta)=\frac{1}{a}\sum_{\mathfrak{s}=1}^{\infty}\frac{(\chi,\mu_{\mathfrak{s}}\rangle\,\mu_{\mathfrak{s}}}{1-E_{\frac{3}{2}}(-\hbar_{\mathfrak{s}})}$$

for $\chi\in\mathfrak{Z}$ in such a way

$$\left|\left(W^{-1}\chi\right)(\tau,\cdot)\right|\le\frac{1}{a(1-v)}|\chi|.$$

We educate that $W^{-1}\chi$ is independent of $\tau\in V$. Furthermore, we get

$$\left\|W^{-1}\right\|_{L_c(\mathfrak{Z},L^2(V,\mathfrak{U})/\operatorname{Ker}W)}\le\frac{1}{a(1-v)}$$

as long as

$$\begin{aligned}
&\chi(\tau)(\eta)=\chi(\tau,\eta),\\
&{}^{C}D_\tau^{\frac{3}{2}}\chi(\tau)(\eta)=\frac{\partial^{\frac{3}{2}}}{\partial\tau^{\frac{3}{2}}}\chi(\tau,\eta),\\
&g\left(\tau,\chi,\int_0^\tau h(\tau,\varrho,\chi)d\varrho,\int_0^1 k(\tau,\varrho,\chi)d\varrho\right)=l_0(\cdot)\sin\chi(\tau,\cdot)+\int_0^\tau h(\tau,\varrho,\chi)d\varrho\\
&\qquad\qquad+\int_0^1 k(\tau,\varrho,\chi)d\varrho,
\end{aligned}$$

$h(\tau,\varrho,\chi) = l_1 e^{-\chi(\varrho,\cdot)}, k(\tau,\varrho,\chi) = l_2 e^{-\chi(\varrho,\cdot)}$ and Ψ is defined by $\Psi(\chi)(\eta) = \int_0^c j(\varrho)\,\text{In}\left(1 + |\chi(\varrho,\eta)|^{\frac{1}{2}}\right) d\varrho$, then it is compact. Therefore, every requirement of Theorem 3.1 is satisfied. Hence, the problem 4.2 is controllable on $[0,1]$.

References

[1] R.P. Agarwal, M. Benchohra and S. Hamanani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109** (2010), 973-1033.

[2] H.M. Ahmed, *Controllability for Sobolev type fractional integro-differential systems in a Banach space*, Advances Diff. Equa., **2012**(1) (2012), 1-10.

[3] A. Abed, M. Younis and A. Hamoud, *Numerical solutions of nonlinear Volterra-Fredholm integro-differential equations by using MADM and VIM*, Nonlinear Funct. Anal. Appl., **27**(1) (2022), 189-201.

[4] H.M. Ahmed, *Approximate controllability via resolvent operators of Sobolev-type fractional stochastic integrodifferential equations with fractional Brownian motion and Poisson jumps*, Bull. Iranian Math. Soc., **45**(4) (2019), 1045-1059.

[5] H.M. Ahmed, M.M. El-Borai, H.M. El-Owaidy and A.S. Ghanem, *Existence solution and controllability of Sobolev type delay nonlinear fractional integro-differential system*, Mathematics, **7**(1) (2019), 79.

[6] H.M. Ahmed, M.M. El-Borai, AS Okb El Bab and M.E. Ramadan, *Controllability and constrained controllability for nonlocal Hilfer fractional differential systems with Clarkes subdifferential*, J. Ineq. Appl., **2019**(1) (2019), 1-23.

[7] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems.* Birkhuser, Berlin (2011).

[8] A. Anguraj, P. Karthikeyan and G.M. N'Guerekata, *Nonlocal Cauchy problem for some fractional abstract integrodifferential equations in Banach space*, Commun. Math. Anal., **6** (2009), 31-35.

[9] K. Balachandran and J.P. Dauer, *Controllability of nonlinear systems via fixed point theorems*, J. Opti. Theory Appl., **53** (1987), 345-352.

[10] K. Balachandran and J.P. Dauer, *Controllability of nonlinear systems in Banach spaces: A survey*, J. Opti. Theory Appl., **115** (2002), 7-28.

[11] K. Balachandran and J.Y. Park, *Controllability of fractional integrodifferential systems in Banach spaces*, Nonlinear Anal. : Hybrid Systems, **3** (2009), 363-367.

[12] B. Bonilla, M. Rivero, L. Rodriguez-Germa and J.J. Trujillo, *Fractional differential equations as alternative models to nonlinear differential equations*, Appl. Math. Comput., **187** (2007), 79-88.

[13] K. Diethelm and A.D. Freed, *On the solution of nonlinear fractional-order differential equation used in the modeling of viscoelasticity.* In: F.keil, W.Mackens, H.Voss, J. Werther (Eds), Scientific Computing in Chemical Engineering IIComputational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag, Heidelberg (1999), 217-224.

[14] A. Hamoud and K. Ghadle, *Existence and uniqueness of solutions for fractional mixed Volterra-Fredholm integro-differential equations*, Indian J. Math., **60**(3) (2018), 375-395.

[15] A. Hamoud, K. Ghadle, M. Bani Issa and Giniswamy, *Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations*, Int. J. Appl. Math., **31**(3) (2018), 333-348.

[16] A. Hamoud and K. Ghadle, *Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations*, J. Appl. Comput. Mech., **5**(1) (2019), 58-69.

[17] A. Hamoud, *Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro-differential equations*, Advan. Theory Nonlinear Anal. Appl., **4**(4) (2020), 321-331.

[18] A. Hamoud, M. SH. Bani Issa and K. Ghadle, *Existence and uniqueness results for nonlinear Volterra-Fredholm integro-differential equations*, Nonlinear Funct. Anal. Appl., **23**(4) (2018), 797-805.

[19] K. Hussain, A. Hamoud and N. Mohammed, *Some new uniqueness results for fractional integro-differential equations*, Nonlinear Funct. Anal. Appl., **24**(4) (2019), 827-836.

[20] A. Hamoud and K. Ghadle, *Some new results on nonlinear fractional iterative Volterra-Fredholm integro differential equations*, TWMS J. Appl. Eng. Math., **12**(4) (2022), 1283-1294.

[21] J.W. Hanneken, D.M. Vaught and B.N. Narahari Achar, Enumeration of the real zeros of the Mittag-Leffler function $E_\alpha(z)$, $1 < \alpha < 2$. In: Sabatier, J., Agrawal, O.P., Machado, J.A.T. (eds.) Advances in Fractional Calculus, pp. 15-26. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-6042-7_2.

[22] P. Karthikeyan, *Some results for boundary value problem of an integrodifferential equations with fractional order*, Dynamic Syst. Appl., **20** (2011), 17-24.

[23] K. Karthikeyan and J.J. Trujillo, *Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions*, Commu. Nonlinear Sci. Numer. Simu., **17** (2012), 4037-4043.

[24] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential equations*, Elsevier, Amsterdam, 2006.

[25] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic, Dordrecht, 1991.

[26] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal., **69** (2008), 3337-3343.

[27] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equation*, Nonlinear Anal., **69** (2008), 2677-2682.

[28] T. Li, N. Pintus and G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., **70** (2019), 86-94.

[29] T. Li and G. Viglialoro, *Analysis and explicit solvability of degenerate tensorial problems*, Bound. Value Probl., **2018** (2018), 21-34.

[30] W. Lin, *Global existence theory and chaos control of fractional differential equation*, J. Math. Anal. Appl., **332** (2007), 709-726.

[31] S. Magar, A. Hamoud, A. Khandagale and K. Ghadle, *Generalized Shehu transform to* Ψ*-Hilfer-Prabhakar fractional derivative and its regularized version*, Advances in the Theory of Nonlinear Analysis and its Application, **6**(3) (2022), 364-379.

[32] A. Pazy, *Semigroups of Linear operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[33] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[34] M. Raja, V. Vijayakumar, A. Shukla, K. Sooppy Nisar and S. Rezapour, *New discussion on nonlocal controllability for fractional evolution system of order* $1 < r < 2$, Advances in Difference Equations, 2021 (2021), 1-19.

[35] A. Sharif, and A. Hamoud, *On* ψ*-Caputo fractional nonlinear Volterra-Fredholm integro-differential equations*, Disconti. Nonlinearity and Complexity, **11**(1) (2022), 97-106.

[36] D.R. Smart, *Fixed point Theorems*, Cambridge University Press, Cambridge **66** 1980.

[37] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[38] D. Tamizharasan and K. Karthikeyan, *Controllability results for fractional integrodifferential systems with boundary conditions*, Indian J. Pure Appl. Math., **52** (2021), 39-45.
[39] P. Wang, Y. Wang, C. Jiang and T. Li, *Convergence of solutions for functional integrodifferential equations with nonlinear boundary conditions*, Adv. Diff. Equ. **2019** (2019), 521.
[40] S. Zhang, *Positive solutions for boundary value problems for nonlinear fractional differential equations*, Elec. J. Diff. Equ., **36** (2006), 1-12.