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$\mathcal{H}(\omega, \theta)$-CONTRACTION AND SOME NEW FIXED POINT RESULTS IN MODIFIED ω-DISTANCE MAPPINGS VIA COMPLETE QUASI METRIC SPACES AND APPLICATION

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Abstract. In this manuscript, we establish the concept of $\mathcal{H}(\omega, \theta)$-contraction which based on modified ω distance mappings which introduced by Alegre and Marin [4] in 2016 and $\mathcal{H}$ simulation functions which introduced by Bataihah et.al. [14] in 2020 and we employ our contraction to prove the existence and uniqueness some new fixed point results. On the other hand, we create some examples and an application to show the importance of our results.

1. Introduction

Banach Contraction principle [10] is one of the most influential theory in pure and applied mathematics and other science, since the time that Banach established his contraction many equations that have no solution now have solution since existence and uniqueness is similar to uniqueness of a solution.

Since then, the mathematicians generalized this theorem in two ways some of them establish a generalize of a metric space such as G-metric spaces, b-metric spaces, quasi metric spaces and ω-modify metric spaces, for examples

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[16]-[18], [20]-[23]. The others generalize Banach contraction, for examples [1]-[14].

In 1931 Wilson [24] introduced the concepts of a quasi metric space and a generalized metric space as follows:.

Definition 1.1. Let X be a nonempty set and $q : X \times X \to [0, +\infty)$ be a given function which satisfy

(i) $q(x, y) = 0$ if and only if $x = y$.
(ii) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

It is obviously that, every metric space is a quasi metric space, but the converse need not be true in general. A quasi metric q induces a metric q_m as follows:

$$q_m(x, y) = \max\{q(x, y), q(y, x)\}.$$

Henceforth, (X, q) refers to a quasi metric space.

Definition 1.2. ([24]) Let (X, q) be a quasi metric space and $\{x_n\}$ be a sequence in X. Then,

(i) the sequence $\{x_n\}$ is left-Cauchy if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $q(x_n, x_m) < \epsilon$ for all $n \geq m > N$,
(ii) the sequence (x_n) is right-Cauchy if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $q(x_n, x_m) < \epsilon$ for all $m \geq n > N$.

Definition 1.3. ([21]) Let (X, q) be a quasi metric space. We say that

(i) (X, q) is left-complete if every left-Cauchy sequence is convergent in X,
(ii) (X, q) is right-complete if every right-Cauchy sequence is convergent in X,
(iii) (X, q) is complete if every Cauchy sequence is convergent in X.

Definition 1.4. ([21]) A modified ω-distance on a quasi metric space (X, q) is a function $\omega : X \times X \to [0, +\infty)$, which satisfy:

(i) $\omega(x, y) \leq p(x, z) + p(z, y), \ \forall \ x, z, y \in X$,
(ii) $\omega(x, .) : X \to [0, +\infty)$ is lower semi-continuous for all $x \in X$,
(iii) for all $\epsilon > 0$, there exist $\alpha > 0$ such that if $\omega(x, y) \leq \alpha$ and $\omega(y, z) \leq \alpha$ then $q(x, z) \leq \epsilon$ for all $x, y, z \in X$.

Definition 1.5. ([14, 19]) Let Θ denotes the class of all continuous and non-decreasing functions, $\theta : [0, +\infty) \to [1, +\infty)$ that satisfies, for all $\{x_n\}$ a sequence in $(0, +\infty)$, $\lim\limits_{n \to +\infty} \theta(x_n) = 1$ if and only if $\lim\limits_{n \to +\infty} x_n = 0$.

Remark 1.6. If $\theta \in \Theta$, then θ^{-1} $(\{1\})$=0.

Definition 1.7. ([10]) A class of functions $H : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ is called $\mathcal{H}$-simulation if

$$H(x,y) \leq \frac{x}{y}, \quad \forall x,y \in [1,+\infty). \tag{1.1}$$

Remark 1.8. Suppose $H \in \mathcal{H}$ and $\{x_n\},\ \{y_n\}$ are sequences in $[1,+\infty)$ with $1 \leq \lim_{n\to+\infty} y_n < \lim_{n\to+\infty} x_n$. Then

$$\limsup_{n\to+\infty} H(x_n,y_n) < 1. \tag{1.2}$$

If X is a nonempty set and $f : X \to X$ is a self-mapping, then the point $u \in X$ is called a fixed point for f if $f(x) = x$. $Fix(f)$ stands for the set of all fixed points of f.

2. Main results

Before we introduce our main result, we establish $\mathcal{H}(\omega,\theta)$-contraction.

Definition 2.1. Assume (X,q) is equipped with modified ω-distance mapping ω. A self-mapping f on X is called $\mathcal{H}(\omega,\theta)$ if there exist $\lambda \in (0,1),\ \theta \in \Theta$ and $H \in \mathcal{H}$ such that for all $x,y \in X$ we have

$$1 \leq H(\theta(\omega(fx,fy)), \theta\lambda N(x,y)), \tag{2.1}$$

where $N(x,y) = \max\{\omega(x,y), \omega(x,fx), \omega(y,fy)\}$.

Lemma 2.2. *Assume that the mapping $f : X \to X$ satisfies $\mathcal{H}(\omega,\theta)$-contraction. Then*

(i) *if $0 < N(x,y)$, then $\omega(fx,fy) < \lambda N(x,y)$,*
(ii) *if $0 = N(x,y)$, then $\omega(fx,fy) = 0$.*

Proof. (i) If $0 < N(x,y)$, then

$$\begin{aligned} 1 &\leq H(\theta(\omega(fx,fy)), \theta\lambda N(x,y)) \\ &\leq \frac{\theta\lambda N(x,y)}{\theta(\omega(fx,fy))}. \end{aligned}$$

Therefore, $\theta\omega(fx,fy) \leq \theta\lambda N(x,y)$. Hence we have the result.

(ii) If $0 = N(x,y)$, then by condition (i), we get that

$$1 \leq \theta\omega(fx,fy) \leq \theta\lambda N(x,y) = 1.$$

Consequently, $\omega(fx,fy) = 0$. □

Lemma 2.3. *Assume that the mapping $f : X \to X$ satisfies $\mathcal{H}(\omega, \theta)$-contraction. Then $Fix(f)$ consists of at most one element.*

Proof. First, we claim that if $v \in Fix(f)$, then $\omega(v, v) = 0$. Assume $\omega(v, v) > 0$. By Lemma 2.2, we get that $\theta\omega(fv, fv) \leq \theta\lambda N(v, v) = \theta\lambda\omega(v, v)$. So, $\omega(fv, fv) \leq \lambda\omega(v, v)$. Thus $\omega(v, v) = \omega(fv, fv) \leq \lambda\omega(v, v) < \omega(v, v)$, which is a contradiction.

Let $u, v \in Fix(f)$. Now to show that $\omega(u, v) = 0$. Assume $\omega(u, v) > 0$ by Lemma 2.2 we get that

$$\begin{aligned}\theta\omega(u, v) = \theta\omega(fu, fv) &\leq \theta\lambda N(u, v)\\ &= \theta\lambda \max\{\omega(u, v), \omega(u, fu), \omega(v, fv)\}\\ &= \theta\lambda\omega(u, v).\end{aligned}$$

Thus $\omega(u, v) = \omega(fu, fv) \leq \lambda\omega(u, v) < \omega(u, v)$, which is a contradiction. Since $\omega(u, v) = 0$ and by the property of modified ω distance and $\omega(v, v) = 0$, we have $q(u, v) = 0$ and so $u = v$. □

Theorem 2.4. *Suppose (X, q) is complete equipped with a modified ω distance mapping ω and suppose there exist $\theta \in \Theta$, $H \in \mathcal{H}$ and $\lambda \in (0, 1)$ such that the self-mapping $f : X \to X$ is a $\mathcal{H}(\omega, \theta)$-contraction. If one of the following satisfied:*

(1) *f is a continuous mapping.*
(2) *If $\beta \neq f\beta$ for $\beta \in X$, then*

$$0 < \inf\{\omega(fx, \beta) + \omega(x, \beta) : x \in X\}. \tag{2.2}$$

Then $Fix(f)$ consist of only one element.

Proof. Construct the sequence $\{x_n\}$ starting at an arbitrary point $x_0 \in X$ by letting $x_{n+1} = f^{n+1}(x_0) = f(x_n)$, for $n \in \mathbb{N}$. By using (2.1), we have

$$\begin{aligned}1 &\leq H(\theta\omega(fx_{n-1}, fx_n), \theta\lambda N(x_{n-1}, x_n))\\ &= H(\theta\omega(x_n, x_{n+1}), \theta\lambda N(x_{n-1}, x_n)).\end{aligned} \tag{2.3}$$

If $N(x_{n_0-1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$, then $\omega(x_{n_0-1}, x_{n_0}) = \omega(x_{n_0}, x_{n_0+1}) = 0$. By part (i) of the definition of ω, we have

$$\omega(x_{n_0-1}, x_{n_0+1}) \leq \omega(x_{n_0-1}, x_{n_0}) + \omega(x_{n_0}, x_{n_0+1}) = 0.$$

By part (iii) of the definition of ω, we have

$$q(x_{n_0-1}, x_{n_0+1}) = 0,$$

and so $x_{n_0-1} = x_{n_0+1}$. Thus, $\omega(x_{n_0-1}, x_{n_0-1}) = 0$. Hence by part (iii) of the definition of ω, we have $q(x_{n_0-1}, x_{n_0}) = 0$. Therefore, $x_{n_0-1} \in Fix(f)$.

Assume $N(x_{n-1}, x_n) > 0$ for each $n \in \mathbb{N}$. Then by Lemma 2.2, we have

$$\omega(x_n, x_{n+1}) = \omega(fx_{n-1}, fx_n) \leq \lambda \max\{\omega(x_{n-1}, x_n), \omega(x_n, x_{n+1})\}.$$

If $\max\{\omega(x_{n-1},x_n),\omega(x_n,x_{n+1})\}=\omega(x_n,x_{n+1})$, we get

$$\omega(x_n,x_{n+1})\le\lambda\omega(x_n,x_{n+1}),$$

which is a contradiction. Therefore, we have

$$\begin{aligned}\omega(x_n,x_{n+1}) &< \lambda\omega(x_{n-1},x_n)\\ &< \lambda^2\omega(x_{n-2},x_{n-1})\\ &\ \vdots\\ &< \lambda^n\omega(x_0,x_1).\end{aligned}\tag{2.4}$$

Now, we also have

$$\begin{aligned}\omega(x_{n+1},x_n) &< \lambda N(x_n,x_{n-1}))\\ &= \max\{\omega(x_n,x_{n-1}),\omega(x_n,x_{n+1}),\omega(x_{n-1},x_n)\}\\ &= \max\{\omega(x_{n-1},x_n),\omega(x_n,x_{n+1})\}\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\omega(x_{n+1},x_n) &< \lambda\max\{\omega(x_n,x_{n-1}),\omega(x_n,x_{n+1}),\omega(x_{n-1},x_n)\}\\ &= \lambda\max\{\omega(x_n,x_{n-1}),\omega(x_n,x_{n+1})\}\\ &< \lambda\max\{\lambda\omega(x_{n-2},x_{n-1}),\lambda\max\{\omega(x_{n-1},x_{n-2}),\omega(x_{n-2},x_{n-1})\}\\ &< \lambda^2\max\{\omega(x_{n-2},x_{n-1}),\omega(x_{n-2},x_{n-1})\}\\ &\ \vdots\\ &< \lambda^n\omega(x_0,x_1),\omega(x_1,x_0)).\end{aligned}\tag{2.6}$$

Now, let $m,n\in\mathbb{N}$ with $m>n$. Then we have

$$\begin{aligned}\omega(x_n,x_m) &< \omega(x_n,x_{n+1})+\omega(x_{n+1},x_{n+2})+\cdots+\omega(x_{m-1},x_m)\\ &< \lambda^n\omega(x_0,x_1)+\lambda^{n+1}\omega(x_0,x_1)+\cdots+\lambda^{m-1}\omega(x_0,x_1)\\ &= \lambda^n\omega(x_0,x_1)[1+\lambda+\lambda^2+\cdots+\lambda^{m-n-1}].\end{aligned}\tag{2.7}$$

Letting $n,m\to\infty$, we get that

$$\lim_{m,n\to\infty}\omega(x_n,x_m)=0.\tag{2.8}$$

Therefore, $\{x_n\}$ is a right Cauchy sequence.

Similarly, to show that (x_n) is a left Cauchy sequence, let $m,n\in\mathbb{N}$ with $m<n$. Then we have

$$\begin{aligned}\omega(x_m,x_n) &< \omega(x_m,x_{m-1})+\omega(x_{m-1},x_n)+\cdots+\omega(x_{n+1},x_n)\\ &< \lambda^{m-1}\max\{\omega(x_0,x_1),\omega(x_1,x_0)\}\\ &\quad+\lambda^{m-2}\max\{\omega(x_0,x_1),\omega(x_1,x_0)\}\\ &\quad+\cdots+\lambda^n\max\{((x_0,x_1)\}\\ &= \lambda^n\max\{((x_0,x_1)\}[\lambda^{m-n-1}+\lambda^{m-n-2}+\cdots+1].\end{aligned}\tag{2.9}$$

Hence, we have

$$\lim_{m,n\to\infty}\omega(x_m,x_n)=0.\tag{2.10}$$

Consequently, $\{x_n\}$ is a Cauchy sequence. Since (X, q) is a complete, there is an element $\beta \in X$ such that $x_n \to \beta$.

Now we want to show that $Fix(f)$ consists of only one element. If f is a continuous function then $\beta = f\beta$. But if f is any mapping, then $\beta \in X$ and $f\beta \neq \beta$. And so,

$$0 < \inf\{\omega(x, \beta) + \omega(fx, \beta) : \forall x \in X\}.$$

Assume $f\beta \neq \beta$, for all $\epsilon > 0$. Since $\lim_{m,n\to\infty} \omega(x_m, x_n) = 0$, choose $K \in \mathbb{N}$ such that $\omega(x_m, x_n) \leq \frac{\epsilon}{2}$ for all $n, m \geq K$. Since ω is lower semi continuous,

$$\omega(x_n, \beta) \leq \lim_{j\to\infty} \inf(\omega(x_n, x_j)) \leq \frac{\epsilon}{2}.$$

If $f\beta \neq \beta$,

$$\begin{aligned}\inf\{\omega(x, \beta) + \omega(f_x, \beta)\} &\leq \inf\{\omega(x_n, \beta) + \omega(f(x_n), \beta) : x \in X\} \\ &= \inf\{\omega(x_n, \beta) + \omega(x_n, \beta)\} \\ &\leq \epsilon,\end{aligned}$$

which is a contradiction. By using Lemma 2.3, we get that, $Fix(f)$ consists of only one element. □

Corollary 2.5. *Suppose (X, q) is complete equipped with a modified ω-distance mapping ω and there exists $\alpha \in (0, 1)$ such that the self-mapping $f : X \to X$ satisfy the following:*

$$2^{\omega(fx,fy)} \leq 2^{\alpha\omega(x,y)}, \ \ \forall x, y \in X$$

and one of the following satisfy:

(1) *f is a continuous mapping.*
(2) *If f is any mapping and $\beta \neq f\beta$ for all $\beta \in X$,*

$$0 < \inf\{\omega(fx, \beta) + \omega(x, \beta)\}. \tag{2.11}$$

Then $Fix(f)$ consists of only one element.

Proof. Define $H : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$, $\theta : [0, \infty) \to [1, \infty)$ by $H(v_1, v_2) = \frac{v_2^\lambda}{v_1}$, $\lambda \in (0, 1)$, $\theta(v) = 2^v$, for all $v \in X$, respectively. Then $H \in \mathcal{H}$ *and* $\theta \in \Theta$.

Now, $2^{\omega(fx,fy)} \leq 2^{\alpha\omega(x,y)} \leq 2^{\alpha N(x,y)}$, then $\theta\omega(fx, fy) \leq \theta\alpha N(x, y)$. If $\alpha = \lambda^2$, then $\lambda \in (0, 1)$. Therefore,

$$1 \leq \frac{(\theta\lambda N(x, y))^\lambda}{\theta\omega(f_x, f_y)} \iff 1 \leq H(\theta\omega(f_x, f_y), \theta\alpha N(x, y)).$$

□

Corollary 2.6. *Suppose (X, q) is complete equipped with a modified ω-distance mapping ω and there exist $\alpha \in (0,1)$ such that the self-mapping $f : X \to X$ satisfy the following:*

$$\omega(fx, fy) \leq \alpha\omega(x, y), \ \ \forall x, y \in X.$$

Then $Fix(f)$ consists of only one element.

Proof. $\omega(f_x, f_y) \leq \alpha\omega(x, y)$ if and only if $2^{\omega(fx,fy)} \leq 2^{\alpha\omega(x,y)}$. Using Corollary 2.5, we get the results. □

Example 2.7. Let $X = \{0, 1, 2, ..., 10\}$, define the self-mapping $f : X \to X$ by

$$f(x) = \begin{cases} 0, & x \in \{0,1\}, \\ 1, & x \in \{2,0,...,5\}, \\ 2, & x \in \{6,7,...,10\}. \end{cases}$$

Then $Fix(f)$ consists of only one element. To show this, define $q : X \times X \to [0, +\infty)$ by

$$q(x, y) = \begin{cases} 0, & x = y, \\ x + 2y, & x \neq y. \end{cases}$$

Also define $\omega : X \times X \to [0, +\infty)$ by $\omega(x, y) = \frac{1}{3}(x + 2y)$ and define $H : X \times X \to [0, +\infty), \theta : X \to [1, +\infty)$ by $H(x, y) = \frac{y^{\frac{1}{\sqrt{2}}}}{x}, \theta(u) = e^u$, respectively. Then

(1) (X, q) is a complete quasi metric space.
(2) ω is a modified $\omega-$distance equipped on q.
(3) f is continuous function.
(4) $1 \leq H(\theta(\omega(fx, fy)), \theta\lambda N(x, y)), \ \lambda = \frac{1}{\sqrt{2}}$.

To show that q is complete, let $\{x_n\}$ be a Cauchy sequence in X. For all $n, m \in \mathbb{N}$, we have

$$\lim_{m,n\to\infty} q(x_m, x_n) = 0. \tag{2.12}$$

Then, there is $K \in \mathbb{N}$ such that $\{X_m\}, \{X_n\}$ for all $n, m \geq K$ therefor $\{x_n\}$ is a convergent in X. Hence (X, q) is complete to prove 4 cases, for all $x, y \in X$, we have the following cases:

Case 1: we have three sub cases:
(i) subcase 1. If $x, y \in \{0, 1\}$, then $\omega(fx, fy) = 0$, so we are done.

(ii) subcase 2. If $x, y \in \{2, 3, ..., 5\}$, then $\omega(fx, fy) = 1$.

$$\lambda^2\omega(x, y) \geq \frac{1}{2}\omega(2, 2) = 1.$$

(iii) subcase 3. If $x, y \in \{6, 7, ..., 10\}$, then $\omega(fx, fy) = 2$.

$$\lambda^2\omega(x, y) \geq \frac{1}{2}\omega(6, 6) = 3.$$

Case 2: If $x \in \{0, 1\}, y \in \{2, 3, ..., 5\}$.

$$\omega(fx, fy) = \omega(0, 1) = \frac{2}{3}.$$

$$\lambda^2\omega(x, y) \geq \frac{1}{2}\omega(0, 2) = \frac{2}{3}.$$

Case 3: If $x \in \{0, 1\}, y \in \{6, 7, ..., 10\}$.

$$\omega(fx, fy) = \omega(0, 2) = \frac{4}{3}.$$

$$\lambda^2\omega(x, y) \geq 2.$$

Case 4: If $x \in \{2, 3, ..., 5\}, y \in \{6, 7, ..., 10\}$.

$$\omega(fx, fy) = \frac{5}{3}.$$

$$\lambda^2\omega(x, y) \geq \frac{14}{6}.$$

Similarly, if $y \in \{0, 1\}, x \in \{2, 3, ..., 5\}$ and $y \in \{0, 1\}$, $x \in \{6, 7, ..., 10\}$ and $y \in \{2, 3, ..., 5\}$, $x \in \{6, 7, ..., 10\}$ sequentially, f satisfy $\mathcal{H}(\omega, \theta)$-contraction. Theorem 2.4 informs us that $Fix(f)$ consists of only one element.

Example 2.8. Consider the following mapping:

$$f(x) = \frac{1 - x^m}{M + x^m}, \quad \text{where } m \in \mathbb{N} - \{1\} \text{ and } M > m.$$

Then $Fix(f)$ consist of only one element on $[0, 1]$. To show this, define the following mapping $H : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$, $\theta : [0, +\infty) \to [1, +\infty)$ by $H(v_1, v_2) = 1 + \ln(\frac{v_2}{v_1})$, $\theta(v) = e^v$, for all $v \in X$, respectively, then $H \in \mathcal{H}$ and $\theta \in \Theta$, also, define: $q : X \times X \to [0, +\infty)$ by $q(x, y) = |x - y|$, then (X, q) is a complete quasi metric space. Furthermore, define $\omega : X \times X \to [0, +\infty)$ by $\omega(x, y) = |x - y|$, then ω is a modified ω distance mapping.

Now, equip (X,q) with ω, for all $x,y\in X$, we have:

$$\begin{aligned}
\omega(fx,fy) &= |\frac{1-x^m}{M+x^m}-\frac{1-y^m}{M+y^m}| \\
&= \frac{1}{(M+x^m)(M+y^m)}\Big[(1-x^m)(M+y^m)-(1-y^m)(M+x^m)\Big] \\
&\le \frac{M-1}{M^2}|x^m-y^m| \\
&= \frac{M-1}{M^2}|x-y|\Big(x^{m-1}+yx^{m-2}+\cdots+xy^{m-2}+y^{m-1}\Big) \\
&\le \frac{(M-1)m}{M^2}|x-y| \\
&\le \frac{M-1}{M}|x-y| \\
&= \lambda\omega(x,y) \\
&\le \lambda N(x,y).
\end{aligned}$$

Hence we have,

$$\omega(fx,fy)\le \lambda N(x,y),$$

this implies that

$$e^{\omega(fx,fy)}\le e^{\lambda N(x,y)}.$$

Thus we have

$$1\le e\le \frac{e^{\lambda N(x,y)}}{e^{\omega(fx,fy)}}.$$

Hence,

$$1\le 1+\ln\frac{e^{\lambda N(x,y)}}{e^{\omega(fx,fy)}}.$$

This means that

$$1\le H(\theta(fx,fy)\theta\lambda N(x,y)).$$

Therefore, f satisfy $\mathcal{H}(\omega,\theta)$-contraction. Theorem 2.4 ensures that $Fix(f)$ consist of only one element.

3. Application

In this application, we will show that the following equation

$$x^{k+1} + x^k + kx - 1, \text{ for } k \geq 2 \tag{3.1}$$

not only has a solution in the unit interval as intermediate value Theorem, but also, the solution is unique. To prove this, it is similar to show that the following mapping has a unique fixed point in the unit interval

$$f(x) = \frac{1 - x^k}{k + x^k}, \text{ for } k \geq 2.$$

Example 2.8 ensures that f has a unique fixed point and so the equation (3.1) has a unique solution.

4. Conclusion

We proved some new fixed point results by employing a new contraction namely, $\mathcal{H}(\omega, \theta)$-contraction and to show that our results are applicable we introduces two examples, one is discrete and the other one is continuous. To show the novelty of our new results we introduced an interesting application.

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