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ERRATUM/ADDENDUM TO "BIISOMETRIC OPERATORS AND BIORTHOGONAL SEQUENCES" [BULL. KOREAN MATH. SOC. 56 (2019), NO. 3, PP. 585–596]

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Abstract. Erratum/Addendum to the paper "Biisometric operators and biorthogonal sequences" [Bull. Korean Math. Soc. **56** (2019), No. 3, pp. 585–596].

Concerning the above paper, the proof of Theorem 3.1 in the original version was incomplete where a relevant argument was missing. This has been fixed now making the proof clearer. Also, the existence assumption for the series expansion in Corollary 3.1 was overlooked in the original version. The corrected statement and proof of both Theorem 3.1 and Corollary 3.1 are worked out below.

Notation and terminology as in [3]: By an operator we mean a bounded linear transformation of a normed space into itself. The kernel and range of an arbitrary operator T are denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. A pair of sequences $\{f_n\}$ and $\{g_n\}$ of vectors in a Hilbert space are said to be *biorthogonal* (to each other) if $\langle f_m; g_n\rangle = \delta_{m,n}$, where δ stands for the Kronecker delta function. If $\{f_n\}$ is such that there exists a sequence $\{g_n\}$ for which $\{f_n\}$ and $\{g_n\}$ are biorthogonal, then it is said that $\{f_n\}$ *admits a biorthogonal sequence* (and so does $\{g_n\}$) and the pair $\{\{f_n\},\{g_n\}\}$ is referred to as a *biorthogonal system*. If, in addition, $\|f_n\| = \|g_n\| = 1$ for all n, then $\{\{f_n\},\{g_n\}\}$ is said to be a *biorthonormal* pair (or system). It was shown in [3, Corollary 2.1] that *there is no distinct pair of biorthonormal sequences*. Two Hilbert-space operators V and W are said to be *biisometric* if $V^*W = I$.

Theorem 3.1. *Let V and W be operators on a Hilbert space $\mathcal{H}$. Suppose their adjoints V^* and W^* are not injective. Equivalently, suppose the ranges of V and W are not dense. Take arbitrary nonzero vectors v and w in $\mathcal{H}$. For each nonnegative integer n consider the vectors*

$$\phi_n = V^n w \quad \textit{and} \quad \psi_n = W^n v$$

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in $\mathcal{H}$. If $\{V, W\}$ is a biisometric pair on $\mathcal{H}$, then there exist

$$v \in \mathcal{N}(V^*) \quad \textit{and} \quad w \in \mathcal{N}(W^*)$$

such that the sequences $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal. Moreover,

$$V\phi_n = \phi_{n+1} \quad \textit{and} \quad W\psi_n = \psi_{n+1},$$

and also

$$V^*\psi_{n+1} = \psi_n \quad \textit{and} \quad W^*\phi_{n+1} = \phi_n.$$

Proof. Suppose V and W have noninjective adjoints. This is equivalent to saying that V and W have nondense ranges (because $\mathcal{N}(T^*)^\perp = \mathcal{R}(T)^-$ for any operator T and $A^\perp = \{0\} \iff A^- = \mathcal{H}$ for any subset A of $\mathcal{H}$; see, e.g., [2, Propositions 5.12 and 5.76]). Since the adjoints V^* and W^* are noninjective, take arbitrary nonzero vectors $v \in \mathcal{N}(V^*)$ and $w \in \mathcal{N}(W^*)$, arbitrary nonnegative integers m, n, and set

$$\phi_n = V^n w \quad \text{and} \quad \psi_n = W^n v.$$

Suppose $n < m$. As $V^*W = I$ a trivial induction leads to $V^{*n}W^n = I$ and so

$$\langle \phi_m\,; \psi_n \rangle = \langle V^m w\,; W^n v \rangle = \langle w\,; V^{*(m-n)} V^{*n} W^n v \rangle = \langle w\,; V^{*(m-n)} v \rangle = 0$$

for $n < m$ because $v \in \mathcal{N}(V^*)$ implies $v \in \mathcal{N}(V^{*m-n})$. On the other hand, suppose $m < n$. As $W^*V = I$ and $w \in \mathcal{N}(W^*)$ a similar argument ensures that for $m < n$,

$$\langle \phi_m\,; \psi_n \rangle = 0.$$

Moreover, since $W^*V = I$, we get $V \neq O$. Thus take any $0 \neq y = \mathcal{R}(V)$ so that $y = Vx$ for some $0 \neq x \in \mathcal{H}$. If $y \in \mathcal{N}(W^*)$, then $0 = W^*y = W^*Vx = x$, which is a contradiction. Hence $\mathcal{R}(V) \cap \mathcal{N}(W^*) = \{0\}$, which implies

$$\mathcal{R}(V)^- \cap \mathcal{N}(W^*) = \{0\}.$$

Therefore $\mathcal{N}(V^*)^\perp \cap \mathcal{N}(W^*) = \{0\}$. Suppose $\mathcal{N}(W^*) \perp \mathcal{N}(V^*)$. This implies that $\mathcal{N}(W^*) \subseteq \mathcal{N}(V^*)^\perp$, and so

$$\mathcal{N}(W^*) = \mathcal{N}(W^*) \cap \mathcal{N}(W^*) \subseteq \mathcal{N}(V^*)^\perp \cap \mathcal{N}(W^*) = \{0\},$$

which contradicts the assumption of W^* being noninjective. Consequently,

$$\mathcal{N}(V^*) \not\perp \mathcal{N}(W^*).$$

Thus there exist (nonzero) $v \in \mathcal{N}(V^*)$ and $w \in \mathcal{N}(W^*)$ such that $\langle w\,; v \rangle \neq 0$, and so we may take $v \in \mathcal{N}(V^*)$ and $w \in \mathcal{N}(W^*)$ for which $\langle\, w\,; v \rangle = 1$. Then

$$\langle \phi_n\,; \psi_n \rangle = \langle\, w\,; v \rangle = 1.$$

(Indeed, since $V^*W = I$ we get $V^{*n}W^n = I$, and so $\langle \phi_n\,; \psi_n \rangle = \langle V^n w\,; W^n v \rangle = \langle w\,; V^{*n} W^n v \rangle = \langle w\,; v \rangle$ for every $n \geq 1$.) Therefore

$$\langle \phi_m\,; \psi_n \rangle = \delta_{m,n}$$

(i.e., $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal sequences). Also, as $\phi_n = V^n w$ and $\psi_n = W^n v$,

$$V\phi_n = V^{n+1}w = \phi_{n+1} \quad \text{and} \quad W\psi_n = W^{n+1}v = \psi_{n+1},$$

and since $V^*W = I = W^*V$ we get

$$V^*\psi_{n+1} = V^*W^{n+1}v = W^n v = \psi_n \ \text{ and } \ W^*\phi_{n+1} = W^*V^{n+1}w = V^n w = \phi_n$$

for every nonnegative integer n. □

Corollary 3.1. *Let $\{V, W\}$ be a biisometric pair on $\mathcal{H}$ and consider the biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ defined in Theorem 3.1 in terms of nonzero vectors $v \in \mathcal{N}(V^*)$ and $w \in \mathcal{N}(W^*)$. If, in addition, these biorthogonal sequences span $\mathcal{H}$, and if $x \in \mathcal{H}$ has series expansion in $\{\phi_n\}$ and in $\{\psi_n\}$, then x can be expressed as*

$$x = \sum_{k=0}^{\infty} \langle x\,;\psi_k\rangle \phi_k = \sum_{k=0}^{\infty} \langle x\,;\phi_k\rangle \psi_k,$$

and therefore

$$Vx = \sum_{k=0}^{\infty} \langle x\,;\psi_k\rangle \phi_{k+1} \quad \textit{and} \quad Wx = \sum_{k=0}^{\infty} \langle x\,;\phi_k\rangle \psi_{k+1},$$

$$V^*x = \sum_{k=0}^{\infty} \langle x\,;\phi_{k+1}\rangle \psi_k \quad \textit{and} \quad W^*x = \sum_{k=0}^{\infty} \langle x\,;\psi_{k+1}\rangle \phi_k.$$

Proof. Take an arbitrary $x \in \mathcal{H}$. First note that if a sequence in a biorthogonal pair is such that it spans $\mathcal{H}$, then it does not necessarily follow that all elements in $\mathcal{H}$ have an expansion as the limit of a linear combination of elements of the sequence (cf. [1, Example 5.4.6]). Thus suppose the biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ span $\mathcal{H}$ (i.e., $\bigvee\{\phi_n\} = \bigvee\{\psi_n\} = \mathcal{H}$) and, in addition, also suppose that

$$x = \sum_{k=0}^{\infty} \alpha_k \phi_k = \sum_{k=0}^{\infty} \beta_k \psi_k$$

for some pair of sequences of scalars $\{\alpha_n\}$ and $\{\beta_n\}$, then

$$\alpha_n = \langle x\,;\psi_n\rangle \ \text{ and } \ \beta_n = \langle x\,;\phi_n\rangle$$

for every $n \geq 0$. Indeed, observe (by the continuity of the inner product and recalling that $\{\phi_k\}$ and $\{\psi_k\}$ are biorthogonal) that

$$\langle x\,;\psi_n\rangle = \sum_{k=0}^{\infty} \alpha_k \langle \phi_k\,;\psi_n\rangle = \alpha_n \ \text{ and } \ \langle x\,;\phi_n\rangle = \sum_{k=0}^{\infty} \alpha_k \langle \psi_k\,;\phi_n\rangle = \beta_n$$

for every $n \geq 0$. Finally, recall from Theorem 3.1 that for each $n \geq 0$

$$V\phi_n = \phi_{n+1}, \ \ W\psi_n = \psi_{n+1}, \ \ V^*\psi_{n+1} = \psi_n, \ \text{ and } \ W^*\phi_{n+1} = \phi_n.$$

So apply V and W^* to the expansion of x in terms of $\{\phi_n\}$ and apply V^* and W to the expansion of x in terms of $\{\psi_n\}$ (using the continuity of the inner product). □

Consequently, the existence assumption for the series expansion should be included in Corollary 4.1 as well. That is, it must be assumed in Corollary 4.1's statement that "$f \in L^2[0,\infty)$ *has an expansion in* $\{\phi_n\}$ *and in* $\{\psi_n\}$".

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