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SYSTEM OF GENERALIZED MULTI-VALUED RESOLVENT EQUATIONS: ALGORITHMIC AND ANALYTICAL APPROACH

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Abstract. In this paper, under some new appropriate conditions imposed on the parameter and mappings involved in the resolvent operator associated with a P-accretive mapping, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. This paper is also concerned with the construction of a new iterative algorithm using the resolvent operator technique and Nadler's technique for solving a new system of generalized multi-valued resolvent equations in a Banach space setting. The convergence analysis of the sequences generated by our proposed iterative algorithm under some appropriate conditions is studied. The final section deals with the investigation and analysis of the notion of $H(\cdot,\cdot)$-co-accretive mapping which has been recently introduced and studied in the literature. We verify that under the conditions considered in the literature, every $H(\cdot,\cdot)$-co-accretive mapping is actually P-accretive and is not a new one. In the meanwhile, some important comments on $H(\cdot,\cdot)$-co-accretive mappings and the results related to them appeared in the literature are pointed out.

1. Introduction

Because of the importance and the wide applications in different areas of science, engineering, social science, economics and management, the theory of variational inequalities has increasingly received much attentions, and has been greatly extended and generalized in various directions to study a wide class of problems arising in physics, nonlinear programming, mechanics, optimization and control, elasticity and applied science, etc., see for example [7,21,26,28,29] and the references therein. Among generalizations of the variational inequality,

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the variational inclusion is very important and useful, and has been studied intensively by many authors in the past years, see, for example, [2, 6, 8–11, 14–16, 19, 23, 31, 33, 36] and the references therein.

The development of numerical methods which provide an efficient and implementable algorithm for solving variational inequality/inclusion and its generalizations, is of the most important and interesting problems in the theory of variational inequalities and inclusions. There are many methods to find solutions of different classes of variational inequality and variational inclusion problems. Among these methods, the resolvent operator technique is very important and interesting. With the development of an iterative algorithm to compute approximate solutions of generalized variational inequalities/inclusions, the technique of resolvent operators has become more and more important and efficient. For this reason, in recent past, the methods based on different classes of resolvent operators have been developed to study the existence of solutions and to discuss convergence analysis of iterative algorithms, for various classes of variational inclusions and their generalizations, see, for example, [2, 14–16, 19, 22, 23] and the references therein.

Ding [12], Ding and Luo [13] and Lee et al. [24] have used subdifferentiability and proximal mapping of a proper function on Hilbert space to study general and generalized quasi-variational-like inclusions, respectively. In 2001, Huang and Fang [17] were the first to introduce generalized m-accretive mapping. They defined the resolvent operator associated with a generalized m-accretive mapping and presented some properties related to it. By introducing a class of generalized monotone operators, the so-called H-monotone operators, and by defining their associated resolvent operators, Fang and Huang [14] studied a class of variational inclusions involving H-monotone operators in the framework of Hilbert spaces. Subsequently, Fang and Huang [15] further introduced a class of generalized accretive operators, called H-accretive operators (also referred to as P-accretive mapping in the literature), which extends the notion of H-monotone operators to the Banach spaces. They defined the resolvent operator associated with an H-accretive operator and gave some its important properties. They also used the defined resolvent operator to construct an iterative algorithm for finding the approximate solution of a class of variational inclusions involving H-accretive operator in a Banach space setting. It is worth mentioning that in most of the resolvent operator methods, the maximal monotonicity has played a key role, but recently introduced notions of H-monotonicity and H-accretivity have not only generalized the maximal monotonicity, but gave a new edge to resolvent operator methods. In 1990, Siddiqi and Ansari [32] introduced and studied another useful and important generalization of variational inequalities the so-called mixed variational inequality. Because of the involvement of nonlinear term in the formulations of mixed variational inequalities, the projection method could not be applied to propose iterative algorithms for solving them. Indeed, the applicability of the projection method is limited due to the fact that it is not easy to find the projection except in very special cases. Making use of the notion of resolvent operator technique, Noor and Noor [27] introduced and studied resolvent equations and proved the equivalence between the mixed variational inequalities and the resolvent equations. Later, Ahmad and Yao [5] considered and studied a system of generalized resolvent equations in the framework of uniformly smooth Banach spaces by verifying its equivalence with a system of variational inclusions. Zou and Huang [38] introduced and studied a class of generalized accretive operators, called $H(\cdot,\cdot)$-accretive operator as an extension of H-accretive operator in a Banach space setting and defined its associated resolvent operator for constructing an iterative algorithm in order to solve a class of variational inclusions. Inspired and motivated by the above research works, Ahmad et al. [3,4] introduced and studied $H(\cdot,\cdot)$-cocoercive and $H(\cdot,\cdot)$-co-accretive mappings, as generalizations of $H(\cdot,\cdot)$-accretive operator, for solving variational inclusion problems.

Recently, Ahmad and Akram [1] employed the notion of $H(\cdot,\cdot)$-co-accretive mapping and established the equivalence between a system of generalized resolvent equations involving generalized pseudocontractive mapping and a system of variational inclusions. They also proved the existence of a solution for the above-mentioned system of resolvent equations and studied the convergence analysis of the sequences generated by their proposed iterative algorithm.

The purpose of this paper is twofold. Our first objective is to study a new system of generalized multi-valued resolvent equations (for short, SGMRE) involving P-accretive mappings in the framework of real Banach spaces. For this end, the Lipschitz continuity of the resolvent operator associated with a P-accretive mapping is proved under some new assumptions imposed on the parameter and mappings involved in it and an estimate of its Lipschitz constant is computed. With the help of the resolvent operator technique, the equivalence between the SGMRE and a system of generalized variational inclusions (for short, SGVI) is established. We apply the obtained equivalence relationship and Nadler's technique to construct a new iterative scheme for finding the approximate solution of the SGMRE. The convergence analysis of the sequences generated by our suggested iterative algorithm under some suitable conditions is studied. The second objective of this paper is to investigate and analyze the concept of $H(\cdot,\cdot)$-co-accretive mapping introduced and studied in [1] and to point out some important comments concerning it. We verify that under the considered conditions by the authors in [1], every $H(\cdot,\cdot)$-co-accretive mapping is actually P-accretive and is not a new one. Some errors and mistakes in the algorithms and main result of [1] are detected and the corresponding correct versions of them are presented. At the same time, we point out that all results given in [1] related to $H(\cdot,\cdot)$-co-accretive mappings can be drawn by using our results regarding P-accretive mappings provided in Sections 2 and 3 of this paper.

2. Preliminary matters and some notion

Before proceeding to the main results of the paper we recall the necessary terminology and notation and few useful results. Throughout the paper, unless otherwise specified, X is a fixed real Banach space with the dual space X^*. The norms of the various Banach spaces that enter our discussion are all denoted by the same symbol $\|\cdot\|$ as there is no occasion for confusion. As usual, the symbol $\langle\cdot,\cdot\rangle$ will represent the duality pairing of X and X^*. The symbol $CB(X)$ (resp., 2^X) is used to represent the set of all nonempty closed and bounded (resp., all nonempty) subsets of X.

For a given multi-valued mapping $\widehat{M}: X \to 2^X$,

(i) the set Range($\widehat{M}$) defined by

$$\text{Range}(\widehat{M}) = \{y \in X : \exists x \in X : (x, y) \in \widehat{M}\} = \bigcup_{x\in X} \widehat{M}(x)$$

is called the range of $\widehat{M}$;

(ii) the set Graph($\widehat{M}$) defined by

$$\text{Graph}(\widehat{M}) = \{(x, u) \in X \times X : u \in \widehat{M}(x)\}$$

is called the graph of $\widehat{M}$.

Let us recall that the normalized duality mapping $\mathcal{F} : X \to 2^{X^*}$ is defined by

$$\mathcal{F}(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \ \forall x \in X.$$

We observe immediately that if $X = \mathcal{H}$, a Hilbert space, then $\mathcal{F}$ is the identity mapping on $\mathcal{H}$. Moreover, it is an immediate consequence of the Hahn-Banach theorem that $\mathcal{F}(x)$ is nonempty for each $x \in X$. In the sequel, j is used to represent a selection of the normalized duality mapping $\mathcal{F}$.

Definition 2.1. Let $T : X \to X$ be a single-valued mapping and $\mathcal{F} : X \to 2^{X^*}$ be the normalized duality mapping. Then T is said to be

(i) accretive if

$$\langle T(x) - T(y), j(x - y)\rangle \geq 0, \ \forall x, \ y \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(ii) strictly accretive if T is accretive and equality holds if and only if $x = y$;

(iii) k-strongly accretive if there exists a constant $k > 0$ such that

$$\langle T(x) - T(y), j(x - y)\rangle \geq k\|x - y\|^2, \ \forall x, y \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(iv) ϱ-relaxed accretive if there exists a constant $\varrho > 0$ such that

$$\langle T(x) - T(y), j(x - y)\rangle \geq -\varrho\|x - y\|^2, \ \forall x, y \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(v) ς-cocoercive if there exists a constant $\varsigma > 0$ such that

$$\langle T(x) - T(y), j(x - y)\rangle \geq \varsigma\|T(x) - T(y)\|^2, \ \forall x, y \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(vi) γ-relaxed cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \geq -\gamma\|T(x) - T(y)\|^2,\ \forall x, y \in X,\ j(x-y)\in\mathcal{F}(x-y);$$

(vii) α-expansive if there exists a constant $\alpha > 0$ such that

$$\|T(x) - T(y)\| \geq \alpha\|x - y\|,\ \forall x, y \in X;$$

(viii) k-contraction if there exists a constant $0 < k < 1$ such that

$$\|T(x) - T(y)\| \leq k\|x - y\|,\ \forall x, y \in X;$$

(ix) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\|,\ \forall x, y \in X.$$

Evidently, every contraction is a Lipschitz continuous mapping with a Lipschitz constant that is smaller than 1.

Definition 2.2. Let $\widehat{M} : X \to 2^X$ be a multi-valued mapping and $\mathcal{F} : X \to 2^{X^*}$ be the normalized duality mapping. Then $\widehat{M}$ is said to be

(i) accretive if

$$\langle u - v, j(x-y)\rangle \geq 0,\ \forall (x,u), (y,v) \in \mathrm{Graph}(\widehat{M}),\ j(x-y) \in \mathcal{F}(x-y);$$

(ii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle u - v, j(x-y)\rangle \geq r\|x-y\|^2,\ \forall (x,u), (y,v) \in \mathrm{Graph}(\widehat{M}),\\ j(x-y) \in \mathcal{F}(x-y);$$

(iii) ξ-relaxed accretive if there exists a constant $\xi > 0$ such that

$$\langle u - v, j(x-y)\rangle \geq -\xi\|x-y\|^2,\ \forall (x,u), (y,v) \in \mathrm{Graph}(\widehat{M}),\\ j(x-y) \in \mathcal{F}(x-y);$$

(iv) m-accretive if $\widehat{M}$ is accretive and $(I + \lambda\widehat{M})(X) = X$ holds for every real constant $\lambda > 0$, where I stands for the identity mapping.

We note that $\widehat{M}$ is an m-accretive mapping if and only if $\widehat{M}$ is accretive and there is no other accretive mapping whose graph contains strictly $\mathrm{Graph}(\widehat{M})$. The m-accretivity is to be understood in terms of inclusion of graphs. If $\widehat{M} : X \to 2^X$ is an m-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the accretivity. If fact, the extended mapping is no longer accretive. In other words, for every pair $(x,u) \in X\times X\setminus \mathrm{Graph}(\widehat{M})$ there exists $(y,v) \in \mathrm{Graph}(\widehat{M})$ and $j(x-y) \in \mathcal{F}(x-y)$ such that $\langle u - v, j(x-y)\rangle < 0$. Thanks to the argument mentioned above, a necessary and sufficient condition for a multi-valued mapping $\widehat{M} : X \to 2^X$ to be m-accretive is that for any $(x,u) \in X \times X$, the property

$$\langle u - v, j(x-y)\rangle \geq 0, \quad \forall (y,v) \in \mathrm{Graph}(\widehat{M}),\ j(x-y) \in \mathcal{F}(x-y)$$

is equivalent to $(x,u)\in \mathrm{Graph}(\widehat{M})$. The above characterization of m-accretive mappings provides a useful and manageable way for recognizing that an element u belongs to $\widehat{M}(x)$.

Definition 2.3. For a given mapping $P : X \to X$, the multi-valued mapping $\widehat{M} : X \to 2^X$ is said to be

(i) P-accretive if $\widehat{M}$ is accretive and $(P+\lambda\widehat{M})(X) = X$ holds for every real constant $\lambda > 0$;

(ii) P-maximal m-relaxed accretive if $\widehat{M}$ is m-relaxed accretive and $(P+\lambda\widehat{M})(X) = X$ holds for every real constant $\lambda > 0$.

The following example illustrates that an m-accretive mapping may not be P-accretive for some single-valued mapping $P : X \to X$.

Example 2.1. Let $\phi : \mathbb{N} \to (0,+\infty)$ and consider the complex linear space l^2_ϕ, the weighted l^2 space, the space of all complex sequences $\{z_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}|z_n|^2\phi(n) < \infty$. It is a well known truth that

$$l^2_\phi = \{z = \{z_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty}|z_n|^2\phi(n) < \infty,\ z_n \in \mathbb{C}\}$$

with respect to the inner product $\langle\cdot,\cdot\rangle : l^2_\phi \times l^2_\phi \to \mathbb{C}$ defined by

$$\langle z,w\rangle = \sum_{n=1}^{\infty} z_n\bar{w}_n\phi(n), \quad \forall z = \{z_n\}_{n=1}^{\infty},\ w = \{w_n\}_{n=1}^{\infty} \in l^2_\phi$$

is a Hilbert space. The inner product defined above induces a norm on l^2_ϕ as follows:

$$\|z\|_{l^2_\phi} = \sqrt{\langle z,z\rangle} = (\sum_{n=1}^{\infty}|z_n|^2\phi(n))^{\frac{1}{2}}, \quad \forall z = \{z_n\}_{n=1}^{\infty} \in l^2_\phi.$$

Any element $z = \{z_n\}_{n=1}^{\infty} = \{x_n + iy_n\}_{n=1}^{\infty} \in l^2_\phi$ can be written as

$$\begin{aligned}
z &= \sum_{k=1}^{\infty}(0,0,\dots,0,x_{2k-1}+iy_{2k-1},x_{2k}+iy_{2k},0,0,\dots)\\
&= \sum_{k=1}^{\infty}\Big[\frac{y_{2k-1}+y_{2k}-i(x_{2k-1}+x_{2k})}{2}(0,0,\dots,0,i_{2k-1},i_{2k},0,0,\dots)\\
&\quad + \frac{y_{2k-1}-y_{2k}-i(x_{2k-1}-x_{2k})}{2}(0,0,\dots,0,i_{2k-1},-i_{2k},0,0,\dots)\Big]\\
&= \sum_{k=1}^{\infty}\Big[\frac{y_{2k-1}+y_{2k}-i(x_{2k-1}+x_{2k})}{2}\omega_{2k-1,2k}\\
&\quad + \frac{y_{2k-1}-y_{2k}-i(x_{2k-1}-x_{2k})}{2}\omega'_{2k-1,2k}\Big],
\end{aligned}$$

where for each $k \in \mathbb{N}$, $\omega_{2k-1,2k} = (0,0,\dots,0,i_{2k-1},i_{2k},0,0,\dots)$, i at the $(2k-1)$th and $(2k)$th coordinates and all other coordinates are zero, and $\omega'_{2k-1,2k} = (0,0,\dots,0,i_{2k-1},-i_{2k},0,0,\dots)$, i and $-i$ at the $(2k-1)$th and $(2k)$th places, respectively, and 0's everywhere else. Therefore, the set $\mathfrak{B} = \{\omega_{2k-1,2k},\omega'_{2k-1,2k} : k \in \mathbb{N}\}$ spans the Banach space l^2_ϕ. It can be easily observed that the set $\mathfrak{B}$ is linearly independent and so it is a basis for l^2_ϕ. Taking

$$\sigma_{2k-1,2k} = (0,0,\dots,0,\frac{1}{\sqrt{2\phi(2k-1)}}i_{2k-1},\frac{1}{\sqrt{2\phi(2k)}}i_{2k},0,0,\dots)$$

and

$$\sigma'_{2k-1,2k} = (0,0,\dots,0,\frac{1}{\sqrt{2\phi(2k-1)}}i_{2k-1},-\frac{1}{\sqrt{2\phi(2k)}}i_{2k},0,0,\dots)$$

for each $k \in \mathbb{N}$, it goes without saying that $\{\sigma_{2k-1,2k},\sigma'_{2k-1,2k} : k \in \mathbb{N}\}$ is also linearly independent such that $\|\sigma_{2k-1,2k}\|_{l^2_\phi} = \|\sigma'_{2k-1,2k}\|_{l^2_\phi} = 1$. Let the mappings $P,\widehat{M} : l^2_\phi \to l^2_\phi$ be defined, respectively, by $P(z) = -\alpha z + \gamma\sigma'_{2\delta-1,2\delta}$ and $\widehat{M}(z) = \alpha z + \beta\sigma_{2\delta-1,2\delta}$ for all $z \in l^2_\phi$, where δ is an arbitrary but fixed natural number, α is an arbitrary positive real constant, and β and γ are two arbitrary nonzero real constants. Then, for all $z,w \in l^2_\phi$ and $j(z-w) \in \mathcal{F}(z-w)$, we yield

$$\begin{aligned}\langle \widehat{M}(z)-\widehat{M}(w), j(z-w)\rangle &= \langle \widehat{M}(z)-\widehat{M}(w), z-w\rangle \\ &= \langle \alpha z + \beta\sigma_{2\delta-1,2\delta} - \alpha w - \beta\sigma_{2\delta-1,2\delta}, z-w\rangle \\ &= \alpha\langle z-w,z-w\rangle = \alpha\|z-w\|^2 \geq 0,\end{aligned}$$

which means that $\widehat{M}$ is an accretive mapping. In the light of the fact that for any $z \in l^2_\phi$ and $\lambda > 0$, $(I+\lambda\widehat{M})(z) = (1+\lambda\alpha)z + \lambda\beta\sigma_{2\delta-1,2\delta}$, where I is the identity mapping on l^2_ϕ, it follows that $(I+\lambda\widehat{M})(l^2_\phi) = l^2_\phi$ for every real constant $\lambda > 0$, that is, the mapping $I+\lambda\widehat{M}$ is surjective for every positive real constant λ. Accordingly, $\widehat{M}$ is an m-accretive mapping.

Thanks to the fact that for any $z \in l^2_\phi$,

$$\begin{aligned}(P+\widehat{M})(z) &= \beta\sigma_{2\delta-1,2\delta} + \gamma\sigma'_{2\delta-1,2\delta} \\ &= (0,0,\dots,0,\frac{\beta}{\sqrt{2\phi(2\delta-1)}}i_{2\delta-1},\frac{\beta}{\sqrt{2\phi(2\delta)}}i_{2\delta},0,0,\dots) \\ &\quad + (0,0,\dots,0,\frac{\gamma}{\sqrt{2\phi(2\delta-1)}}i_{2\delta-1},-\frac{\gamma}{\sqrt{2\phi(2\delta)}}i_{2\delta},0,0,\dots) \\ &= (0,0,\dots,0,\frac{\beta+\gamma}{\sqrt{2\phi(2\delta-1)}}i_{2\delta-1},\frac{\beta-\gamma}{\sqrt{2\phi(2\delta)}}i_{2\delta},0,0,\dots),\end{aligned}$$

we conclude that for any $z \in l^2_\phi$,

$$\|(P+\widehat{M})(z)\|_{l^2_\phi} = \|\beta\sigma_{2\delta-1,2\delta} + \gamma\sigma'_{2\delta-1,2\delta}\|_{l^2_\phi} = (\beta+\gamma)^2 + (\beta-\gamma)^2$$

$$= 2(\beta^2 + \gamma^2) > 0.$$

This fact guarantees that $\mathbf{0} \notin (P + \widehat{M})(l_\phi^2)$, where $\mathbf{0}$ is the zero vector of the space l_ϕ^2. Consequently, $P + \widehat{M}$ is not surjective and so the mapping $\widehat{M}$ is not P-accretive.

Denoting the set of all functions $\phi : \mathbb{N} \to (0,1]$ by Φ and $l_\Phi^2 = \{l_\phi^2 : \phi \in \Phi\}$, it is easy to see that $l^2 \subseteq l_\phi^2$ for each $\phi \in \Phi$ so that for some $\phi_0 \in \Phi$, we have $l^2 \subset l_{\phi_0}^2$, that is, l^2 is strictly contained within $l_{\phi_0}^2$. We recall that $l^2 = \{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^2 < \infty,\ x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$ denotes the real or complex linear space consisting of all square-summable sequences $x = \{x_n\}_{n=1}^\infty$ for which $\|x\|_{l^2} < \infty$. Evidently, if $\phi(n) = 1$ for all $n \in \mathbb{N}$, then the weight space l_ϕ^2 coincides exactly with the linear space l^2. It is significant to mention that the two Hilbert spaces l^2 and l_ϕ^2 need not be the same for all $\phi \in \Phi$. In order to show this fact, we consider the two cases when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\mathbb{F} = \mathbb{R}$, letting $x_n = \ln n$ for all $n \in \mathbb{N}$, we have $\sum_{n=1}^\infty |x_n|^2 = \sum_{n=1}^\infty (\ln n)^2 = \infty$, i.e., $x = \{x_n\}_{n=1}^\infty \notin l^2$. Defining $\phi_1 : \mathbb{N} \to (0, +\infty)$ by $\phi_1(n) = \frac{1}{n^4}$ for all $n \in \mathbb{N}$, we have $\phi_1 \in \Phi$ and $\sum_{n=1}^\infty |x_n|^2 \phi_1(n) = \sum_{n=1}^\infty \frac{(\ln n)^2}{n^4}$. Since $\sum_{n=1}^\infty \frac{(\ln n)^2}{n^4}$ is convergent, it follows that $x \in l_{\phi_1}^2$. In the case where $\mathbb{F} = \mathbb{C}$, letting $z_n = \frac{n!}{\sqrt{2}} + i\frac{n!}{\sqrt{2}}$ for all $n \in \mathbb{N}$, we have $\sum_{n=1}^\infty |z_n|^2 = \sum_{n=1}^\infty (n!)^2 = \infty$ which ensures that $z = \{z_n\}_{n=1}^\infty \notin l^2$. Now, let $\phi_2 : \mathbb{N} \to (0, +\infty)$ be defined by $\phi_2(n) = e^{-n^2}$ for all $n \in \mathbb{N}$. Then we have $\phi_2 \in \Phi$ and $\sum_{n=1}^\infty |z_n|^2 \phi_2(n) = \sum_{n=1}^\infty \frac{(n!)^2}{e^{n^2}}$. Since $\sum_{n=1}^\infty \frac{(n!)^2}{e^{n^2}}$ is convergent, we conclude that $z = \{z_n\}_{n=1}^\infty \in l_{\phi_2}^2$. Therefore, for some $\phi \in \Phi$, l_ϕ^2 is a proper superset of the Hilbert space l^2.

Example 2.2. Let $m, n \in \mathbb{N}$ be arbitrary but fixed and let $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = (a_{ij}) \mid a_{ij} \in \mathbb{F}, i = 1, 2, \dots, m; j = 1, 2, \dots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^* B) = \sum_{i=1}^m \sum_{j=1}^n \bar{a}_{ij} b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A, and the bar denotes complex conjugation and superscript denotes the transpose of the entries. Let us denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the (i,j)-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then

$$D_n(\mathbb{R}) = \{A = (a_{ij}) \mid a_{ij} \in \mathbb{R},\ a_{ij} = 0 \text{ if } i \neq j;\ i, j = 1, 2, \ldots, n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$, and the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$, and the Hilbert-Schmidt norm induced by it become as

$$\langle A, B\rangle = tr(A^*B) = tr(AB)$$

and

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(AA)} = \Big(\sum_{i=1}^{n} a_{ii}^2\Big)^{\frac{1}{2}},$$

respectively. Let the mappings $P_1, P_2, \widehat{M} : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined by $P_1(A) = P_1((a_{ij})) = \left(a'_{ij}\right), P_2(A) = P_2((a_{ij})) = \left(a''_{ij}\right)$ and $\widehat{M}(A) = \widehat{M}((a_{ij})) = \left(a'''_{ij}\right)$ for all $A = (a_{ij}) \in D_n(\mathbb{R})$, respectively, where for each $i, j \in \{1, 2, \ldots, n\}$,

$$a'_{ij} = \begin{cases} |a_{ii} - \beta| + |a_{ii} - \gamma| - \varrho a_{ii}^k - \mu\sqrt[l]{a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a''_{ij} = \begin{cases} \frac{\varsigma^{a_{ii}} - 1}{\varsigma^{a_{ii}} + 1}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$a'''_{ij} = \begin{cases} \varrho a_{ii}^k + \mu\sqrt[l]{a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

where ϱ, μ and ς are arbitrary positive real constants, β and γ are arbitrary real constants such that $\beta \geq \gamma$, and k, l are arbitrary but fixed odd natural numbers. Then, for any $A = (a_{ij}), B = (b_{ij}) \in D_n(\mathbb{R})$ and $j(A - B) \in \mathcal{F}(A - B)$, we yield

$$\begin{aligned}
\langle \widehat{M}(A) - \widehat{M}(B), j(A - B)\rangle &= \langle \widehat{M}(A) - \widehat{M}(B), A - B\rangle \\
&= tr\Big(\left(a'''_{ij} - b'''_{ij}\right)(a_{ij} - b_{ij})\Big) \\
&= \sum_{i=1}^{n} \big(\varrho(a_{ii}^k - b_{ii}^k) + \mu(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})\big)(a_{ii} - b_{ii}) \\
&= \varrho \sum_{i=1}^{n} (a_{ii} - b_{ii})^2 \sum_{t=1}^{k} a_{ii}^{k-t} b_{ii}^{t-1} \\
&\quad + \mu \sum_{i=1}^{n} (\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}).
\end{aligned}$$

For any $i \in \{1, 2, \ldots, n\}$,

(i) if $a_{ii} = b_{ii} = 0$, then $(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}) = 0$;

(ii) if $a_{ii} \neq 0$ and $b_{ii} = 0$, then $(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}) = a_{ii}\sqrt[l]{a_{ii}} = \sqrt[l]{a_{ii}^{l+1}}$;

(iii) if $a_{ii} = 0$ and $b_{ii} \neq 0$, then $(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}) = b_{ii}\sqrt[l]{b_{ii}} = \sqrt[l]{b_{ii}^{l+1}}$;

(iv) if $a_{ii}, b_{ii} \neq 0$, then $\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}} = \frac{a_{ii}-b_{ii}}{\sum_{r=1}^{l}\sqrt[l]{a_{ii}^{l-r}b_{ii}^{r-1}}}$.

Since l is an odd natural number, we conclude that $\sqrt[l]{a_{ii}^{l+1}}, \sqrt[l]{b_{ii}^{l+1}} > 0$ and $\sum_{r=1}^{l}\sqrt[l]{a_{ii}^{l-r}b_{ii}^{r-1}} > 0$. In view of these facts, we infer that $(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}) > 0$ and

$$\sum_{i=1}^{n}(\sqrt[l]{a_{ii}} - \sqrt[l]{b_{ii}})(a_{ii} - b_{ii}) = \sum_{i=1}^{n}\frac{(a_{ii}-b_{ii})^2}{\sum_{r=1}^{l}\sqrt[l]{a_{ii}^{l-r}b_{ii}^{r-1}}} > 0.$$

In the meanwhile, due to the fact that k is an odd natural number, it can be easily seen that for each $i \in \{1, 2, \ldots, n\}$, $\sum_{t=1}^{k} a_{ii}^{k-t}b_{ii}^{t-1} \geq 0$. In the light of the arguments mentioned above and the fact that ϱ and μ are positive, we deduce that for any $A = (a_{ij}), B = (b_{ij}) \in D_n(\mathbb{R})$ and $j(A-B) \in \mathcal{F}(A-B)$,

$$\langle \widehat{M}(A) - \widehat{M}(B), j(A-B)\rangle \geq 0,$$

that is, $\widehat{M}$ is an accretive mapping.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) := |x-\beta| + |x-\gamma|$ for all $x \in \mathbb{R}$. Then, for any $A = (a_{ij}) \in D_n(\mathbb{R})$, we have

$$(P_1 + \widehat{M})(A) = (P_1 + \widehat{M})((a_{ij})) = \left(a'_{ij} + a'''_{ij}\right) = (\hat{a}_{ij}),$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\hat{a}_{ij} = \begin{cases} |a_{ii}-\beta| + |a_{ii}-\gamma|, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} f(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

The fact that $f(\mathbb{R}) = [\beta - \gamma, +\infty)$ implies that $(P_1 + \widehat{M})(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, i.e., $P_1 + \widehat{M}$ is not surjective, and so $\widehat{M}$ is not a P_1-η-accretive mapping. Now, assume that the real constant $\lambda > 0$ is chosen arbitrarily but fixed and let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \frac{\varsigma^x - 1}{\varsigma^x + 1} + \lambda\varrho x^k + \lambda\mu\sqrt[l]{x}$ for all $x \in \mathbb{R}$. Then, for any $A = (a_{ij}) \in D_n(\mathbb{R})$, yields

$$(P_2 + \lambda\widehat{M})(A) = (P_2 + \lambda\widehat{M})((a_{ij})) = \left(a''_{ij} + \lambda a'''_{ij}\right) = (\widetilde{a}_{ij}),$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} \frac{\varsigma^{a_{ii}}-1}{\varsigma^{a_{ii}}+1} + \lambda\varrho a_{ii}^k + \lambda\mu\sqrt[l]{a_{ii}}, & i = j, \\ 0, & i \neq j \end{cases} = \begin{cases} g(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Relying on the fact that k and l are odd natural numbers, it is easy to observe that $g(\mathbb{R}) = \mathbb{R}$, which guarantees that $(P_2 + \lambda\widehat{M})(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $P_2 + \lambda\widehat{M}$ is a surjective mapping. Taking into account the arbitrariness in the choice of $\lambda > 0$, we conclude that $\widehat{M}$ is a P_2-accretive mapping.

It should be pointed out that if $P = I$, then the definition of I-accretive mappings is that of m-accretive mappings. In fact, the class of P-accretive mappings has close relation with that of m-accretive mappings.

Proposition 2.1. *Let $P : X \to X$ be a strictly accretive mapping and let $\widehat{M} : X \to 2^X$ be a P-accretive mapping. Then $\widehat{M}$ is m-accretive.*

Proof. Taking into account of the fact that $\widehat{M}$ is accretive, in order to show that $\widehat{M}$ is m-accretive, it is sufficient to prove that for any given points $x, u \in X$, the inequality $\langle u - v, j(x - y)\rangle \geq 0$ holds for all $(y, v) \in \text{Graph}(\widehat{M})$ and $j(x - y) \in \mathcal{F}(x - y)$ implies that $(x, u) \in \text{Graph}(\widehat{M})$. For this end, by contradiction, let us suppose that there exists some $(x_0, u_0) \notin \text{Graph}(\widehat{M})$ such that

$$\langle u_0 - v, j(x_0 - y)\rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(\widehat{M}),\ j(x_0 - y) \in \mathcal{F}(x_0 - y). \tag{2.1}$$

In the light of the fact that $\widehat{M}$ is P-accretive we have $(P + \lambda\widehat{M})(X) = X$ for every constant $\lambda > 0$. Consequently, there exists $(x_1, u_1) \in \text{Graph}(\widehat{M})$ such that

$$P(x_1) + \lambda u_1 = P(x_0) + \lambda u_0. \tag{2.2}$$

Since $(x_1, u_1) \in \text{Graph}(\widehat{M})$, replacing (y, v) by (x_1, u_1) in (2.1) and with the help of (2.2),

$$-\langle P(x_0) - P(x_1), j(x_0 - x_1)\rangle = \lambda\langle u_0 - u_1, x_0 - x_1\rangle \geq 0, \quad \forall j(x_0 - x_1) \in \mathcal{F}(x_0 - x_1),$$

which implies that

$$\langle P(x_0) - P(x_1), j(x_0 - x_1)\rangle \leq 0,\ \forall j(x_0 - x_1) \in \mathcal{F}(x_0 - x_1). \tag{2.3}$$

On the other hand, from accretivity of P it follows that

$$\langle P(x_0) - P(x_1), j(x_0 - x_1)\rangle \geq 0,\ \forall j(x_0 - x_1) \in \mathcal{F}(x_0 - x_1). \tag{2.4}$$

By (2.3) and (2.4) and the strict accretivity of P we conclude that $x_0 = x_1$. Then (2.2) implies that $u_0 = u_1$ and so $(x_0, u_0) \in \text{Graph}(\widehat{M})$, which contradicts with our assumption. Accordingly for any given points $x, u \in X$, if the inequality $\langle u - v, j(x - y)\rangle \geq 0$ holds for all $(y, v) \in \text{Graph}(\widehat{M})$ and $j(x - y) \in \mathcal{F}(x - y)$, then we have $(x, u) \in \text{Graph}(\widehat{M})$. This property guarantees that $\widehat{M}$ is m-accretive. This completes the proof. □

Theorem 2.1. *Let $P : X \to X$ be an accretive mapping and $\widehat{M} : X \to 2^X$ be a ϱ-strongly accretive mapping. Then, the mapping $(P + \lambda\widehat{M})^{-1} : \text{Range}(P + \lambda\widehat{M}) \to X$ is single-valued for every constant $\lambda > 0$.*

Proof. Choose real constant $\lambda > 0$ arbitrarily. For any given point $u \in \text{Range}(P + \lambda\widehat{M})$, let $x, y \in (P + \lambda\widehat{M})^{-1}(u)$. Then, we have

$$u = (P + \lambda\widehat{M})(x) = (P + \lambda\widehat{M})(y),$$

form which we yield $\lambda^{-1}(u-P(x)) \in \widehat{M}(x)$ and $\lambda^{-1}(u-P(y)) \in \widehat{M}(y)$. Considering the fact that P is accretive and $\widehat{M}$ is ϱ-strongly accretive, it follows that

$$\begin{aligned}\lambda\varrho\|x-y\|^2 \leq\ & \lambda\langle\lambda^{-1}(u-P(x))-\lambda^{-1}(u-P(y)), j(x-y)\rangle \\ & + \langle P(x)-P(y), j(x-y)\rangle = 0\end{aligned}$$

for all $j(x-y) \in \mathcal{F}(x-y)$. Since $\lambda, \varrho > 0$, from the last inequality we deduce that $x=y$, which implies that the mapping $(P+\lambda\widehat{M})^{-1}$ from Range$(P+\lambda\widehat{M})$ into X is single-valued. This gives us the desired result. □

We should remark that in the rest of the paper, whenever we say that $\widehat{M}$ is a ϱ-strongly P-accretive mapping, our mean is that $\widehat{M}$ is a ϱ-strongly accretive mapping and $(P+\lambda\widehat{M})(X) = X$ holds for every constant $\lambda > 0$.

We obtain the following assertion as an immediate consequence of the last result.

Corollary 2.1. *Suppose that $P : X \to X$ is an accretive mapping and $\widehat{M} : X \to 2^X$ is a ϱ-strongly P-accretive mapping. Then, the mapping $(P+\lambda\widehat{M})^{-1} : X \to X$ is single-valued for every constant $\lambda > 0$.*

Clearly, Corollary 2.1 enables us to define the notion of resolvent operator $R^P_{\lambda,\widehat{M}}$ associated with $P, \widehat{M}$ and an arbitrary positive real constant λ as follows.

Definition 2.4. Assume that $P : X \to X$ is an accretive mapping and $\widehat{M} : X \to 2^X$ is a γ-strongly P-accretive mapping. For every real constant $\lambda > 0$, the resolvent operator $R^P_{\lambda,\widehat{M}}$ is defined by

$$R^P_{\lambda,\widehat{M}}(x) = (P+\lambda\widehat{M})^{-1}(x), \ \forall x \in X.$$

We now conclude this section with the following theorem in which the appropriate conditions for the resolvent operator $R^P_{\lambda,\widehat{M}}$ to be Lipschitz continuous are stated and an estimate of its Lipschitz constant is also given.

Theorem 2.2. *Let $P : X \to X$ be a ξ-strongly accretive mapping and $\widehat{M} : X \to 2^X$ be a ϱ-strongly P-accretive mapping. Then, for any real constant $\lambda > 0$, the resolvent operator $R^P_{\lambda,\widehat{M}} : X \to X$ is $\frac{1}{\lambda\varrho+\xi}$-Lipschitz continuous, i.e.,*

$$\|R^P_{\lambda,\widehat{M}}(x) - R^P_{\lambda,\widehat{M}}(y)\| \leq \frac{1}{\lambda\varrho+\xi}\|x-y\|, \ \forall x, y \in X.$$

Proof. Taking into consideration the fact that the mapping $\widehat{M}$ is P-accretive, for any given points $x, y \in X$ with $\|R^P_{\lambda,\widehat{M}}(x) - R^P_{\lambda,\widehat{M}}(y)\| \neq 0$, we have

$$R^P_{\lambda,\widehat{M}}(x) = (P+\lambda\widehat{M})^{-1}(x) \text{ and } R^P_{\lambda,\widehat{M}}(y) = (P+\lambda\widehat{M})^{-1}(y),$$

which imply that

$$\lambda^{-1}(x-P(R^P_{\lambda,\widehat{M}}(x))\in\widehat{M}(R^P_{\lambda,\widehat{M}}(x))) \text{ and } \lambda^{-1}(y-P(R^P_{\lambda,\widehat{M}}(y))\in\widehat{M}(R^P_{\lambda,\widehat{M}}(y))).$$

Since $\widehat{M}$ is ϱ-strongly accretive, it follows that

$$\begin{aligned}&\lambda^{-1}\langle x-P(R^P_{\lambda,\widehat{M}}(x))-(y-P(R^P_{\lambda,\widehat{M}}(y))),j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\rangle\\ &\geq \varrho\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2\end{aligned}$$

for all $j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\in\mathcal{F}(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))$. Owing to the fact that $\lambda^{-1}>0$, using the preceding inequality we yield

$$\begin{aligned}&\langle x-y,j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\rangle\\ &\geq \lambda\varrho\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2\\ &\quad+\langle P(R^P_{\lambda,\widehat{M}}(x))-P(R^P_{\lambda,\widehat{M}}(y)),j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\rangle\end{aligned}\tag{2.5}$$

for all $j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\in\mathcal{F}(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))$.

From ξ-strong accretivity of P and by means of (2.5), it follows that for all $j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\in\mathcal{F}(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))$,

$$\begin{aligned}&\|x-y\|\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|\\ &=\|x-y\|\|j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\|\\ &\geq\langle x-y,j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\rangle\\ &\geq \lambda\varrho\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2\\ &\quad+\langle P(R^P_{\lambda,\widehat{M}}(x))-P(R^P_{\lambda,\widehat{M}}(y)),j(R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y))\rangle\\ &\geq \lambda\varrho\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2+\xi\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2\\ &=(\lambda\varrho+\xi)\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|^2.\end{aligned}$$

In virtue of the fact that $\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|\neq 0$, we conclude that

$$\|R^P_{\lambda,\widehat{M}}(x)-R^P_{\lambda,\widehat{M}}(y)\|\leq\frac{1}{\lambda\varrho+\xi}\|x-y\|.$$

This completes the proof. □

3. Formulations, iterative algorithms and convergence results

Let for each $i\in\{1,2\}$, X_i be a real Banach space and $P_i:X_i\to X_i$ be an accretive mapping. Suppose that $S:X_1\times X_2\to X_1$ and $T:X_1\times X_2\to X_2$ are any nonlinear mappings, and $E:X_1\to 2^{X_1}$ and $F:X_2\to 2^{X_2}$ are multivalued mappings. Furthermore, let for each $i\in\{1,2\}$, the multi-valued mapping $\widehat{M_i}:X_i\to 2^{X_i}$ be a ϱ_i-strongly P_i-accretive mapping. For given two arbitrary real constants $\lambda_1, \lambda_2 > 0$, we consider the problem of finding $(x, y) \in X_1 \times X_2$, $u \in E(x)$, $v \in F(y)$, $z' \in X_1$ and $z'' \in X_2$ such that

$$\left\{ \begin{array}{l} S(x,v) + \lambda_1^{-1} J^{P_1}_{\lambda_1,\widehat{M}_1}(z') = 0, \\ T(u,y) + \lambda_2^{-1} J^{P_2}_{\lambda_2,\widehat{M}_2}(z'') = 0, \end{array} \right. \tag{3.1}$$

where for each $i \in \{1, 2\}$, $J^{P_i}_{\lambda_i,\widehat{M}_i} = I_i - P_i \circ R^{P_i}_{\lambda_i,\widehat{M}_i} = I_i - P_i(R^{P_i}_{\lambda_i,\widehat{M}_i}(\cdot))$, I_i is the identity mapping on X_i, $R^{P_i}_{\lambda_i,\widehat{M}_i}$ is the resolvent operator associated with P_i, positive real constant λ_i and strongly P_i-accretive mapping $\widehat{M_i}$, and $P_i \circ R^{P_i}_{\lambda_i,\widehat{M}_i}$ denotes P_i composition $R^{P_i}_{\lambda_i,\widehat{M}_i}$. The problem (3.1) is called a *system of generalized multi-valued resolvent equations* (SGMRE).

Let X_i, P_i, $\widehat{M_i}$ $(i = 1, 2)$, S, T, E and F be the same as in the SGMRE (3.1). Corresponding to the SGMRE (3.1), we now consider the following system of generalized variational inclusions (SGVI): find $(x, y) \in X_1 \times X_2$, $u \in E(x)$ and $v \in F(y)$ such that

$$\left\{ \begin{array}{l} 0 \in S(x,v) + \widehat{M}_1(x), \\ 0 \in T(u,y) + \widehat{M}_2(y). \end{array} \right. \tag{3.2}$$

We remark that for appropriate choices of the mappings P_i, $\widehat{M_i}$ $(i = 1, 2)$, S, T, E and F, and the underlying spaces X_i $(i = 1, 2)$, the SGVI (3.2) includes various systems of variational inequalities/inclusions and many classes of variational inequality/inclusion problems, see, for example, [18, 20, 34, 35, 37] and the references therein.

The following conclusion, which has a prominent role in getting the main results of this paper, follows directly from Definition 2.1 and some simple arguments.

Lemma 3.1. *Let X_i, P_i, $\widehat{M_i}$ $(i = 1, 2)$, S, T, E and F be the same as in the* SGMRE (3.1). *Then $(x, y, u, v) \in X_1 \times X_2 \times E(x) \times F(y)$ is a solution of the* SGVI (3.2) *if and only if (x, y, u, v) satisfies the relations*

$$\left\{ \begin{array}{l} x = R^{P_1}_{\lambda_1,\widehat{M}_1}[P_1(x) - \lambda_1 S(x,v)], \\ y = R^{P_2}_{\lambda_2,\widehat{M}_2}[P_2(y) - \lambda_2 T(u,y)], \end{array} \right.$$

where $\lambda_i > 0$ and $R^{P_i}_{\lambda_i,\widehat{M}_i}$ $(i = 1, 2)$ are the same as in the SGMRE (3.1).

The next assertion, which tells the SGMRE (3.1) and the SGVI (3.2) are equivalent, plays a key role in proposing algorithms and in the study of our suggested iterative algorithms.

Proposition 3.1. *Assume that X_i, P_i, $\widehat{M_i}$ $(i = 1, 2)$, S, T, E and F are the same as in the* SGMRE (3.1). *Then (x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in E(x)$ and $v \in F(y)$ is a solution of the* SGVI (3.2) *if and only if (x, y, u, v, z', z''),* *where* $(z', z'') \in X_1 \times X_2$*, is a solution of the* SGMRE (3.1) *satisfying*

$$\begin{cases} x = R^{P_1}_{\lambda_1, \widehat{M}_1}(z'), \\ y = R^{P_2}_{\lambda_2, \widehat{M}_2}(z''), \\ z' = P_1(x) - \lambda_1 S(x, v), \\ z'' = P_2(y) - \lambda_2 T(u, y), \end{cases} \tag{3.3}$$

where $\lambda_i > 0$ *and* $R^{P_i}_{\lambda_i, \widehat{M}_i}$ $(i = 1, 2)$ *are the same as in the* SGMRE (3.1)*.*

Proof. In the light of Lemma 3.1, $(x, y, u, v) \in X_1 \times X_2 \times E(x) \times F(y)$ is a solution of the SGVI (3.2) if and only if

$$\begin{cases} x = R^{P_1}_{\lambda_1, \widehat{M}_1}[P_1(x) - \lambda_1 S(x, v)], \\ y = R^{P_2}_{\lambda_2, \widehat{M}_2}[P_2(y) - \lambda_2 T(u, y)] \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x = R^{P_1}_{\lambda_1, \widehat{M}_1}(z'), \\ y = R^{P_2}_{\lambda_2, \widehat{M}_2}(z''), \\ z' = P_1(x) - \lambda_1 S(x, v), \\ z'' = P_2(y) - \lambda_2 T(u, y) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} z' = P_1(R^{P_1}_{\lambda_1, \widehat{M}_1}(z')) - \lambda_1 S(x, v), \\ z'' = P_2(R^{P_2}_{\lambda_2, \widehat{M}_2}(z'')) - \lambda_2 T(u, y) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \big(I_1 - P_1 \circ R^{P_1}_{\lambda_1, \widehat{M}_1}\big)(z') = -\lambda_1 S(x, v), \\ \big(I_2 - P_2 \circ R^{P_2}_{\lambda_2, \widehat{M}_2}\big)(z'') = -\lambda_2 T(u, y) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} S(x, v) + \lambda_1^{-1} J^{P_1}_{\lambda_1, \widehat{M}_1}(z') = 0, \\ T(u, y) + \lambda_2^{-1} J^{P_2}_{\lambda_2, \widehat{M}_2}(z'') = 0 \end{cases}$$

where for $i = 1, 2$, $J^{P_i}_{\lambda_i, \widehat{M}_i} = I_i - P_i \circ R^{P_i}_{\lambda_i, \widehat{M}_i}$. Consequently, $(x, y, u, v, z', z'') \in X_1 \times X_2 \times E(x) \times F(y) \times X_1 \times X_2$ is a solution of the SGMRE (3.1). Accordingly, the solution sets of the two systems (3.1) and (3.2) are the same. The proof is finished. $\square$

According to the remark which followed the proof of Theorem 5 in [25], for any $A, B \in CB(X)$, $a \in A$ and $\eta > 0$, there exists a $b \in B$ such that $d(a, b) \leq D(A, B) + \eta$ ([25, page 480, lines 12–14]). As a direct consequence of this fact, for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, taking $\eta = \varepsilon D(T(x), T(y))$, we have the following assertion which plays a key role in the sequel.

Lemma 3.2. *Let X be a complete metric space, and $T : X \to CB(X)$ be a multi-valued mapping. Then for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\|u - v\| \leq (1 + \varepsilon) D(T(x), T(y)),$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \quad \forall A, B \in CB(X).$$

Based on Proposition 3.1 and with the help of Nadler's technique [25], we are able to construct an iterative algorithm for approximating the solution of the SGMRE (3.1) as follows.

Algorithm 3.1. *Let X_i, P_i, $\widehat{M_i}$ $(i = 1, 2)$, S, T, E and F be the same as in the* SGMRE (3.1). *For any given $(x_0, y_0), (z'_0, z''_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$ and $v_0 \in F(y_0)$, define the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ in the following way:*

$$\begin{cases} x_n = R^{P_1}_{\lambda_1, \widehat{M}_1}(z'_n), \\ y_n = R^{P_2}_{\lambda_2, \widehat{M}_2}(z''_n), \\ u_n \in E(x_n); \|u_{n+1} - u_n\|_1 \leq (1 + \frac{1}{1+n}) D_1(E(x_{n+1}), E(x_n)), \\ v_n \in F(y_n); \|v_{n+1} - v_n\|_2 \leq (1 + \frac{1}{1+n}) D_2(F(y_{n+1}), F(y_n)), \\ z'_{n+1} = P_1(x_n) - \lambda_1 S(x_n, v_n), \\ z''_{n+1} = P_2(y_n) - \lambda_2 T(u_n, y_n), \end{cases} \tag{3.4}$$

where $n = 0, 1, 2, \ldots$; $\lambda_i > 0$ $(i = 1, 2)$ are positive real constants, and for $i = 1, 2$, D_i is the Hausdorff metric on $CB(X_i)$.

Before stating the main result of this section, we need to define and recall some specific notions and an efficient lemma.

Definition 3.1. A multi-valued mapping $S : X \to CB(X)$ is said to be D-Lipschitz continuous with constant λ_S (or λ_S-D-Lipschitz continuous) if there exits a constant $\lambda_S > 0$ such that

$$D(S(x), S(y)) \leq \lambda_S \|x - y\|, \ \forall x, y \in X,$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

Definition 3.2. Let X be a real Banach space and $\mathcal{F}$ be the normalized duality mapping from X into X^*. A mapping $P : X \to X$ is said to be ς-generalized pseudocontractive if there exits a constant $\varsigma > 0$ such that for any $x, y \in X$,

$$\langle P(x) - P(y), j(x - y) \rangle \leq \varsigma \|x - y\|^2, \ \forall j(x - y) \in \mathcal{F}(x - y).$$

Definition 3.3. A mapping $T : X \times X \to X$ is said to be

(i) λ_{T_1}-Lipschitz continuous in the first argument if there exits a constant $\lambda_{T_1} > 0$ such that

$$\|T(x, u) - T(y, u)\| \leq \lambda_{T_1} \|x - y\|^2, \ \forall x, y, u \in X;$$

(ii) λ_{T_2}-Lipschitz continuous in the second argument if there exits a constant $\lambda_{T_2} > 0$ such that

$$\|T(u,x) - T(u,y)\| \leq \lambda_{T_2}\|x-y\|^2, \ \forall x,y,u \in X.$$

Lemma 3.3 ([30])**.** *Let X be a real Banach space and $\mathcal{F}$ be the normalized duality mapping from X into X^*. Then, for any $x,y \in X$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall j(x+y) \in \mathcal{F}(x+y).$$

Theorem 3.1. *Let for each $i \in \{1,2\}$, X_i be a real Banach space, $P_i : X_i \to X_i$ be a ζ_i-strongly accretive, r_i-contraction, and ς_i-generalized pseudocontractive mapping, and $\widehat{M}_i : X_i \to 2^{X_i}$ be a ϱ_i-strongly P_i-accretive mapping. Assume that $S : X_1 \times X_2 \to X_1$ is λ_{S_1}-Lipschitz continuous and λ_{S_2}-Lipschitz continuous in the first and second arguments, respectively, $T : X_1 \times X_2 \to X_2$ is λ_{T_1}-Lipschitz continuous and λ_{T_2}-Lipschitz continuous in the first and second arguments, respectively, and the multi-valued mappings $E : X_1 \to CB(X_1)$ and $F : X_2 \to CB(X_2)$ are λ_{D_E}-D_1-Lipschitz continuous and λ_{D_F}-D_2-Lipschitz continuous, respectively. If there exist constants $\lambda_i > 0$ $(i = 1,2)$ such that*

$$\begin{cases} 0 < \hat{L}_1(\hat{K}_1 + \sqrt{\theta_1} + \sqrt{\theta_3}) < 1, \\ 0 < \hat{L}_2(\hat{K}_2 + \sqrt{\theta_2} + \sqrt{\theta_4}) < 1, \end{cases} \tag{3.5}$$

where

$$\begin{aligned}
&\hat{K}_i = \sqrt{\frac{1+2\varsigma_i+3r_i}{1-r_i}}, \ \hat{L}_i = \frac{1}{\lambda_i\varrho_i+\zeta_i}, \ (i=1,2),\\
&\theta_1 = \frac{1+\lambda_1\lambda_{S_1}}{1-\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{D_F})}, \ \theta_2 = \frac{\lambda_1\lambda_{S_2}\lambda_{D_F}}{1-\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{D_F})},\\
&\theta_3 = \frac{\lambda_2\lambda_{T_1}\lambda_{D_E}}{1-\lambda_2(\lambda_{T_1}\lambda_{D_E}+\lambda_{T_2})}, \ \theta_4 = \frac{1+\lambda_2\lambda_{T_2}}{1-\lambda_2(\lambda_{T_1}\lambda_{D_E}+\lambda_{T_2})},\\
&\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{D_F}) < 1, \ \lambda_2(\lambda_{T_1}\lambda_{D_E}+\lambda_{T_2}) < 1,
\end{aligned}$$

then, the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ generated by Algorithm 3.1 converge strongly to x,y,u,v,z' and z'', respectively, and (x,y,u,v,z',z'') is a solution of the SGMRE (3.1).

Proof. Making use of (3.3), it follows that

$$\begin{aligned}
&\|z'_{n+1} - z'_n\|_1\\
&= \|P_1(x_n) - \lambda_1 S(x_n,v_n) - (P_1(x_{n-1}) - \lambda_1 S(x_{n-1},v_{n-1}))\|_1\\
&\leq \|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})\|_1\\
&\quad + \|x_n - x_{n-1} + \lambda_1(S(x_n,v_n) - S(x_{n-1},v_{n-1}))\|_1.
\end{aligned} \tag{3.6}$$

Utilizing Lemma 3.3 and taking into account that the mapping P_1 is r_1-contraction and ς_1-generalized pseudocontractive, we yield

$$\begin{aligned}
&\|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})\|_1^2\\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\langle P_1(x_n) - P_1(x_{n-1}), j_1(x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1}))\rangle_1
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - x_{n-1}\|_1^2 + 2\langle P_1(x_n) - P_1(x_{n-1}), j_1(x_n - x_{n-1})\rangle_1 \\
&\quad + 2\langle P_1(x_n) - P_1(x_{n-1}), j_1(x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})) \\
&\quad - j_1(x_n - x_{n-1})\rangle_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\langle P_1(x_n) - P_1(x_{n-1}), j_1(x_n - x_{n-1})\rangle_1 \\
&\quad + 2\big(\|P_1(x_n) - P_1(x_{n-1})\|_1(\|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})\|_1 \\
&\quad + \|x_n - x_{n-1}\|_1)\big) \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\varsigma_1\|x_n - x_{n-1}\|_1^2 + 2\big(r_1\|x_n - x_{n-1}\|_1(\|x_n - x_{n-1} \\
&\quad + P_1(x_n) - P_1(x_{n-1})\|_1 + \|x_n - x_{n-1}\|_1)\big) \\
&= \|x_n - x_{n-1}\|_1^2 + 2\varsigma_1\|x_n - x_{n-1}\|_1^2 + r_1\big(2\|x_n - x_{n-1}\|_1\|x_n - x_{n-1} \\
&\quad + P_1(x_n) - P_1(x_{n-1})\|_1\big) + 2r_1\|x_n - x_{n-1}\|_1^2 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\varsigma_1\|x_n - x_{n-1}\|_1^2 + r_1(\|x_n - x_{n-1}\|_1^2 \\
&\quad + \|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})\|_1^2) + 2r_1\|x_n - x_{n-1}\|_1^2 \\
&= (1 + 2\varsigma_1 + 3r_1)\|x_n - x_{n-1}\|_1^2 + r_1\|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n_1})\|_1^2,
\end{aligned}$$

which implies that

$$\|x_n - x_{n-1} + P_1(x_n) - P_1(x_{n-1})\|_1 \leq \hat{K}_1\|x_n - x_{n-1}\|_1, \tag{3.7}$$

where $\hat{K}_1 = \sqrt{\frac{1+2\varsigma_1+3r_1}{1-r_1}}$. Again, by using Lemma 3.3, and in virtue of the facts that the mapping S is λ_{S_1}-Lipschitz continuous in the first argument and λ_{S_2}-Lipschitz continuous in the second argument, and the mapping F is λ_{D_F}-D_2-Lipschitz continuous, we obtain

$$\begin{aligned}
&\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1^2 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\lambda_1\langle S(x_n, v_n) - S(x_{n-1}, v_{n-1}), \\
&\quad j_1(x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1})))\rangle_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\lambda_1\|S(x_n, v_n) - S(x_{n-1}, v_{n-1})\|_1 \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\lambda_1\big(\|S(x_n, v_n) - S(x_{n-1}, v_n)\|_1 \\
&\quad + \|S(x_{n-1}, v_n) - S(x_{n-1}, v_{n-1})\|_1\big) \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\lambda_1\big(\lambda_{S_1}\|x_n - x_{n-1}\|_1 + \lambda_{S_2}\|v_n - v_{n-1}\|_2\big) \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\leq \|x_n - x_{n-1}\|_1^2 + 2\lambda_1(\lambda_{S_1}\|x_n - x_{n-1}\|_1 \\
&\quad + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})\|y_n - y_{n-1}\|_2) \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_{n-1}\|_1^2 + 2\lambda_1\lambda_{S_1}\|x_n - x_{n-1}\|_1 \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\quad + 2\lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})\|y_n - y_{n-1}\|_2 \\
&\quad \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\le \|x_n - x_{n-1}\|_1^2 + \lambda_1\lambda_{S_1}\big(\|x_n - x_{n-1}\|_1^2 \\
&\quad + \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1^2\big) \\
&\quad + \lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})\big(\|y_n - y_{n-1}\|_2^2 \\
&\quad + \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1^2\big) \\
&= \|x_n - x_{n-1}\|_1^2 + \lambda_1\lambda_{S_1}\|x_n - x_{n-1}\|_1^2 \\
&\quad + \lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})\|y_n - y_{n-1}\|_2^2 + (\lambda_1\lambda_{S_1} \\
&\quad + \lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n}))\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1^2,
\end{aligned}
$$

form which we conclude that

$$
\begin{aligned}
&\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1^2 \\
&\le \frac{1 + \lambda_1\lambda_{S_1}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n}))}\|x_n - x_{n-1}\|_1^2 \\
&\quad + \frac{\lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n}))}\|y_n - y_{n-1}\|_2^2 \\
&= \theta_1^n\|x_n - x_{n-1}\|_1^2 + \theta_2^n\|y_n - y_{n-1}\|_2^2 \\
&\le \theta_1^n\|x_n - x_{n-1}\|_1^2 + 2\sqrt{\theta_1^n\theta_2^n}\|x_n - x_{n-1}\|_1\|y_n - y_{n-1}\|_2 \\
&\quad + \theta_2^n\|y_n - y_{n-1}\|_2^2 \\
&= (\sqrt{\theta_1^n}\|x_n - x_{n-1}\|_1 + \sqrt{\theta_2^n}\|y_n - y_{n-1}\|_2)^2,
\end{aligned}
\tag{3.8}
$$

where for each $n \in \mathbb{N}$,

$$
\theta_1^n = \frac{1 + \lambda_1\lambda_{S_1}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n}))}
$$

and

$$
\theta_2^n = \frac{\lambda_1\lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n})}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}(1 + \frac{1}{1+n}))}.
$$

Making use of (3.8), we deduce that

$$
\begin{aligned}
&\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\
&\le \sqrt{\theta_1^n}\|x_n - x_{n-1}\|_1 + \sqrt{\theta_2^n}\|y_n - y_{n-1}\|_2.
\end{aligned}
\tag{3.9}
$$

Substituting (3.7) and (3.9) into (3.6), it follows that

$$\|z'_{n+1}-z'_n\|_1 \leq (\hat{K}_1+\sqrt{\theta_1^n})\|x_n-x_{n-1}\|_1+\sqrt{\theta_2^n}\|y_n-y_{n-1}\|_2. \tag{3.10}$$

Following the same arguments, taking into consideration the facts that the mapping P_2 is r_2-Lipschitz continuous and ς_2-generalized pseudocontractive, T is λ_{T_1}-Lipschitz continuous and λ_{T_2}-Lipschitz continuous in the first and second arguments, respectively, and E is λ_{D_E}-D_1-Lipschitz continuous, and using Lemma 3.3 and (3.4), we can show that

$$\|z''_{n+1}-z''_n\|_2 \leq \sqrt{\theta_3^n}\|x_n-x_{n-1}\|_1+(\hat{K}_2+\sqrt{\theta_4^n})\|y_n-x_{n-1}\|_2, \tag{3.11}$$

where for each $n\in\mathbb{N}$,

$$\hat{K}_2=\sqrt{\frac{1+2\varsigma_2+3r_2}{1-r_2}},\ \theta_3^n=\frac{\lambda_2\lambda_{T_1}\lambda_{D_E}(1+\frac{1}{1+n})}{1-\lambda_2^n(\lambda_{T_2}+\lambda_{T_1}\lambda_{D_E}(1+\frac{1}{1+n}))},$$
$$\theta_4^n=\frac{1+\lambda_2\lambda_{T_2}}{1-\lambda_2^n(\lambda_{T_2}+\lambda_{T_1}\lambda_{D_E}(1+\frac{1}{1+n}))}.$$

Let us now define a norm $\|\cdot\|_*$ on $X_1\times X_2$ by

$$\|(x,y)\|_*=\|x\|_1+\|y\|_2,\ \forall(x,y)\in X_1\times X_2.$$

It can be easily seen that $(X_1\times X_2,\|\cdot\|_*)$ is a Banach space. Then, employing (3.10) and (3.11), for each $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\|(z'_{n+1},z''_{n+1})-(z'_n,z''_n)\|_* &= \|z'_{n+1}-z'_n\|_1+\|z''_{n+1}-z''_n\|_2\\ &\leq (\hat{K}_1+\sqrt{\theta_1^n}+\sqrt{\theta_3^n})\|x_n-x_{n-1}\|_1\\ &\quad+(\hat{K}_2+\sqrt{\theta_2^n}+\sqrt{\theta_4^n})\|y_n-y_{n-1}\|_2.\end{aligned} \tag{3.12}$$

From (3.4) and Theorem 2.2, it follows that for each $n\in\mathbb{N}$,

$$\begin{aligned}\|x_n-x_{n-1}\|_1 &= \|R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n)-R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_{n-1})\|_1\\ &\leq \frac{1}{\lambda_1\varrho_1+\zeta_1}\|z'_n-z'_{n-1}\|_1=L_1\|z'_n-z'_{n-1}\|_1,\end{aligned} \tag{3.13}$$

where $L_1=\frac{1}{\lambda_1\varrho_1+\zeta_1}$. By an argument analogous to the previous one, applying (3.4) and utilizing Theorem 2.2, for each $n\in\mathbb{N}$, we deduce that

$$\|y_n-y_{n-1}\|_2 \leq L_2\|z''_n-z''_{n-1}\|_2, \tag{3.14}$$

where $L_2=\frac{1}{\lambda_2\varrho_2+\zeta_2}$. Combining (3.12)–(3.14), we yield

$$\begin{aligned}\|(z'_{n+1},z''_{n+1})-(z'_n,z''_n)\|_* &\leq L_1(\hat{K}_1+\sqrt{\theta_1^n}+\sqrt{\theta_3^n})\|z'_n-z'_{n-1}\|_1\\ &\quad+L_2(\hat{K}_2+\sqrt{\theta_2^n}+\sqrt{\theta_4^n})\|z''_n-z''_{n-1}\|_2\\ &\leq \vartheta^n(\|z'_n-z'_{n-1}\|_1+\|z''_n-z''_{n-1}\|_2)\\ &= \vartheta^n\|(z'_n,z''_n)-(z'_{n-1},z''_{n-1})\|_*,\end{aligned} \tag{3.15}$$

where for each $n \in \mathbb{N}$,

$$\vartheta^n = \max\{L_1(\hat{K}_1 + \sqrt{\theta_1^n} + \sqrt{\theta_3^n}), L_2^n(\hat{K}_2 + \sqrt{\theta_2^n} + \sqrt{\theta_4^n})\}.$$

The fact that for each $i \in \{1, 2, 3, 4\}$, $\theta_i^n \to \theta_i$, where

$$\theta_1 = \frac{1 + \lambda_1 \lambda_{S_1}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F})},\ \theta_2 = \frac{\lambda_1 \lambda_{S_2} \lambda_{D_F}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F})},$$
$$\theta_3 = \frac{\lambda_2 \lambda_{T_1} \lambda_{D_E}}{1 - \lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2})},\ \theta_4 = \frac{1 + \lambda_2 \lambda_{T_2}}{1 - \lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2})},$$

implies that $\vartheta^n \to \vartheta$ as $n \to \infty$, where

$$\vartheta = \max\{L_1(\hat{K}_1 + \sqrt{\theta_1} + \sqrt{\theta_3}), L_2(\hat{K}_2 + \sqrt{\theta_2} + \sqrt{\theta_4})\}.$$

Evidently, (3.5) guarantees that $\vartheta \in (0, 1)$. Therefore, there exist $n_0 \in \mathbb{N}$ and $\hat{\vartheta} \in (\vartheta, 1)$ such that $\vartheta^n \leq \hat{\vartheta}$ for all $n \geq n_0$. Hence, for all $n > n_0$, by (3.15), we get

$$\begin{aligned}
&\|(z'_{n+1}, z''_{n+1}) - (z'_n, z''_n)\|_* \\
&\leq \vartheta^n \|(z'_n, z''_n) - (z'_{n-1}, z''_{n-1})\|_* \\
&\leq \hat{\vartheta} \|(z'_n, z''_n) - (z'_{n-1}, z''_{n-1})\|_* \\
&\leq \hat{\vartheta}[\hat{\vartheta}\|(z'_{n-1}, z''_{n-1}) - (z'_{n-2}, z''_{n-2})\|_*] \\
&= \hat{\vartheta}^2 \|(z'_{n-1}, z''_{n-1}) - (z'_{n-2}, z''_{n-2})\|_* \\
&\leq \ \cdots \leq \hat{\vartheta}^{n-n_0} \|(z'_{n_0+1}, z''_{n_0+1}) - (z'_{n_0}, z''_{n_0})\|_*.
\end{aligned} \tag{3.16}$$

Making use of (3.16), it follows that for any $m \geq n > n_0$,

$$\begin{aligned}
\|(z'_m, z''_m) - (z'_n, z''_n)\|_* &\leq \sum_{k=n}^{m-1} \|(z'_{k+1}, z''_{k+1}) - (z'_k, z''_k)\|_* \\
&\leq \sum_{k=n}^{m-1} \hat{\vartheta}^{k-n_0} \|(z'_{n_0+1}, z''_{n_0+1}) - (z'_{n_0}, z''_{n_0})\|_*.
\end{aligned} \tag{3.17}$$

Taking into account that $\hat{\vartheta} \in (0, 1)$, it follows that the right-hand side of (3.17) tends to zero, as $n \to \infty$, that is, $\|(z'_m, z''_m) - (z'_n, z''_n)\|_* \to 0$ as $n \to \infty$, consequently, $\{(z'_n, z''_n)\}_{n=0}^{\infty}$ is a Cauchy sequence in $X_1 \times X_2$. In view of the completeness of $X_1 \times X_2$, $(z'_n, z''_n) \to (z', z'')$ for some $(z', z'') \in X_1 \times X_2$ as $n \to \infty$. The two inequalities (3.13) and (3.14) imply that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then, by using (3.4) and thanks to the fact that the mappings E and F are λ_{D_E}-D_1-Lipschitz and λ_{D_F}-D_2-Lipschitz continuous, respectively, we observe that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are also Cauchy sequences in X_1 and X_2, respectively. Accordingly, there are $u \in X_1$ and $v \in X_2$ such that $u_n \to u$ and $v_n \to v$, as $n \to \infty$. Since $u_n \in E(x_n)$ for each $n \geq 0$, we have

$$\begin{aligned}
d_1(u, E(x)) &= \inf\{\|u - q\|_1 : q \in E(x)\} \\
&\leq \|u - u_n\|_1 + d_1(u_n, E(x))
\end{aligned}$$

$$\leq \|u - u_n\|_1 + D_1(E(x_n), E(x))$$
$$\leq \|u - u_n\|_1 + \lambda_{D_E}\|x_n - x\|_1.$$

Obviously, the right-hand side of the last inequality approaches zero, as $n \to \infty$, and so in virtue of the fact that $E(x)$ is closed, it follows that $u \in E(x)$. In a similar fashion to the preceding analysis, one can show that $v \in F(y)$. Since $x_n \to x$, $y_n \to y$, $u_n \to u$ and $v_n \to v$ as $n \to \infty$, in the light of the Lipschitz continuity of S and T in both the arguments, with the help of (3.4), we deduce that $z'_n \to z' = P_1(x) - \lambda_1 S(x, v)$ and $z''_n \to z'' = P_2(y) - \lambda_2 T(u, y)$. At the same time, by using Theorem 2.2, for each $n \geq 0$, we yield

$$\|R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n) - R^{P_1}_{\lambda_1,\widehat{M}_1}(z')\|_1 \leq \frac{1}{\lambda_1\varrho_1 + \zeta_1}\|z'_n - z'\|_1. \tag{3.18}$$

Now, taking into account that $z'_n \to z'$ as $n \to \infty$, from (3.18) we deduce that

$$\|R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n) - R^{P_1}_{\lambda_1,\widehat{M}_1}(z')\|_1 \to 0 \text{ as } n \to \infty,$$

and so

$$R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n) \to R^{P_1}_{\lambda_1,\widehat{M}_1}(z') \text{ as } n \to \infty.$$

Following the same argument, we can prove that $R^{P_2}_{\lambda_2,\widehat{M}_2}(z''_n) \to R^{P_2}_{\lambda_2,\widehat{M}_2}(z'')$, as $n \to \infty$. Now, (3.4) implies that $x = R^{P_1}_{\lambda_1,\widehat{M}_1}(z')$ and $y = R^{P_2}_{\lambda_2,\widehat{M}_2}(z'')$. In the light of the above-mentioned discussion, $(x, y, u, v, z', z'') \in X_1 \times X_2 \times E(x) \times F(y) \times X_1 \times X_2$ is a solution of the SGMRE (3.1). This completes the proof. $\square$

It should be pointed out that the SGMRE (3.1) can also be written as follows:

$$\begin{cases} z' = P_1(x) - S(x, v) + (I_1 - \lambda_1^{-1})J^{P_1}_{\lambda_1,\widehat{M}_1}(z'), \\ z'' = P_2(y) - T(u, y) + (I_2 - \lambda_2^{-1})J^{P_2}_{\lambda_2,\widehat{M}_2}(z''), \end{cases} \tag{3.19}$$

where for $i = 1, 2$, I_i is the identity mapping on X_i.

The fixed point formulation (3.19) enables also us to suggest the following iterative algorithm for approximating a solution of the SGMRE (3.1).

Algorithm 3.2. *Suppose that X_i, P_i, $\widehat{M}_i$ $(i = 1, 2)$, S, T, E and F are the same as in the* SGMRE (3.1). *For any given $(x_0, y_0), (z'_0, z''_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$ and $v_0 \in F(y_0)$, compute the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ by the following iterative schemes*:

$$\begin{cases} x_n = R^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n), \\ y_n = R^{P_2}_{\lambda_2,\widehat{M}_2}(z''_n), \\ u_n \in E(x_n); \|u_{n+1} - u_n\|_1 \leq (1 + \frac{1}{1+n})D_1(E(x_{n+1}), E(x_n)), \\ v_n \in F(y_n); \|v_{n+1} - v_n\|_2 \leq (1 + \frac{1}{1+n})D_2(F(y_{n+1}), F(y_n)), \\ z'_{n+1} = P_1(x_n) - S(x_n, v_n) + (I_1 - \lambda_1^{-1})J^{P_1}_{\lambda_1,\widehat{M}_1}(z'_n), \\ z''_{n+1} = P_2(y_n) - T(u_n, y_n) + (I_2 - \lambda_2^{-1})J^{P_2}_{\lambda_2,\widehat{M}_2}(z''_n), \end{cases}$$

where $n = 0, 1, 2, \dots$; $\lambda_i > 0$ $(i = 1, 2)$ *are positive real constants, and for* $i = 1, 2$, D_i *is the Hausdorff metric on* $CB(X_i)$.

It is worth mentioning that by the argument similar to that of Theorem 3.1, one can study the convergence analysis of the iterative sequences generated by Algorithm 3.2.

4. Some comments on $H(\cdot, \cdot)$-co-accretive mappings

This section deals with the investigation and analysis of the $H(\cdot, \cdot)$-co-accretive mapping introduced and studied in [1]. By pointing out some important remarks on $H(\cdot, \cdot)$-co-accretive mapping, we show that the main results in [1] can be easily deduced by using the assertions presented in Sections 2 and 3.

Definition 4.1 ([1, Definition 2.2])**.** Let $H : X \times X \to X$, $A, B : X \to X$ be the mappings and $\mathcal{F} : X \to 2^{X^*}$ be the normalized duality mapping. Then

(i) $H(A, \cdot)$ is said to be cocoercive (or μ-cocoercive) with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), j(x - y)\rangle \geq \mu \|Ax - Ay\|^2, \quad \forall x, y, u \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(ii) $H(\cdot, B)$ is said to be relaxed-cocoercive (or γ-relaxed cocoercive) with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), j(x - y)\rangle \geq -\gamma \|Bx - By\|^2, \quad \forall x, y, u \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(iii) $H(A, B)$ is said to be symmetric cocoercive (or $\mu\gamma$-symmetric cocoercive) with respect to A and B, if $H(A, \cdot)$ is μ-cocoercive with respect to A and $H(\cdot, B)$ is γ-relaxed cocoercive with respect to B;

(iv) $H(A, B)$ is said to be mixed Lipschitz continuous with respect to A and B, if there exists a constant $r > 0$ such that

$$\|H(Ax, Bx) - H(Ay, By)\| \leq r\|x - y\|, \quad \forall x, y \in X.$$

Proposition 4.1. *Let X be a real Banach space and $\mathcal{F}$ be the normalized duality mapping from X into X^*. Assume that $A, B : X \to X$ and $H : X \times X \to X$ are the mappings such that A is η-expansive, B is σ-Lipschitz continuous and $\eta > \sigma$. Suppose further that the mapping $H(A, B)$ is $\mu\gamma$-symmetric cocoercive with respect to mappings A and B and $P : X \to X$ is a mapping defined by $P(x) := H(Ax, Bx)$ for all $x \in X$. Then*

(i) *If $\mu\eta^2 = \gamma\sigma^2$, then P is an accretive mapping*;
(ii) *If $\mu\eta^2 > \gamma\sigma^2$, then P is a $(\mu\eta^2 - \gamma\sigma^2)$-strongly accretive mapping*;
(iii) *If $\mu\eta^2 < \gamma\sigma^2$, then P is a $(\mu\eta^2 - \gamma\sigma^2)$-relaxed accretive mapping.*

Proof. In the light of the assumptions, for any $x, y \in X$ and for all $j(x-y) \in \mathcal{F}(x-y)$, we have

$$\begin{aligned}\langle P(x) - P(y), j(x-y)\rangle &= \langle H(Ax, Bx) - H(Ay, By), j(x-y)\rangle \\ &= \langle H(Ax, Bx) - H(Ay, Bx), j(x-y)\rangle \\ &\quad + \langle H(Ay, Bx) - H(Ay, By), j(x-y)\rangle \\ &\geq \mu\|Ax - Ay\|^2 - \gamma\|Bx - By\|^2 \\ &\geq \mu\eta^2\|x-y\|^2 - \gamma\sigma^2\|x-y\|^2 \\ &= (\mu\eta^2 - \gamma\sigma^2)\|x-y\|^2.\end{aligned}$$

Now, the assertions (i)–(iii) follow respectively from parts (i), (ii) and (iv) of Definition 2.1. □

Thanks to Proposition 4.1, every symmetric cocoercive bifunction $H : X \times X \to X$ with respect to mappings $A, B : X \to X$, under the conditions imposed on A and B mentioned in it, is actually a univariate accretive, strongly accretive, or relaxed accretive mapping and is not a new one. In fact, the notion $\mu\gamma$-symmetric cocoercivity of bifunction H with respect to mappings A and B presented in Definition 4.1(iii), where A and B are η-expansive and σ-Lipschitz continuous, respectively, is exactly the same concept of accretivity, $k = (\mu\eta^2 - \gamma\sigma^2)$-strong accretivity, or $\varrho = (\mu\eta^2 - \gamma\sigma^2)$-relaxed accretivity of the univariate mapping $P := H(A, B)$ given in parts (i), (iii) and (iv) of Definition 2.1.

Definition 4.2 ([1, Definition 2.4])**.** Let $M : X \times X \to 2^X$ be a multi-valued mapping, $f, g : X \to X$ be the mappings and $\mathcal{F} : X \to 2^{X^*}$ be the normalized duality mapping. Then

(i) $M(f, \cdot)$ is said to be α-strongly accretive with respect to f, if there exists a constant $\alpha > 0$ such that

$$\begin{aligned}&\langle u - v, j(x-y)\rangle \geq \alpha\|x-y\|^2, \ \forall x, y, w \in X, \\ &\quad u \in M(f(x), w), \ v \in M(f(y), w), \ j(x-y) \in \mathcal{F}(x-y);\end{aligned}$$

(ii) $M(\cdot, g)$ is said to be β-relaxed accretive with respect to g, if there exists a constant $\beta > 0$ such that

$$\begin{aligned}&\langle u - v, j(x-y)\rangle \geq -\beta\|x-y\|^2, \ \forall x, y, w \in X, \\ &\quad u \in M(w, g(x)), \ v \in M(w, g(y)), \ j(x-y) \in \mathcal{F}(x-y);\end{aligned}$$

(iii) $M(\cdot, \cdot)$ is said to be symmetric (or $\alpha\beta$-symmetric) accretive with respect to f and g, if $M(f, \cdot)$ is α-strongly accretive with respect to f and $M(\cdot, g)$ is β-relaxed accretive with respect to g.

Proposition 4.2. *Let X be a real Banach space and $\mathcal{F}$ be the normalized duality mapping from X into X^*. Suppose that $f, g : X \to X$ and $M : X \times X \to 2^X$ are the mappings, and let the mapping $\widehat{M} : X \to 2^X$ be defined by*

$\widehat{M}(x) := M(f(x), g(x))$ *for all* $x \in X$. *Moreover, let* $M(\cdot,\cdot)$ *be* $\alpha\beta$*-symmetric accretive with respect to* f *and* g. *Then,*

(i) *If* $\alpha = \beta$, *then* $\widehat{M}$ *is accretive*;
(ii) *If* $\alpha > \beta$, *then* $\widehat{M}$ *is* $(\alpha-\beta)$*-strongly accretive*;
(iii) *If* $\alpha < \beta$, *then* $\widehat{M}$ *is* $(\alpha-\beta)$*-relaxed accretive.*

Proof. In virtue of the fact that $M(\cdot,\cdot)$ is $\alpha\beta$-symmetric accretive with respect to f and g, for any $x, y \in X$, $u \in \widehat{M}(x)$, $v \in \widehat{M}(y)$ and $j(x-y) \in \mathcal{F}(x-y)$, we yield

$$\begin{aligned}\langle u-v, j(x-y)\rangle &= \langle u-w+w-v, j(x-y)\rangle \\ &= \langle u-w, j(x-y)\rangle + \langle w-v, j(x-y)\rangle \\ &\geq \alpha\|x-y\|^2 - \beta\|x-y\|^2 \\ &= (\alpha-\beta)\|x-y\|^2.\end{aligned}$$

In the light of the last inequality and invoking Definition 2.2(i)–(iii), the statements follow immediately. □

Based on Proposition 4.2, the concept $\alpha\beta$-symmetric accretivity of $M : X\times X \to 2^X$ with respect to mappings $f, g : X \to X$ given in Definition 4.2(iii) is actually the same notion accretivity, $r = (\alpha-\beta)$-strong accretiviy, or $\xi = (\alpha-\beta)$-relaxed accretivity of the univariate multi-valued mapping $\widehat{M} := M(f, g) : X \to 2^X$ presented in Definition 2.2(i)–(iii). In other words, every symmetric accretive mapping is exactly an accretive, strongly accretive or relaxed accretive mapping, and is not a new one.

Ahmad and Akram [1] considered and studied a class of accretive mappings the so-called $H(\cdot,\cdot)$-co-accretive mappings as a generalization of P-accretive mappings as follows.

Definition 4.3 ([1, Definition 2.7])**.** Let $A, B, f, g : X \to X$ and $H : X\times X \to X$ be the mappings. A multi-valued mapping $M : X \times X \to 2^X$ is said to be $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g, if $H(A, B)$ is symmetric cocoercive with respect to A and B, $M(f, g)$ is symmetric accretive with respect to f and g and $(H(A, B) + \lambda M(f, g))(X) = X$ for every real constant $\lambda > 0$.

The next assertion tells us that the class of $H(\cdot,\cdot)$-co-accretive mappings coincides exactly with one of the classes of P-accretive mappings, or $(\alpha-\beta)$-strongly P-accretive mappings, or P-maximal $(\alpha-\beta)$-relaxed accretive mappings and in contrary of the claim in [1] is not a new one.

Proposition 4.3. *Suppose that* X *is a real Banach space and* $\mathcal{F}$ *is the normalized duality mapping from* X *into* X^*. *Let* $A, B, f, g : X \to X$ *and* $H : X \times X \to X$ *be the mappings and let* $M : X \times X \to 2^X$ *be an* $H(\cdot,\cdot)$*-co-accretive mapping with respect to* A, B, f *and* g *with constants* $\mu, \gamma, \alpha, \beta > 0$. *Furthermore, let the mappings* $P : X \to X$ *and* $\widehat{M} : X \to 2^X$ *be defined by* $P(x) := H(Ax, Bx)$ *and* $\widehat{M}(x) := M(f(x), g(x))$ *for all* $x \in X$, *respectively.*

(i) *If $\alpha = \beta$, then $\widehat{M}$ is a P-accretive mapping*;
(ii) *If $\alpha > \beta$, then $\widehat{M}$ is an $(\alpha - \beta)$-strongly P-accretive mapping*;
(iii) *If $\alpha < \beta$, then $\widehat{M}$ is a P-maximal $(\alpha - \beta)$-relaxed accretive mapping.*

Proof. Taking into account that M is $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g, accordance with Definition 4.3, $M(f,g)$ is $\alpha\beta$-symmetric accretive with respect to f and g with constants $\alpha, \beta > 0$, respectively, and $(H(A,B) + \lambda M(f,g))(X) = X$ for every real constant $\lambda > 0$. Utilizing Proposition 4.2, $\widehat{M}$ is accretive (resp. $(\alpha-\beta)$-strongly accretive, $(\alpha-\beta)$-relaxed accretive) if $\alpha = \beta$ (resp. $\alpha > \beta$, $\alpha < \beta$). At the same time, thanks to the assumptions, for every real constant $\lambda > 0$, we have $(P + \lambda\widehat{M})(X) = (H(A,B) + \lambda M(f,g))(X) = X$. Now, the conclusions (i)–(iii) follow immediately. $\square$

Note. In the rest of the paper, we say that M is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g, means that $H(A,B)$ is $\mu\gamma$-symmetric cocoercive with respect to A and B, and $M(f,g)$ is $\alpha\beta$-symmetric accretive with respect to f and g, where $\mu > \gamma$ and $\alpha > \beta$.

In order to define the resolvent operator associated with an $H(\cdot,\cdot)$-co-accretive mapping, the authors [1] presented the following assertion in which the required conditions for the operator $(H(A,B)+\lambda M(f,g))^{-1}$ to be single-valued for every real constant $\lambda > 0$, are provided.

Lemma 4.1 ([1, Theorem 2.1])**.** *Let X be a real Banach space and let A, B, f, $g : X \to X$ and $H : X \times X \to X$ be the mappings. Let $M : X \times X \to 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous such that $\eta > \sigma$. Then the mapping $(H(A,B) + \lambda M(f,g))^{-1}$ is single-valued for every real constant $\lambda > 0$.*

Proof. Let us define the mappings $P : X \to X$ and $\widehat{M} : X \to 2^X$ as $P(x) := H(Ax, Bx)$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$. Since $\alpha > \beta$, Proposition 4.3(ii) implies that $\widehat{M}$ is an $(\alpha - \beta)$-strongly P-accretive mapping. In the meanwhile, taking into consideration the facts that $H(A,B)$ is $\mu\gamma$-symmetric cocoercive with respect to mappings A and B, and $\mu > \gamma$ and $\eta > \sigma$, thanks to Proposition 4.1(ii) it follows that P is a $(\mu\eta^2 - \gamma\sigma^2)$-strongly accretive mapping. Now, according to Corollary 2.1, the mapping $(P + \lambda\widehat{M})^{-1} = (H(A,B) + \lambda M(f,g))^{-1} : X \to X$ is single-valued for every real constant $\lambda > 0$. This gives the desired result. $\square$

Remark 4.1. Due to the proof of [1, Theorem 2.1] is similar to that of Theorem 3.8 in [4], the authors [1] deleted its proof. However, by a careful reading the proof of [4, Theorem 3.8], we found that there is a small mistake in the contexts of [1, Theorem 2.1] and [4, Theorem 3.8]. In fact, in the light of the proof of [4, Theorem 3.8], in the contexts of [1, Theorem 2.1] and [4, Theorem 3.8], the two constants η and σ in addition to being positive must be satisfied the condition $\eta > \sigma$, as we have added the aforesaid condition to the context of Lemma 4.1.

By virtue of Lemma 4.1 (that is, [1, Theorem 2.1]), Ahmad and Akram [1] defined the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ associated with an $H(\cdot,\cdot)$-co-accretive mapping $M : X \times X \to 2^X$ and an arbitrary real constant $\lambda > 0$ as follows.

Definition 4.4 ([1, Definition 2.8])**.** Let $A, B, f, g : X \to X$ and $H : X \times X \to X$ be the mappings, and let $M : X \times X \to 2^X$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Suppose further that A is η-expansive and B is σ-Lipschitz continuous such that $\eta > \sigma$. The resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) = (H(A,B) + \lambda M(f,g))^{-1}(x), \quad \forall x \in X \text{ and } \lambda > 0.$$

Note, in particular, that the authors [1] defined the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ associated with an $H(\cdot,\cdot)$-co-accretive mapping $M : X \times X \to 2^X$ and an arbitrary real constant $\lambda > 0$ based on Lemma 4.1 (that is, [1, Theorem 2.1]). According to Lemma 4.1, the two mappings A and B must be η-expansive and σ-Lipschitz continuous, respectively. At the same time, as we have pointed out in Remark 4.1, the two constants η and σ in addition to being positive must be satisfied the condition $\eta > \sigma$. In the light of the above-mentioned arguments, there is a small mistake in the context of Definition 2.8 of [1]. In fact, the conditions η-expansivity of A and σ-Lipschitz continuity of B such that $\eta > \sigma$, must be added to the context of [1, Definition 2.8], as we have done in the context of Definition 4.4. On the other hand, by defining the two mappings $P : X \to X$ and $\widehat{M} : X \to 2^X$ as $P(x) := H(Ax, Bx)$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$, considering the facts that $\widehat{M}$ is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g with constants $\mu, \gamma, \alpha, \beta > 0$, and $\alpha > \beta$, Proposition 4.3(ii) implies that $\widehat{M}$ is an $(\alpha - \beta)$-strongly P-accretive mapping. In the meanwhile, relying on Note 4, $H(A, B)$ is $\mu\gamma$-symmetric cocoercive with respect to A and B. Since $\mu > \gamma$ and $\eta > \sigma$, according to Proposition 4.1(ii), P is a $(\mu\eta^2 - \gamma\sigma^2)$-strongly accretive mapping. Then, invoking Definition 2.4, for every real constant $\lambda > 0$, the resolvent operator $R^P_{\lambda,\widehat{M}} = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ associated with an $(\alpha - \beta)$-strongly P-accretive mapping $\widehat{M} : X \to 2^X$ is defined by

$$\begin{aligned} R^P_{\lambda,\widehat{M}}(x) = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) &= (P + \lambda\widehat{M})^{-1}(x) \\ &= (H(A,B) + \lambda M(f,g))^{-1}(x), \quad \forall x \in X. \end{aligned}$$

In fact, the notion of the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ associated with an arbitrary real constant $\lambda > 0$ and an $H(\cdot,\cdot)$-co-accretive mapping $M : X \times X \to 2^X$ presented in Definition 4.4, is actually the same concept of the resolvent operator $R^P_{\lambda,\widehat{M}}$ associated with $(\alpha - \beta)$-strongly $P = H(\cdot,\cdot)$-accretive mapping

$\widehat{M} = M(\cdot,\cdot)$ and real constant $\lambda > 0$, given in Definition 2.4, and is not a new one.

Section 2 in [1] is closed with an assertion on the Lipschitz continuity of the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}$ along with an estimate of its Lipschitz constant as follows.

Lemma 4.2 ([1, Theorem 2.2])**.** *Let $A, B, f, g : X \to X$ and $H : X\times X \to X$ be the mappings. Suppose that $M : X\times X \to 2^X$ is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g with constants μ, γ, α and β, respectively. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta$, $\mu > \gamma$ and $\eta > \sigma$. Then, for every real constant $\lambda > 0$, the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ is Lipschitz continuous with constant $L = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$, i.e.,*

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(y)\| \leq L\|x - y\|, \quad \forall x, y \in X.$$

Proof. Define the two mappings $P : X \to X$ and $\widehat{M} : X \to 2^X$ by $P(x) := H(Ax, Bx)$ and $\widehat{M}(x) = M(f(x), g(x))$ for all $x \in X$, respectively. Since M is an $H(\cdot,\cdot)$-co-accretive mapping with respect to mappings A, B, f and g, in view of Note 4, $H(A, B)$ is a $\mu\gamma$-symmetric cocoercive mapping with respect to A and B, and $M(f, g)$ is an $\alpha\beta$-symmetric accretive mapping with respect to f and g such that $\mu > \gamma$ and $\alpha > \beta$. Owing to these facts and in the light of Propositions 4.1(ii) and 4.3(ii), with the help of the fact that $\eta > \sigma$ it follows that P is a $(\mu\eta^2 - \gamma\sigma^2)$-strongly accretive mapping and $\widehat{M}$ is an $(\alpha-\beta)$-strongly P-accretive mapping. Then, taking $\zeta = \mu\eta^2 - \gamma\sigma^2$ and $\varrho = \alpha - \beta$, Theorem 2.2 implies that for any real constant $\lambda > 0$, the resolvent operator $R^P_{\lambda,\widehat{M}} = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : X \to X$ is $\frac{1}{\lambda\varrho+\zeta} = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$-Lipschitz continuous, i.e., for any $x, y \in X$,

$$\begin{aligned}\|R^P_{\lambda,\widehat{M}}(x) - R^P_{\lambda,\widehat{M}}(y)\| &= \|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(y)\| \\ &\leq \frac{1}{\lambda\varrho + \zeta}\|x - y\| \\ &= \frac{1}{\lambda(\alpha - \beta) + \mu\eta^2 - \gamma\sigma^2}\|x - y\|.\end{aligned}$$

This completes the proof. □

Let for each $i \in \{1, 2\}$, X_i be a real Banach space, $A_i, B_i, f_i, g_i : X_i \to X_i$, $H_i : X_i \times X_i \to X_i$, $S : X_1 \times X_2 \to X_1$ and $T : X_1 \times X_2 \to X_2$ be the mappings, and let $E : X_1 \to 2^{X_1}$ and $F : X_2 \to 2^{X_2}$ be multi-valued mappings. Suppose further that for each $i \in \{1, 2\}$, $M_i : X_i \times X_i \to 2^{X_i}$ is an $H_i(A_i, B_i)$-co-accretive mapping with respect to A_i, B_i, f_i and g_i. Recently, for given two arbitrary real constants $\lambda_1, \lambda_2 > 0$, Ahmad and Akram [1] considered and studied the problem of finding $(x, y) \in X_1 \times X_2$, $u \in E(x)$, $v \in F(y)$,

$(z', z'') \in X_1 \times X_2$ such that

$$\left\{ \begin{array}{l} S(x,v) + \lambda_1^{-1} J^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}(z') = 0, \\ T(u,y) + \lambda_2^{-1} J^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}(z'') = 0, \end{array} \right. \tag{4.1}$$

where for each $i \in \{1,2\}$, $J^{H_i(\cdot,\cdot)}_{\lambda_i, M_i(\cdot,\cdot)} = I_i - H_i[A_i(R^{H_i(\cdot,\cdot)}_{\lambda_i, M_i(\cdot,\cdot)}(\cdot)), B_i(R^{H_i(\cdot,\cdot)}_{\lambda_i, M_i(\cdot,\cdot)}(\cdot))]$, $R^{H_i(\cdot,\cdot)}_{\lambda_i, M_i(\cdot,\cdot)}$ is the resolvent operator associated with the $H_i(A_i, B_i)$-co-accretive mapping M_i. Corresponding to the system of generalized resolvent equations (4.1), they considered and studied a system of variational inclusions as follows: find $(x,y) \in X_1 \times X_2$, $u \in E(x)$ and $v \in F(y)$ such that

$$\left\{ \begin{array}{l} 0 \in S(x,v) + M_1(f_1(x), g_1(x)), \\ 0 \in T(u,y) + M_2(f_2(y), g_2(y)). \end{array} \right. \tag{4.2}$$

By providing an alternative equivalence formulation in which the equivalence between the system (4.2) and a fixed point problem is established, they presented a characterization of a solution of the system (4.2) as follows.

Lemma 4.3 ([1, Lemma 3.1])**.** *Let X_i, A_i, B_i, f_i, g_i, H_i, M_i $(i = 1,2)$, S, T, E and F be the same as in the system* (4.1)*. Then $(x,y,u,v) \in X_1 \times X_2 \times E(x) \times F(y)$ is a solution of the system of variational inclusions* (4.2) *if and only if (x,y,u,v) satisfies*

$$\left\{ \begin{array}{l} x = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 S(x,v)], \\ y = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 T(u,y)], \end{array} \right.$$

where $\lambda_1, \lambda_2 > 0$ are two arbitrary real constants.

Proof. Let us define for each $i \in \{1,2\}$, the mappings $P_i : X_i \to X_i$ and $\widehat{M}_i : X_i \to 2^{X_i}$ by $P_i(x) = H_i(A_i(x_i), B_i(x_i))$ and $\widehat{M}_i(x_i) = M_i(f_i(x_i), g_i(x_i))$ for all $x_i \in X_i$, respectively. Taking into account that for each $i \in \{1,2\}$, M_i is an $H_i(\cdot,\cdot)$-co-accretive mapping with respect to mappings A_i, B_i, f_i and g_i, according to Note (4.1), for each $i \in \{1,2\}$, $H_i(A_i, B_i)$ is a $\mu_i\gamma_i$-symmetric cocoercive mapping with respect to A_i and B_i, and $M_i(f_i, g_i)$ is an $\alpha_i\beta_i$-symmetric accretive mapping with respect to f_i and g_i such that $\mu_i > \gamma_i$ and $\alpha_i > \beta_i$. In the light of these facts, from Propositions 4.1(ii) and 4.3(ii) and the fact that $\eta_i > \sigma_i$, it follows that for each $i \in \{1,2\}$, P_i is a $(\mu_i\eta_i^2 - \gamma_i\sigma_i^2)$-strongly accretive mapping and $\widehat{M}_i$ is an $(\alpha_i - \beta_i)$-strongly P_i-accretive mapping. Then, invoking Lemma 3.1, $(x,y,u,v) \in X_1 \times X_2 \times E(x) \times F(y)$ is a solution of the system

$$\left\{ \begin{array}{l} 0 \in S(x,v) + \widehat{M}_1(x) \\ 0 \in T(u,y) + \widehat{M}_2(y) \end{array} \right. = \left\{ \begin{array}{l} 0 \in S(x,v) + M_1(f_1(x), g_1(x)), \\ 0 \in T(u,y) + M_2(f_2(y), g_2(y)), \end{array} \right.$$

if and only if

$$\left\{ \begin{array}{l} x = R^{P_1}_{\lambda_1, \widehat{M}_1}[P_1(x) - \lambda_1 S(x,v)] \\ y = R^{P_2}_{\lambda_2, \widehat{M}_2}[P_2(y) - \lambda_2 T(u,y)] \end{array} \right.$$

$$= \begin{cases} x = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 S(x, v)], \\ y = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 T(u, y)], \end{cases}$$

where $\lambda_i > 0$ $(i = 1, 2)$ are real constants. This ends the proof of the lemma. □

It is worthwhile to stress that in virtue of the above-mentioned discussion, in contrary of the claim in [1], they, instead of presentation a characterization of a solution for the system (4.2) involving $H_i(\cdot,\cdot)$-co-accretive mappings M_i $(i = 1, 2)$, gave actually a characterization of a solution of the system (3.2) involving P_i-accretive mappings $\widehat{M}_i$ $(i = 1, 2)$.

Employing Lemma 4.3 (that is, [1, Lemma 3.1]), they proved the equivalence between the two systems (4.1) and (4.2).

Lemma 4.4 ([1, Proposition 3.1]). *Let X_i, A_i, B_i, f_i, g_i, H_i, M_i $(i = 1, 2)$, S, T, E and F be the same as in the system* (4.1). *Then the system of variational inclusions* (4.2) *has a solution (x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in E(x)$ and $v \in F(y)$ if and only if the system of generalized resolvent equations* (4.1) *has a solution (z', z'', x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in E(x)$, $v \in F(y)$, $z'_1 \in X_1$ and $z'' \in X_2$, where*

$$\begin{cases} x = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z'), \\ y = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z''), \\ z' = H_1(A_1(x), B_1(x)) - \lambda_1 S(x, v), \\ z'' = H_2(A_2(y), B_2(y)) - \lambda_2 T(u, y). \end{cases}$$

Proof. Let us define the mappings $P_i : X_i \to X_i$ and $\widehat{M}_i : X_i \to 2^{X_i}$ for $i = 1, 2$, the same as in the proof of Lemma 4.3. By a same argument as the proof of Lemma 4.3, it follows that for each $i \in \{1, 2\}$, P_i is a $(\mu_i\eta_i^2 - \gamma_i\sigma_i^2)$-strongly accretive mapping and $\widehat{M}_i$ is an $(\alpha_i - \beta_i)$-strongly P_i-accretive mapping. Then, we observe that all the conditions of Proposition 3.1 hold and so Proposition 3.1 implies that (x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in E(x)$ and $v \in F(y)$ is a solution of the system

$$\begin{cases} 0 \in S(x, v) + \widehat{M}_1(x) \\ 0 \in T(u, y) + \widehat{M}_2(y) \end{cases} = \begin{cases} 0 \in S(x, v) + M_1(f_1(x), g_1(x)), \\ 0 \in T(u, y) + M_2(f_2(y), g_2(y)), \end{cases}$$

if and only if (z', z'', x, y, u, v), where $(z', z'') \in X_1 \times X_2$ is a solution of the system

$$\begin{cases} S(x, v) + \lambda_1^{-1} J^{P_1}_{\lambda_1,\widehat{M}_1}(z') = 0 \\ T(u, y) + \lambda_2^{-1} J^{P_2}_{\lambda_2,\widehat{M}_2}(z'') = 0 \end{cases} = \begin{cases} S(x, v) + \lambda_1^{-1} J^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z') = 0, \\ T(u, y) + \lambda_2^{-1} J^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z'') = 0, \end{cases}$$

satisfying

$$\begin{cases} x = R^{P_1}_{\lambda_1,\widehat{M}_1}(z') = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z'), \\ y = R^{P_2}_{\lambda_2,\widehat{M}_2}(z'') = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z''), \\ z' = P_1(x) - \lambda_1 S(x,v) = H_1(A_1(x),B_1(x)) - \lambda_1 S(x,v), \\ z'' = P_2(y) - \lambda_2 T(u,y) = H_2(A_2(y),B_2(y)) - \lambda_2 T(u,y), \end{cases}$$

where $\lambda_i > 0$ and $R^{P_i}_{\lambda_i,\widehat{M}_i} = R^{H_i(\cdot,\cdot)}_{\lambda_i,M_i(\cdot,\cdot)}$ $(i = 1,2)$ are the same as in the system of generalized resolvent equations (4.1). This gives the desired result. □

In order to approximate a solution of the system (4.1), the authors proposed an iterative algorithm based on Lemma 4.4 (that is, [1, Proposition 3.1]) as follows.

Algorithm 4.1 ([1, Algorithm 3.1]). *Suppose that X_i, A_i, B_i, f_i, g_i, H_i, M_i $(i = 1,2)$, S, T, E and F are the same as in the system* (4.1). *For any given $(x_0,y_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$, $v_0 \in F(y_0)$, $z'_0 \in X_1$ and $z''_0 \in X_2$, compute the sequences $\{z'_n\}_{n=0}^{\infty}$, $\{z''_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by the following iterative schemes*:

$$x_n = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z'_n), \tag{4.3}$$

$$y_n = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z''_n), \tag{4.4}$$

$$u_n \in E(x_n); \|u_{n+1} - u_n\| \le D(E(x_{n+1}),E(x_n)), \tag{4.5}$$

$$v_n \in F(y_n); \|v_{n+1} - v_n\| \le D(F(y_{n+1}),F(y_n)), \tag{4.6}$$

$$z'_{n+1} = H_1(A_1(x_n),B_1(x_n)) - \lambda_1 S(x_n,v_n), \tag{4.7}$$

$$z''_{n+1} = H_2(A_2(y_n),B_2(y_n)) - \lambda_2 T(u_n,y_n), \tag{4.8}$$

where $n = 0,1,2,\ldots$ and $\lambda_1,\lambda_2 > 0$ are two real constants.

By a careful reading Algorithm 4.1, we found that the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ generated by Algorithm 4.1 are not well defined necessarily. In fact, for any given $(x_0,y_0),(z'_0,z''_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$ and $v_0 \in F(y_0)$, the authors [1] computed $(x_n,y_n) \in X_1 \times X_2$ by induction on n using the iterative schemes (4.3) and (4.4), and then they claimed that one can choose $u_{n+1} \in F(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that the relations $\|u_{n+1} - u_n\| \le D(E(x_{n+1}),E(x_n))$ and $\|v_{n+1} - v_n\| \le D(F(y_{n+1}),F(y_n))$ hold. However, if X is a complete metric space and $T : X \to CB(X)$ is a multi-valued mapping, then Lemma 3.2 tells that for any $\varepsilon > 0$ and for any given $x,y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that $d(u,v) \le (1+\varepsilon)D(T(x),T(y))$. Whereas, the following example shows that for any given $x,y \in X$ and $u \in T(x)$, there may not be a point $v \in T(y)$ such that $d(u,v) \le D(T(x),T(y))$.

Example 4.1. Consider $X = l^{\infty}(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} | \sup\limits_{n\in\mathbb{Z}} |z_n| < \infty, z_n \in \mathbb{C}\}$, the Banach space consisting of all bounded complex sequences $z = \{z_n\}_{n=-\infty}^{\infty}$

with the supremum norm $\|z\|_\infty = \sup\limits_{n\in\mathbb{Z}} |z_n|$. Any element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^\infty(\mathbb{Z})$ can be written as follows:

$$\begin{aligned}
z = &\sum_{t\in\{\pm1,\pm3,\dots\}} \Big[(\dots,0,\dots,0,x_{2t-1}+iy_{2t-1},0,x_{2t+1}+iy_{2t+1},0,\dots) \\
&\quad + (\dots,0,\dots,0,x_{2t}+iy_{2t},0,x_{2t+2}+iy_{2t+2},0,\dots)\Big] \\
= &\sum_{t\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2t-1}+y_{2t+1}-i(x_{2t-1}+x_{2t+1})}{2}(\dots,0,\dots,0,i_{2t-1},0,i_{2t+1},0,\dots) \\
&\quad + \frac{y_{2t-1}-y_{2t+1}-i(x_{2t-1}-x_{2t+1})}{2}(\dots,0,\dots,0,i_{2t-1},0,-i_{2t+1},0,\dots) \\
&\quad + \frac{y_{2t}+y_{2t+2}-i(x_{2t}+x_{2t+2})}{2}(\dots,0,\dots,0,i_{2t},0,i_{2t+2},0,\dots) \\
&\quad + \frac{y_{2t}-y_{2t+2}-i(x_{2t}-x_{2t+2})}{2}(\dots,0,\dots,0,i_{2t},0,-i_{2t+2},0,\dots)\Big] \\
= &\sum_{t\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2t-1}+y_{2t+1}-i(x_{2t-1}+x_{2t+1})}{2}\sigma_{2t-1,2t+1} \\
&\quad + \frac{y_{2t-1}-y_{2t+1}-i(x_{2t-1}-x_{2t+1})}{2}\sigma'_{2t-1,2t+1} \\
&\quad + \frac{y_{2t}+y_{2t+2}-i(x_{2t}+x_{2t+2})}{2}\sigma_{2t,2t+2} \\
&\quad + \frac{y_{2t}-y_{2t+2}-i(x_{2t}-x_{2t+2})}{2}\sigma'_{2t,2t+2}\Big],
\end{aligned}$$

where for each $t \in \{\pm1,\pm3,\dots\}$, $\sigma_{2t-1,2t+1} = (\dots,0,\dots,0,i_{2t-1},0,i_{2t+1},0,\dots)$, i at the $(2t-1)$th and $(2t+1)$th coordinates, and all other coordinates are zero, $\sigma'_{2t-1,2t+1} = (\dots,0,\dots,0,i_{2t-1},0,\ -i_{2t+1},0,\dots)$, i and $-i$ at the $(2t-1)$th and $(2t+1)$th places, respectively, and 0's everywhere else, $\sigma_{2t,2t+2} = (\dots,0,\dots,0,i_{2t},0,i_{2t+2},0,\dots)$, i at the $(2t)$th and $(2t+2)$th coordinates, and all other coordinates are zero, and $\sigma'_{2t,2t+2} = (\dots,0,\dots,0,i_{2t},0,-i_{2t+2},0,\dots)$, i and $-i$ in the $(2t)$th and $(2t+2)$th positions, respectively, and 0's elsewhere. Accordingly, the set

$$\mathfrak{B} = \Big\{\sigma_{2t-1,2t+1},\sigma'_{2t-1,2t+1},\sigma_{2t,2t+2},\sigma'_{2t,2t+2} : t = \pm1,\pm3,\dots\Big\}$$

spans the Banach space $l^\infty(\mathbb{Z})$. It can be easily seen that the set $\mathfrak{B}$ is linearly independent and so it is a Schauder basis for the Banach space $l^\infty(\mathbb{Z})$. Let us define the multi-valued mapping $T : X \to CB(X)$ as

$$T(x) = \begin{cases} \{\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}i\}_{n=-\infty}^{\infty}, \sigma_{2t,2t}, \delta'_{2t,2t+2} : & \\ \quad t = \pm1,\pm3,\dots\}, & x \neq \sigma'_{2s,2s+2}, \\ \{\sigma_{2t-1,2t+1}, \sigma'_{2t-1,2t+1} : t = \pm1,\pm3,\dots\}, & x = \sigma'_{2s,2s+2}, \end{cases}$$

where $\gamma \in [-1,0)$ is an arbitrary but fixed real number, α, β, θ are arbitrary but fixed even natural numbers, and $s \in \{\pm 1, \pm 3, \dots\}$ is chosen arbitrarily but fixed. Take $\sigma'_{2s,2s+2} \neq x \in X$ arbitrarily but fixed, $y = \sigma'_{2s,2s+2}$ and $u = \{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i\}_{n=-\infty}^{\infty}$. If $a = \{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i\}_{n=-\infty}^{\infty}$, then taking into account that $\gamma < 0$, for any $t \in \{\pm 1, \pm 3, \dots\}$, we obtain

$$
\begin{aligned}
& d(a, \sigma_{2t-1,2t+1}) \\
&= \|\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i\}_{n=-\infty}^{\infty} - \sigma_{2t-1,2t+1}\|_\infty \\
&= \sup\{|\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}|, |\frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}} - 1|, \\
&\quad |\frac{\gamma}{(2t+1)^{((2t+1)^\beta+1)!}\sqrt[((2t+1)^\theta+2)!]{(2t+1)^\alpha!}} - 1| : n \in \mathbb{Z}, n \neq 2t-1, 2t+1\} \\
&= \begin{cases} |\frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}} - 1|, & \text{if } t \in \{2k+1 | k \in \mathbb{N} \cup \{0\}\}, \\ |\frac{\gamma}{(2t+1)^{((2t+1)^\beta+1)!}\sqrt[((2t+1)^\theta+2)!]{(2t+1)^\alpha!}} - 1|, & \text{if } t \in \{-(2k+1) | k \in \mathbb{N} \cup \{0\}\}, \end{cases} \\
&= \begin{cases} 1 - \frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}}, & \text{if } t \in \{2k+1 | k \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\gamma}{(2t+1)^{((2t+1)^\beta+1)!}\sqrt[((2t+1)^\theta+2)!]{(2t+1)^\alpha!}}, & \text{if } t \in \{-(2k+1) | k \in \mathbb{N} \cup \{0\}\}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& d(a, \sigma'_{2t-1,2t+1}) \\
&= \|\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i\}_{n=-\infty}^{\infty} - \sigma'_{2t-1,2t+1}\|_\infty \\
&= \sup\{|\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}|, |\frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}} - 1|, \\
&\quad |\frac{\gamma}{(2t+1)^{((2t+1)^\beta+1)!}\sqrt[((2t+1)^\theta+2)!]{(2t+1)^\alpha!}} + 1| : n \in \mathbb{Z}, n \neq 2t-1, 2t+1\} \\
&= |\frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}} - 1| \\
&= 1 - \frac{\gamma}{(2t-1)^{((2t-1)^\beta+1)!}\sqrt[((2t-1)^\theta+2)!]{(2t-1)^\alpha!}}.
\end{aligned}
$$

The fact that $\gamma \in [-1,0)$ implies that

$$
\begin{aligned}
d(a, T(y)) = \inf_{b \in T(y)} d(a,b) = \inf \Big\{ 1 - \frac{\gamma}{(2t+\delta)^{((2t+\delta)^\beta+1)!}\sqrt[((2t+\delta)^\theta+2)!]{(2t+\delta)^\alpha!}} : \\
\delta = \pm 1; t = \pm 1, \pm 3, \dots \Big\} = 1.
\end{aligned}
$$

In the case where $a = \sigma_{2m,2m+2}$ for some $m \in \{\pm 1, \pm 3, \dots\}$, then for each $t \in \{\pm 1, \pm 3, \dots\}$, we get

$$d(a, \sigma_{2t-1,2t+1}) = \|\sigma_{2m,2m+2} - \sigma_{2t-1,2t+1}\|_\infty = 1$$

and

$$d(a, \sigma'_{2t-1,2t+1}) = \|\sigma_{2m,2m+2} - \sigma'_{2t-1,2t+1}\|_\infty = 1.$$

Thereby,

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = 1.$$

If $a = \sigma'_{2r,2r+2}$ for some $r \in \{\pm 1, \pm 3, \dots\}$, in virtue of the facts that for each $t \in \{\pm 1, \pm 3, \dots\}$,

$$d(a, \delta_{2t-1,2t+1}) = \|\sigma'_{2r,2r+2} - \sigma_{2t-1,2t+1}\|_\infty = 1$$

and

$$d(a, \delta'_{2t-1,2t+1}) = \|\sigma'_{2r,2r+2} - \sigma'_{2t-1,2t+1}\|_\infty = 1,$$

it follows that

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = 1.$$

These facts ensure that

$$\sup_{a \in T(x)} d(a, T(y)) = 1.$$

If $b = \sigma_{2j-1,2j+1}$ for some $j \in \{\pm 1, \pm 3, \dots\}$, since $\gamma \in [-1, 0)$, we get

$$\begin{aligned}
&d(\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i\}_{n=-\infty}^{\infty}, \sigma_{2j-1,2j+1}) \\
&= \|(\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}} i)_{n=-\infty}^{\infty} - \delta_{2j-1,2j+1}\|_\infty \\
&= \sup\{|\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}|, |\frac{\gamma}{(2j-1)^{((2j-1)^\beta+1)!}\sqrt[((2j-1)^\theta+2)!]{(2j-1)^\alpha!}} - 1|, \\
&\quad |\frac{\gamma}{(2j+1)^{((2j+1)^\beta+1)!}\sqrt[((2j+1)^\theta+2)!]{(2j+1)^\alpha!}} - 1| : n \in \mathbb{Z}, n \neq 2j-1, 2j+1\} \\
&= \begin{cases} |\frac{\gamma}{(2j-1)^{((2j-1)^\beta+1)!}\sqrt[((2j-1)^\theta+2)!]{(2j-1)^\alpha!}} - 1|, & \text{if } j \in \{2k+1 | k \in \mathbb{N} \cup \{0\}\}, \\ |\frac{\gamma}{(2j+1)^{((2j+1)^\beta+1)!}\sqrt[((2j+1)^\theta+2)!]{(2j+1)^\alpha!}} - 1|, & \text{if } t \in \{-(2k+1) | k \in \mathbb{N} \cup \{0\}\}, \end{cases} \\
&= \begin{cases} 1 - \frac{\gamma}{(2j-1)^{((2j-1)^\beta+1)!}\sqrt[((2j-1)^\theta+2)!]{(2j-1)^\alpha!}}, & \text{if } j \in \{2k+1 | k \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\gamma}{(2j+1)^{((2j+1)^\beta+1)!}\sqrt[((2j+1)^\theta+2)!]{(2j+1)^\alpha!}}, & \text{if } j \in \{-(2k+1) | k \in \mathbb{N} \cup \{0\}\}, \end{cases}
\end{aligned}$$

and for each $t \in \{\pm 1, \pm 3, \dots\}$,

$$d(\sigma_{2t,2t+2}, \sigma_{2j-1,2j+1}) = \|\sigma_{2t,2t+2} - \sigma_{2j-1,2j+1}\|_\infty = 1$$

and

$$d(\sigma'_{2t,2t+2},\sigma_{2j-1,2j+1})=\|\sigma'_{2t,2t+2}-\sigma_{2j-1,2j+1}\|_\infty=1.$$

Since $\gamma<0$, we infer that

$$d(T(x),b)=\inf_{a\in T(x)}d(a,b)=1.$$

For the case when $b=\sigma'_{2q-1,2q+1}$ for some $q\in\{\pm1,\pm3,\dots\}$, in the light of the fact that $\gamma\in[-1,0)$, we yield

$$\begin{aligned}
&d(\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}i\}_{n=-\infty}^{\infty},\sigma'_{2q-1,2q+1})\\
=&\ \|(\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}i)_{n=-\infty}^{\infty}-\sigma'_{2q-1,2q+1}\|_\infty\\
=&\ \sup\{|\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}|,|\frac{\gamma}{(2q-1)^{((2q-1)^\beta+1)!}\sqrt[((2q-1)^\theta+2)!]{(2q-1)^\alpha!}}-1|,\\
&|\frac{\gamma}{(2q+1)^{((2q+1)^\beta+1)!}\sqrt[((2q+1)^\theta+2)!]{(2q+1)^\alpha!}}+1|:n\in\mathbb{Z},n\neq 2q-1,2q+1\}\\
=&\ |\frac{\gamma}{(2q-1)^{((2q-1)^\beta+1)!}\sqrt[((2q-1)^\theta+2)!]{(2q-1)^\alpha!}}-1|\\
=&\ 1-\frac{\gamma}{(2q-1)^{((2q-1)^\beta+1)!}\sqrt[((2q-1)^\theta+2)!]{(2q-1)^\alpha!}},
\end{aligned}$$

and for each $t\in\{\pm1,\pm3,\dots\}$,

$$d(\sigma_{2t,2t+2},\sigma'_{2q-1,2q+1})=\|\sigma_{2t,2t+2}-\sigma'_{2q-1,2q+1}\|_\infty=1$$

and

$$d(\sigma'_{2t,2t+2},\sigma'_{2q-1,2q+1})=\|\sigma'_{2t,2t+2}-\sigma'_{2q-1,2q+1}\|_\infty=1.$$

Considering the fact that $\gamma<0$, it follows that

$$d(T(x),b)=\inf_{a\in T(x)}d(a,b)=1.$$

Hence,

$$\sup_{b\in T(y)}d(T(x),b)=1.$$

Thanks to the above-mentioned discussion, we deduce that

$$D(T(x),T(y))=\max\Big\{\sup_{a\in T(x)}d(a,T(y)),\sup_{b\in T(y)}d(T(x),b)\Big\}=1.$$

Owing to the fact that $\gamma\in[-1,0)$, we derive that for each $t\in\{\pm1,\pm3,\dots\}$,

$$\|\{\frac{\gamma}{n^{(n^\beta+1)!}\sqrt[(n^\theta+2)!]{n^\alpha!}}i\}_{n=-\infty}^{\infty}-\sigma_{2t-1,2t+1}\|_\infty$$

$$=\begin{cases} 1-\dfrac{\gamma}{(2t-1)^{((2t-1)^{\beta}+1)!\ ((2t-1)^{\theta}+2)!}\sqrt{(2t-1)^{\alpha}!}} > 1, & \text{if } t\in\{2k+1\,|\,k\in\mathbb{N}\cup\{0\}\},\\ 1-\dfrac{\gamma}{(2t+1)^{((2t+1)^{\beta}+1)!\ ((2t+1)^{\theta}+2)!}\sqrt{(2t+1)^{\alpha}!}} > 1, & \text{if } t\in\{-(2k+1)\,|\,k\in\mathbb{N}\cup\{0\}\},\end{cases}$$

and

$$\begin{aligned}&\|\{\frac{\gamma}{n^{(n^{\beta}+1)!\ (n^{\theta}+2)!}\sqrt{n^{\alpha}!}}i\}_{n=-\infty}^{\infty}-\sigma'_{2t-1,2t+1}\|_{\infty}\\ &=1-\frac{\gamma}{(2t-1)^{((2t-1)^{\beta}+1)!\ ((2t-1)^{\theta}+2)!}\sqrt{(2t-1)^{\alpha}!}} > 1.\end{aligned}$$

These facts imply that for any $v\in T(y)$,

$$d(u,v)=\|u-v\|_{\infty} > D(T(x),T(y)).$$

It should be remarked that if $T(y)$ is a compact subset of X, then such a point v does exist. In fact, if $T: X\to C(X)$, where $C(X)$ denotes the family of all the nonempty compact subsets of X, then for any given $x,y\in X$ and $u\in T(x)$, there exists $v\in T(y)$ such that $d(u,v)\le D(T(x),T(y))$. Due to the above-mentioned arguments, by rewriting the two relations (4.5) and (4.6), we now present the correct version of Algorithm 4.1 as follows.

Algorithm 4.2. *Let X_i, A_i, B_i, f_i, g_i, H_i, M_i $(i=1,2)$, S, T, E and F be the same as in the system* (4.1). *For any given $(x_0,y_0)\in X_1\times X_2$, $u_0\in E(x_0)$, $v_0\in F(y_0)$, $z'_0\in X_1$ and $z''_0\in X_2$, compute the sequences $\{z'_n\}_{n=0}^{\infty}$, $\{z''_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by the following iterative schemes*:

$$\begin{cases} x_n=R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z'_n),\\ y_n=R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z''_n),\\ u_n\in E(x_n); \|u_{n+1}-u_n\|_1\le (1+\frac{1}{1+n})D_1(E(x_{n+1}),E(x_n)),\\ v_n\in F(y_n); \|v_{n+1}-v_n\|_2\le (1+\frac{1}{1+n})D_2(F(y_{n+1}),F(y_n)),\\ z'_{n+1}=H_1(A_1(x_n),B_1(x_n))-\lambda_1 S(x_n,v_n),\\ z''_{n+1}=H_2(A_2(y_n),B_2(y_n))-\lambda_2 T(u_n,y_n),\end{cases}$$

where $n=0,1,2,\dots$; $\lambda_1,\lambda_2>0$ are positive real constants and for $i=1,2$, D_i is the Hausdorff metric on $CB(X_i)$.

By rewriting the system of generalized resolvent equations (4.1) and obtaining a new format of it as

$$\begin{aligned} z'&=H_1(A_1(x),B_1(x))-S(x,v)+(I_1-\lambda_1^{-1})J^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(z'),\\ z''&=H_2(A_2(y),B_2(y))-T(u,y)+(I_2-\lambda_2^{-1})J^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(z''),\end{aligned}$$

Ahmad and Akram [1] used the above new fixed point formulations and constructed Algorithm 3.2 in [1]. But, by an argument analogous to the previous one, mentioned for Algorithm 4.1, one can deduce that Algorithm 3.2 in [1] does not work. Indeed, because of (4.5) and (4.6) have been also used in [1, Algorithm 3.2] to define the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$,

$\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$, by a similar argument given as above, we observe that the aforesaid iterative sequences generated by [1, Algorithm 3.2] are not well defined. By editing (4.5) and (4.6) and rewriting them the same as in Algorithm 4.2, we now present the correct version of [1, Algorithm 3.2] as follows.

Algorithm 4.3. *Assume that X_i, A_i, B_i, f_i, g_i, H_i, M_i $(i = 1, 2)$, S, T, E and F are the same as in the system (4.1). For any given $(x_0, y_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$, $v_0 \in F(y_0)$, $z'_0 \in X_1$ and $z''_0 \in X_2$, compute the sequences $\{z'_n\}_{n=0}^{\infty}$, $\{z''_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by the following iterative schemes*:

$$\begin{cases} x_n = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}(z'_n), \\ y_n = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}(z''_n), \\ u_n \in E(x_n); \|u_{n+1} - u_n\|_1 \leq (1 + \frac{1}{1+n})D_1(E(x_{n+1}), E(x_n)), \\ v_n \in F(y_n); \|v_{n+1} - v_n\|_2 \leq (1 + \frac{1}{1+n})D_2(F(y_{n+1}), F(y_n)), \\ z'_{n+1} = H_1(A_1(x_n), B_1(x_n)) - S(x_n, v_n) + (I_1 - \lambda_1^{-1})J^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}(z'_n), \\ z''_{n+1} = H_2(A_2(y_n), B_2(y_n)) - T(u_n, y_n) + (I_2 - \lambda_2^{-1})J^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}(z''_n), \end{cases}$$

where for $i = 1, 2$, I_i is the identity mapping on X_i, and λ_i, D_i are the same as in Algorithm 4.2.

Note, in particular, that by defining the mappings $P_i : X_i \to X_i$ and $\widehat{M_i} : X_i \to 2^{X_i}$, for each $i \in \{1, 2\}$, by $P_i(x_i) := H_i(A_i x_i, B_i x_i)$ and $\widehat{M_i}(x_i) = M_i(f_i(x_i), g_i(x_i))$ for all $x_i \in X_i$, respectively, the same argument used in the proof of Lemma 4.2 shows that for each $i \in \{1, 2\}$, P_i is a $(\mu_i \eta_i^2 - \gamma_i \sigma_i^2)$-strongly accretive mapping and $\widehat{M_i}$ is an $(\alpha_i - \beta_i)$-strongly P_i-accretive mapping. Then, Algorithms 4.2 and 4.3 become actually the same Algorithms 3.1 and 3.2, respectively, and are not new ones.

Finally, the authors [1] concluded their paper with an assertion regarding the existence of a solution for the system (4.1) and the convergence of the iterative sequences generated by their suggested algorithm to the solution of the above-mentioned system. Before proceeding to the main result in [1], we need to recall the following notion presented in [1].

Definition 4.5 ([1, Definition 2.6])**.** Let $A, B : X \to X$ and $H : X \times X \to X$ be the mappings. Then

(i) $H(A, \cdot)$ is said to be generalized pseudocontractive with respect to A, if there exists a constant $s > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), j(x - y)\rangle \leq s\|x - y\|^2, \quad \forall x, y, u \in X, \ j(x - y) \in \mathcal{F}(x - y);$$

(ii) $H(\cdot, B)$ is said to be generalized pseudocontractive with respect to B, if there exits a constant $t > 0$ such that

$$\langle H(u, Bx) - H(u, By), j(x-y)\rangle \le t\|x-y\|^2, \quad \forall x, y, u \in X,\ j(x-y) \in \mathcal{F}(x-y).$$

Remark 4.2. It should be pointed out that if bifunction $H : X \times X \to X$ is s-generalized pseudocontractive and t-generalized pseudocontractive with respect to mappings $A : X \to X$ and $B : X \to X$, respectively, then the univariate mapping $P : X \to X$ defined by $P(x) := H(Ax, Bx)$ for all $x \in X$ is $(s+t)$-generalized pseudocontractive, because of

$$\begin{aligned}\langle P(x) - P(y), j(x-y)\rangle &= \langle H(Ax, Bx) - H(Ay, By), j(x-y)\rangle \\ &= \langle H(Ax, Bx) - H(Ay, Bx), j(x-y)\rangle \\ &\quad + \langle H(Ay, Bx) - H(Ay, By), j(x-y)\rangle \\ &\le s\|x-y\|^2 + t\|x-y\|^2 \\ &= (s+t)\|x+y\|^2.\end{aligned}$$

In the light of this fact, we found that the notion of generalized pseudocontractiveness of the mapping $H : X \times X \to X$ with respect to A and B given in Definition 4.5 is exactly the same concept of generalized pseudocontractiveness of the mapping $P : X \to X$ defined as above, presented in Definition 3.2 and is not a new one.

Ahmad and Akram [1] finally proved the existence of a solution for the system of resolvent equations (4.1) and studied the convergence analysis of the sequences generated by their proposed iterative algorithm under some appropriate conditions imposed on the parameters and mappings. Before dealing with the main result given in [1], we need to define the following notion.

Definition 4.6. For given mappings $A, B : X \to X$, a mapping $H : X \times X \to X$ is said to be r-mixed contraction with respect to A and B if there exists a constant $r \in (0, 1)$ such that

$$\|H(Ax, Bx) - H(Ay, By)\| \le r\|x-y\|, \ \forall x, y \in X.$$

Theorem 4.1 ([1, Theorem 3.1])**.** *Let for each $i \in \{1, 2\}$, X_i be a real Banach space and let $H_i : X_i \times X_i \to X_i$ be r_i-mixed contraction with respect to A_i and B_i, s_i-generalized pseudocontractive with respect to A_i and t_i-generalized pseudocontractive with respect to B_i. Suppose that $S : X_1 \times X_2 \to X_1$ is λ_{S_1}-Lipschitz continuous and λ_{S_2}-Lipschitz continuous in the first and second arguments, respectively, and $T : X_1 \times X_2 \to X_2$ is λ_{T_1}-Lipschitz continuous and λ_{T_2}-Lipschitz continuous in the first and second arguments, respectively. Let $E : X_1 \to CB(X_1)$ and $F : X_2 \to CB(X_2)$ be λ_{D_E}-D_1-Lipschitz continuous and λ_{D_F}-D_2-Lipschitz continuous, respectively, and let for each $i \in \{1, 2\}$, $M_i : X_i \times X_i \to 2^{X_i}$ be an $H_i(A_i, B_i)$-co-accretive mapping with respect to $A_i, B_i, f_i, g_i : X_i \to X_i$ with constants $\mu_i, \gamma_i, \alpha_i, \beta_i$, respectively, such that $\alpha_i >$*

β_i and $\mu_i > \gamma_i$. Suppose further that for each $i \in \{1,2\}$, A_i is η_i-expansive and B_i is σ_i-Lipschitz continuous such that $\eta_i > \sigma_i$. If there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$\left\{ \begin{array}{l} 0 < L_1(K_1 + \sqrt{\theta_1} + \sqrt{\theta_3}) < 1, \\ 0 < L_2(K_2 + \sqrt{\theta_2} + \sqrt{\theta_4}) < 1, \end{array} \right. \tag{4.9}$$

where

$$K_i = \sqrt{\frac{1 + 2(s_i + t_i) + 3r_i}{1 - r_i}}, \; L_i = \frac{1}{\lambda_i(\alpha_i - \beta_i) + \mu_i \eta_i^2 - \gamma_i \sigma_i^2} \; (i = 1, 2),$$

$$\theta_1 = \frac{1 + \lambda_1 \lambda_{S_1}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F})}, \; \theta_2 = \frac{\lambda_1 \lambda_{S_2} \lambda_{D_F}}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F})},$$

$$\theta_3 = \frac{\lambda_2 \lambda_{T_1} \lambda_{D_E}}{1 - \lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2})}, \; \theta_4 = \frac{1 + \lambda_2 \lambda_{T_2}}{1 - \lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2})},$$

$$\lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}) < 1, \; \lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2}) < 1,$$

then the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{z'_n\}_{n=0}^{\infty}$ and $\{z''_n\}_{n=0}^{\infty}$ generated by Algorithm 4.2 converge strongly to x, y, u, v, z' and z'', respectively, and (x, y, u, v, z', z'') is a solution of the system (4.1).

Proof. Let us define the mappings $P_i : X_i \to X_i$ and $\widehat{M_i} : X_i \to 2^{X_i}$ for each $i \in \{1,2\}$ as $P_i(x_i) = H_i(A(x_i), B(x_i))$ and $\widehat{M_i}(x_i) = M_i(f_i(x_i), g_i(x_i))$ for all $x_i \in X_i$, respectively. By the argument similar to that of Lemma 4.2, one can deduce that for each $i \in \{1,2\}$, P_i is a $(\mu_i\eta_i^2 - \gamma_i\sigma_i^2)$-strongly accretive mapping and $\widehat{M_i}$ is an $(\alpha_i - \beta_i)$-strongly P_i-accretive mapping. Since for each $i \in \{1,2\}$, H_i is r_i-mixed contraction with respect to mappings A_i and B_i, s_i-generalized pseudocontractive with respect to A_i and t_i-generalized pseudocontractive with respect to B_i, it follows that for each $i \in \{1,2\}$, P_i is r_i-contraction, and thanks to Remark 4.2, P_i is $(s_i + t_i)$-generalized pseudocontractive. Then, taking $\zeta_i = \mu_i\eta_i^2 - \gamma_i\sigma_i^2$, $\varrho_i = \alpha_i - \beta_i$ and $\varsigma_i = s_i + t_i$ for each $i \in \{1,2\}$, we observe that (4.9) becomes actually the same (3.5). Furthermore, Algorithm 4.2 coincides exactly with Algorithm 3.1. Now, all the conditions of Theorem 3.1 are satisfied, and so the conclusion follows from Theorem 3.1 immediately. □

Remark 4.3. It is important to mention that by a careful reading the proof of [1, Theorem 3.1], we found that there are some errors and small mistakes in its context which must be resolved. Firstly, the authors assumed that for each $i \in \{1,2\}$, the mapping H_i is r_i-mixed Lipschitz continuous with respect to mappings A_i and B_i. Obviously, the fact for each $i \in \{1,2\}$, $K_i = \sqrt{\frac{1+2(s_i+t_i)+3r_i}{1-r_i}}$ implies that $r_i \in (0,1)$. Owing to the fact that every r-Lipschitz continuous mapping with $r \in (0,1)$ is contraction, it follows that for each $i \in \{1,2\}$, the mapping H_i is actually a r_i-mixed contraction with respect to A_i and B_i. In virtue of this fact, in context of [1, Theorem 3.1], the assumption of r_i-mixed Lipschitz continuity of the mapping H_i with respect to mappings A_i and B_i $(i = 1, 2)$ must be replaced by the assumption that for each $i \in \{1, 2\}$, the mapping H_i is r_i-mixed contraction with respect to A_i and B_i, as we have done in the context of Theorem 4.1. Secondly, the authors [1] assumed that the mappings M_1 and M_2 are Lipschitz continuous with constants $L_1 = \frac{1}{\lambda_1(\alpha_1-\beta_1)+\mu_1\eta_1^2-\gamma_1\sigma_1^2}$ and $L_2 = \frac{1}{\lambda_2(\alpha_2-\beta_2)+\mu_2\eta_2^2-\gamma_2\sigma_2^2}$, respectively, whereas the sufficient conditions for the multi-valued mappings M_1 and M_2 to be Lipschitz continuous with constants L_1 and L_2, where L_1 and L_2 are the same as in above, are stated in Lemma 4.2. According to Lemma 4.2, for each $i \in \{1, 2\}$, the multi-valued mapping $M_i : X_i \times X_i \to 2^{X_i}$ needs to be an $H_i(\cdot, \cdot)$-co-accretive mapping with respect to A_i, B_i, f_i and g_i with constants $\mu_i, \gamma_i, \alpha_i$ and β_i, respectively, such that $\alpha_i > \beta_i$ and $\mu_i > \gamma_i$, and the mappings A_i and B_i need to be η_i-expansive and σ_i-Lipschitz continuous, respectively, such that $\eta_i > \sigma_i$. Hence, in the context of [1, Theorem 3.1], the assumption that the mappings M_1 and M_2 are Lipschitz continuous with constants $L_1 = \frac{1}{\lambda_1(\alpha_1-\beta_1)+\mu_1\eta_1^2-\gamma_1\sigma_1^2}$ and $L_2 = \frac{1}{\lambda_2(\alpha_2-\beta_2)+\mu_2\eta_2^2-\gamma_2\sigma_2^2}$, respectively, must be replaced by the above-mentioned conditions related to the mappings A_i, B_i, f_i, g_i and M_i, as we have done in the context of Theorem 4.1. Thirdly, in view of (4.9), the assumptions of $\lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{D_F}) < 1$ and $\lambda_2(\lambda_{T_1}\lambda_{D_E} + \lambda_{T_2}) < 1$ must be added to the context of Theorem 3.1 of [1], as we have added to the context of Theorem 4.1. At the same time, as we have already shown, Algorithm 4.1 (that is, [1, Algorithm 3.1]) does not work and Algorithm 4.2 is its correct version. Hence, in the context of Theorem 3.1 of [1], Algorithm 4.1 must be replaced by Algorithm 4.2, as we have done in the context of Theorem 4.1.

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