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BACH ALMOST SOLITONS IN PARASASAKIAN GEOMETRY

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Abstract. If a paraSasakian manifold of dimension $(2n+1)$ represents Bach almost solitons, then the Bach tensor is a scalar multiple of the metric tensor and the manifold is of constant scalar curvature. Additionally it is shown that the Ricci operator of the metric g has a constant norm. Next, we characterize 3-dimensional paraSasakian manifolds admitting Bach almost solitons and it is proven that if a 3-dimensional paraSasakian manifold admits Bach almost solitons, then the manifold is of constant scalar curvature. Moreover, in dimension 3 the Bach almost solitons are steady if $r = -6$; shrinking if $r > -6$; expanding if $r < -6$.

1. Introduction

ParaSasakian (in short, ps) manifolds were introduced by Adati and Matsumoto in 1977 [1] as a special case of an almost paracontact (in short, apc) manifold introduced by Sato [26]. However, in [21] Kaneyuki and Kozai defined an apc manifold on pseudo-Riemannian manifold $\mathcal{N}$ of dimension $(2n+1)$ and constructed the almost paracomplex shape on $\mathcal{N}^{2n+1} \times \mathbb{R}$. The primary difference among an apc manifold within the sense of Sato [26] and Kaneyuki et al. [22] is the signature of the metric. In 2009, Zamkovoy [29] defined a ps manifold as a normal paracontact manifold by taking pseudo-Riemannian metric and the author obtains a condition for a paracontact manifold to be a paraSasakian manifold.

Bach tensor was introduced [2] by Bach to observe conformal geometry in early 1920's and showed that Bach tensor is a trace-free tensor of rank 2 which is conformally invariant in dimension 4. Therefore, as an alternative of the Hilbert-Einstein functional, one chooses the functional

$$\mathcal{W}(g) = \int_{\mathcal{N}} \| \mathbf{C} \|_g^2 \, d\mu_g \tag{1}$$

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for a 4-dimensional manifold, where $\mathbf{C}$ denotes the Weyl tensor of type (1,3) defined by

$$(2)\qquad \mathbf{C}(Z_1, Z_2)Z_3 = \mathbf{R}(Z_1, Z_2)Z_3 - \frac{1}{2n-1}[\mathbf{S}(Z_2, Z_3)Z_1 - \mathbf{S}(Z_1, Z_3)Z_2 + g(Z_2, Z_3)\mathbf{Q}Z_1 - g(Z_1, Z_3)\mathbf{Q}Z_2] + \frac{r}{2n(2n-1)}[g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2],$$

where $\mathbf{R}$ denotes the Riemannian curvature tensor, $\mathbf{S}$ being the Ricci tensor and $\mathbf{Q}$ indicates the Ricci operator defined by $g(\mathbf{Q}Z_1, Z_2) = \mathbf{S}(Z_1, Z_2)$ for all smooth vector fields Z_1, Z_2 and Z_3. On any Riemannian manifold $(\mathcal{N}, g)$ of dimension $(2n+1)$, the Bach tensor $\mathcal{B}$ of type (0,2) is defined by

$$(3)\qquad \mathcal{B}(Z_1, Z_2) = \frac{1}{2n-2}\sum_{i,j=1}^{n}((\nabla_{v_i}\nabla_{v_j}\mathbf{C})(Z_1, v_i)v_j, Z_2) + \frac{1}{2n-1}\sum_{i,j=1}^{n}\mathbf{S}(v_i, v_j)\mathbf{C}(Z_1, v_i, v_j, Z_2),$$

$\{v_i\}_{i=1}^{2n+1}$ being a local orthonormal basis of the tangent space at each point of the manifold.

Utilizing the expression of Cotton tensor [28]

$$(4)\qquad \mathbf{C}_0(Z_1, Z_2)Z_3 = (\nabla_{Z_1}\mathbf{S})(Z_2, Z_3) - (\nabla_{Z_2}\mathbf{S})(Z_1, Z_3) - \frac{1}{4n}[(Z_1 r)g(Z_2, Z_3) - (Z_2 r)g(Z_1, Z_3)],$$

and the Weyl tensor (2), the Bach tensor can be expressed as

$$(5)\qquad \mathcal{B}(Z_1, Z_2) = \frac{1}{2n-1}\sum_{i=1}^{n}[(\nabla_{v_i}\mathbf{C}_0)(v_i, Z_1)Z_2) + \mathbf{S}(v_i, v_i)\mathbf{C}(Z_1, v_i, v_i, Z_2)]$$

for all smooth vector fields Z_1 and Z_2.

For more information about Bach tensor, we confer the reader ([5, 12, 14, 16, 18, 27]) and references therein.

In [11], Das and Kar investigate the various aspects of Bach flows on product manifolds. They compare their results with Ricci flow. In general relativity, such flow is proposed in [4] to characterize the Harava-Lifschitz gravity and short time existence of such flow was investigated by Bahuaud and Helliwell in [3]. In 2013, Cao and Chen published their work on Bach flat gradient shrinking Ricci solitons in [10]. Later, the author Ho [20] studied depthly the solitons of the Bach flow and bach flows on a four dimensional Lie group. On four dimension the Bach flow was also characterized by Helliwell on locally homogeneous product manifolds in 2020 [19]. Recently Ghosh [17] studied Bach almost solitons in Riemannian geometry, defined by

$$(6)\qquad (\pounds_V g)(Z_1, Z_2) + 2\mathcal{B}(Z_1, Z_2) = 2\lambda g(Z_1, Z_2),$$

$\pounds_V$ being the Lie derivative along the vector field V and V is the potential vector field; λ is the soliton constant.

The present paper is structured as follows: Section 2, contains some basic formulas of paraSasakian manifolds. Section 3 deals with the study of Bach almost solitons in a paraSasakian manifold. Finally, we consider 3-dimensional paraSasakian manifolds admitting Bach almost solitons.

2. ParaSasakian manifolds

Let $\mathcal{N}^{2n+1}$ be a $(2n+1)$-dimensional smooth manifold. If there exit a tensor field ϕ of type $(1,1)$, a vector field ζ and a 1-form τ on $\mathcal{N}^{2n+1}$ fulfilling the following relation ([6, 7, 26])

$$\phi^2 = I - \tau \otimes \zeta, \quad \eta(\zeta) = 1, \ \phi\zeta = 0, \ \tau \circ \phi = 0, \tag{7}$$

where I is the identity transformation, then the triplet (ϕ, ζ, τ) is an apc structure and the manifold is an apc manifold.

If an apc manifold $\mathcal{N}^{2n+1}$ with an apc structure (ϕ, ζ, τ) admits a pseudo-Riemannian metric g such that [21]

$$g(Z_1, Z_2) = -g(\phi Z_1, \phi Z_2) + \tau(Z_1)\tau(Z_2) \tag{8}$$

for all vector fields Z_1 and Z_2, then we say that $\mathcal{N}^{2n+1}$ is an apc structure (ϕ, ζ, τ, g) and such a metric g is called a compatible metric. The fundamental 2-form of $\mathcal{N}^{2n+1}$ is defined by

$$\Phi(Z_1, Z_2) = g(Z_1, \phi Z_2).$$

An apc metric structure becomes a paracontact metric structure if

$$d\tau(Z_1, Z_2) = g(Z1, \phi Z_2)$$

for all vector fields Z_1 and Z_2, where

$$d\tau(Z_1, Z_2) = \frac{1}{2}[Z_1\tau(Z_2) - Z_2\tau(Z_1) - \tau([Z_1, Z_2])],$$

$[Z_1, Z_2]$ being the Lie bracket of Z_1 and Z_2.

Paracontact manifolds have been studied by numerous authors such as Calvaruso ([8, 9]), Mertin-Molina [25], Kaneyuki and Willams [22], Zamkovoy et al. [30] and lots of others.

An apc manifold is called normal ([23, 29]) if and only if the tensor $N_\phi - 2d\tau \otimes \xi$ vanishes identically, N_ϕ being the Nijenhuis tensor of $\phi : N_\phi(Z_1, Z_2) = [\phi, \phi](Z_1, Z_2) = \phi^2[Z_1, Z_2] + [\phi Z_1, \phi Z_2] - \phi[\phi Z_1, Z_2] - \phi[Z_1, \phi Z_2]$. A normal paracontact metric manifold is known as a paraSasakian manifold. It is known [29] that an apc manifold is a paraSasakian manifold if and only if

$$(\nabla_{Z_1}\phi)Z_2 = -g(Z_1, Z_2)\zeta + \tau(Z_2)Z_1 \tag{9}$$

for all vector fields Z_1, Z_2, where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric. From the above equation it follows that

$$\nabla_{Z_1}\zeta = -\phi Z_1. \tag{10}$$

Moreover, in a ps manifold the curvature tensor $\mathbf{R}$, the Ricci tensor $\mathbf{S}$ and the Ricci operator $\mathbf{Q}$ defined by $g(\mathbf{Q}Z_1, Z_2) = \mathbf{S}(Z_1, Z_2)$ satisfy [29]

$$\mathbf{R}(Z_1, Z_2)\zeta = -(\tau(Z_2)Z_1 - \tau(Z_1)Z_2), \tag{11}$$

$$\mathbf{R}(\zeta, Z_1)Z_2 = -g(Z_1, Z_2) + \tau(Z_2)Z_1, \tag{12}$$

$$\mathbf{S}(Z_1, \zeta) = -2n\tau(Z_1), \tag{13}$$

$$\mathbf{Q}\zeta = -2n\zeta. \tag{14}$$

ParaSasakian manifolds have been studied by several authors such as Ghosh et al. [13], De and Sarkar [15], Erken [24], Zamkovoy [29] and many others.

Zamkovoi [29] proved the following:

Proposition 2.1. *Let $\mathcal{N}^{2n+1}$ be a paraSasakian manifold. Then*

$$\boldsymbol{S}(Z_1, \phi Z_2) = -\boldsymbol{S}(\phi Z_1, Z_2) - g(Z_1, \phi Z_2) \tag{15}$$

for all smooth vector fields Z_1 and Z_2.

3. Bach almost solitons and paraSasakian manifolds

Let (g, ζ, λ) be a Bach almost solitons in a $(2n+1)$-dimensional ps manifold $\mathcal{N}$. Then

$$(\pounds_\zeta g)(Z_1, Z_2) + 2B(Z_1, Z_2) = 2\lambda g(Z_1, Z_2). \tag{16}$$

Using (10), we obtain

$$(\pounds_\zeta g)(Z_1, Z_2) = g(\nabla_{Z_1}\zeta, Z_2) + g(Z_1, \nabla_{Z_2}\zeta) = 0. \tag{17}$$

Using (17) in (16) yields

$$B(Z_1, Z_2) = \lambda g(Z_1, Z_2).$$

This leads to the following:

Theorem 3.1. *Let (g, ζ, λ) be a Bach almost solitons in a $(2n+1)$-dimensional paraSasakian manifold $\mathcal{N}$. Then the Bach tensor is a scalar multiple of the metric tensor.*

Setting Z_3 by ζ in (2) yields

$$\begin{aligned}\mathbf{C}(Z_1, Z_2)\zeta = \mathbf{R}(Z_1, Z_2)\zeta - \frac{1}{2n-1}[&\mathbf{S}(Z_2, \zeta)Z_1 - \mathbf{S}(Z_1, \zeta)Z_2 + \tau(Z_2)\mathbf{Q}Z_1 \\ &- \tau(Z_1)\mathbf{Q}Z_2] + \frac{r}{2n(2n-1)}[\tau(Z_2)Z_1 - \tau(Z_1)Z_2]\end{aligned} \tag{18}$$

for all smooth vector fields Z_1 and Z_2.

Operating $\mathbf{Q}$ on both sides of (18) and using (11), (13), we have

$$\begin{aligned}\mathbf{Q}(\mathbf{C}(Z_1, Z_2)\zeta) = [1 - \frac{2n}{2n-1} + \frac{r}{2n(2n-1)}](&\tau(Z_2)\mathbf{Q}Z_1 - \tau(Z_1)\mathbf{Q}Z_2) \\ &- \frac{1}{2n-1}(\tau(Z_2)\mathbf{Q}^2 Z_1 - \tau(Z_1)\mathbf{Q}^2 Z_2).\end{aligned} \tag{19}$$

Taking an orthonormal basis $\{v_i\}$ and replacing Z_2 and Z_4 by v_i, we obtain

$$\sum_{i=1}^{2n+1} g(\mathbf{Q}(\mathbf{C}(Z_1, v_i)\zeta), v_i) = -\frac{r^2 - 4n^2}{2n(2n-1)}\tau(Z_1) + [\frac{\mid \mathbf{Q} \mid^2 - 4n^2}{2n-1}]\tau(Z_1). \tag{20}$$

Substituting ζ for Z_3 in (4) we get

$$\begin{aligned} \mathbf{C}_0(Z_1, Z_2)\zeta = {} & g((\nabla_{Z_1}\mathbf{Q})Z_2, \zeta) - g((\nabla_{Z_2}\mathbf{Q})Z_1, \zeta) \\ & - \frac{1}{4n}[(Z_1 r)\tau(Z_2) - (Z_2 r)\tau(Z_1)]. \end{aligned} \tag{21}$$

From Proposition 2.1, it follows that

$$\phi\mathbf{Q}Z_1 = \mathbf{Q}\phi Z_1 - \phi Z_1. \tag{22}$$

Combining (10) and (14), we have

$$(\nabla_{Z_1}\mathbf{Q})\zeta = 2n\phi Z_1 + \mathbf{Q}\phi Z_1. \tag{23}$$

From the preceding equation, it follows that

$$g((\nabla_{Z_1}\mathbf{Q})Z_2, \zeta) = 2ng(\phi Z_1, Z_2) + g(\mathbf{Q}\phi Z_1, Z_2). \tag{24}$$

Using (24) in (21) yields

$$\begin{aligned} \mathbf{C}_0(Z_1, Z_2)\zeta = {} & 2ng(\phi Z_1, Z_2) + g(\mathbf{Q}\phi Z_1, \phi Z_2) \\ & - 2ng(\phi Z_2, Z_1) - g(\mathbf{Q}\phi Z_2, Z_1) + g(\mathbf{Q}Z_2, \phi Z_1) \\ & + g(Z_2, \phi Z_1) - \frac{1}{4n}[(Z_1 r)\tau(Z_2) - (Z_2 r)\tau(Z_1)]. \end{aligned} \tag{25}$$

Differentiating (25) along the vector field Z_4, provides

$$\begin{aligned} (\nabla_{Z_4}\mathbf{C}_0)(Z_1, Z_2)\zeta = {} & \nabla_{Z_4}\mathbf{C}_0(Z_1, Z_2)\zeta - \mathbf{C}_0(\nabla_{Z_4}Z_1, Z_2)\zeta \\ & - \mathbf{C}_0(Z_1, \nabla_{Z_4}Z_2)\zeta - \mathbf{C}_0(Z_1, Z_2)\nabla_{Z_4}\zeta. \end{aligned} \tag{26}$$

Utilizing (10) and (25) in (26), we obtain

$$\begin{aligned} (\nabla_{Z_4}\mathbf{C}_0)(Z_1, Z_2)\zeta = {} & 2ng((\nabla_{Z_4}\phi)Z_1, Z_2) - g((\nabla_{Z_4}\mathbf{Q})Z_1, \phi Z_2) \\ & - g(\mathbf{Q}X, (\nabla_{Z_4}\phi)Z_2) - g(Z_1, (\nabla_{Z_4}\phi)Z_2) \\ & - 2ng((\nabla_{Z_4}\phi)Z_2, Z_1) + g((\nabla_{Z_4}\mathbf{Q})Z_2, \phi Z_4) \\ & + g(\mathbf{Q}Z_2, (\nabla_{Z_4}\phi)Z_1) + g(Z_2, (\nabla_{Z_4}\phi)Z_1) \\ & - \frac{1}{4n}[g(\nabla_{Z_4}Dr, Z_1)\tau(Z_2) - g(\nabla_{Z_4}Dr, Z_2)\tau(Z_1) \\ & - g(\phi Z_4, Z_2)(Z_1 r) + g(\phi Z_4, Z_1)(Z_2 r)], \end{aligned} \tag{27}$$

where D denotes the gradient operator.

From (25) we can easily obtain the following

$$\begin{aligned} \mathbf{C}_0(\nabla_{Z_4}Z_1, Z_2)\zeta = {} & 2ng(\phi\nabla_{Z_4}Z_1, Z_2) - g(\mathbf{Q}\nabla_{Z_4}Z_1, \phi Z_2) \\ & - g(\nabla_{Z_4}Z_1, \phi Z_2) - 2ng(\phi Z_2, \nabla_{Z_4}Z_1) \\ & + g(\mathbf{Q}Z_2, \phi\nabla_{Z_4}Z_1) + g(Z_2, \phi\nabla_{Z_4}Z_1) \end{aligned} \tag{28}$$

$$- \frac{1}{4n}[((\nabla_{Z_4} Z_1)r)\tau(Z_2) - (Z_2 r)\tau(\nabla_{Z_4} Z_1)].$$

Similarly, from (25), we can obtain

$$\begin{aligned} \mathbf{C}_0(Z_1, \nabla_{Z_4} Z_2)\zeta = {} & 2ng(\phi Z_1, \nabla_{Z_4} Z_2) - g(\mathbf{Q}Z_1, \phi\nabla_{Z_4} Z_2) \\ & - g(Z_1, \phi\nabla_{Z_4} Z_2) - 2ng(\phi\nabla_{Z_4} Z_2, Z_1) \\ & + g(\mathbf{Q}\nabla_{Z_4} Z_2, \phi Z_1) + g(\nabla_{Z_4} Z_2, \phi Z_1) \\ & - \frac{1}{4n}[(Z_1 r)\tau(\nabla_{Z_4} Z_2) - ((\nabla_{Z_4} Z_2)r)\tau(Z_1)]. \end{aligned} \tag{29}$$

Again from (4), we infer

$$\begin{aligned} \mathbf{C}_0(Z_1, Z_2)\nabla_{Z_4}\zeta = {} & (\nabla_{Z_1}\mathbf{S})(Z_2, \phi Z_4) - (\nabla_{Z_2}\mathbf{S})(Z_1, \phi Z_4) \\ & - \frac{1}{4n}[(Z_1 r)g(Z_2, \phi Z_4) - (Z_2 r)g(Z_1, \phi Z_4)]. \end{aligned} \tag{30}$$

Utilizing (27), (28), (29) and (30) in (26) we have

$$\begin{aligned} & (\nabla_{Z_4}\mathbf{C}_0)(Z_1, Z_2)\zeta \\ & = 2ng((\nabla_{Z_4}\phi)Z_1, Z_2) - g((\nabla_{Z_4}\mathbf{Q})Z_1, \phi Z_2) \\ & \quad - g(\mathbf{Q}Z_1, (\nabla_{Z_4}\phi)Z_2) - g(Z_1, (\nabla_{Z_4}\phi)Z_2) - 2ng((\nabla_{Z_4}\phi)Z_2, Z_1) \\ & \quad + g((\nabla_{Z_4}\mathbf{Q})Z_2, \phi Z_1) + g(\mathbf{Q}Z_2, (\nabla_{Z_4}\phi)Z_1) + g(Z_2, (\nabla_{Z_4}\phi)Z_1) \\ & \quad - \frac{1}{4n}[g(\nabla_{Z_4} Dr, Z_1)\tau(Z_2) - g(\nabla_{Z_4} Dr, Z_2)\tau(Z_1) - g(\phi Z_4, Z_2)(Z_1 r) \\ & \quad + g(\phi Z_4, Z_1)(Z_2 r)] - 2ng(\phi\nabla_{Z_4} Z_1, Z_2) + g(\mathbf{Q}\nabla_{Z_4} Z_1, \phi Z_2) \\ & \quad + g(\nabla_{Z_4} Z_1, \phi Z_2) + 2ng(\phi Z_2, \nabla_{Z_4} Z_1) - g(\mathbf{Q}Z_2, \phi\nabla_{Z_4} Z_1) \\ & \quad - g(Z_2, \phi\nabla_{Z_4} Z_1) + \frac{1}{4n}[((\nabla_{Z_4} Z_1)r)\tau(Z_2) - (Z_2 r)\tau(\nabla_{Z_4} Z_1)] \\ & \quad - 2ng(\phi Z_1, \nabla_{Z_4} Z_2) + g(\mathbf{Q}Z_1, \phi\nabla_{Z_4} Z_2) + g(Z_1, \phi\nabla_{Z_4} Z_2) \\ & \quad + 2ng(\phi\nabla_{Z_4} Z_2, Z_1) - g(\mathbf{Q}\nabla_{Z_4} Z_2, \phi Z_1) - g(\nabla_{Z_4} Z_2, \phi Z_1) \\ & \quad + \frac{1}{4n}[(Z_1 r)\tau(\nabla_{Z_4} Z_2) - ((\nabla_{Z_4} Z_2)r)\tau(Z_1)] - (\nabla_{Z_1}\mathbf{S})(Z_2, \phi Z_4) \\ & \quad + (\nabla_Y \mathbf{S})(Z_1, \phi Z_4) + \frac{1}{4n}[(Z_1 r)g(Z_2, \phi Z_4) - (Z_2 r)g(Z_1, \phi Z_4)]. \end{aligned} \tag{31}$$

Substituting $Z_1 = Z_4 = v_i$ in (31), where $\{v_i\}$ is an orthonormal basis, we have

$$\begin{aligned} & \sum_{i=1}^{2n+1} (\nabla_{v_i}\mathbf{C}_0)(v_i, Z_2)\zeta \\ & = 2ng(v_i, Z_2)\tau(v_i) + g((\nabla_{v_i}\mathbf{Q})\phi v_i, Z_2) + g(\mathbf{Q}v_i, Z_2)\tau(v_i) \\ & \quad - \frac{1}{4n}[g(\nabla_{v_i} Dr, v_i)\tau(Z_2) - g(\nabla_{v_i} Dr, Z_2)\tau(v_i) - g(\phi v_i, Z_2)(v_i r)]. \end{aligned} \tag{32}$$

From Proposition 2.1, it follows

$$\phi\mathbf{Q}Z_1 = \mathbf{Q}\phi Z_1 - \phi Z_1. \tag{33}$$

Now

$$g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_4) + g((\nabla_{Z_1}\mathbf{Q})Z_2, \phi Z_4) = g((\nabla_{Z_1}\mathbf{Q}\phi Z_2 - \mathbf{Q}\nabla_{Z_1}\phi Z_2), Z_4) + g((\nabla_{Z_1}\mathbf{Q}Z_2 - \mathbf{Q}\nabla_{Z_1}Z_2), \phi Z_4). \tag{34}$$

Making use of (9) and (33) in (34) implies

$$g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_4) + g((\nabla_{Z_1}\mathbf{Q})Z_2, \phi Z_4) = g((\nabla_{Z_1}\phi)\mathbf{Q}Z_2, Z_4) + g(\mathbf{Q}(\nabla_{Z_1}\phi)Z_2, Z_4).$$

Using (9) and (13) in the foregoing equation, we have

$$\begin{aligned} g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_4) &+ g((\nabla_{Z_1}\mathbf{Q})Z_2, \phi Z_4) \\ = &-g(Z_1, \mathbf{Q}Z_2)\tau(Z_4) - (2n-1)g(Z_1, Z_4)\tau(Z_2) \\ &+ (2n-1)g(Z_1, Z_2)\tau(Z_4) + g(\mathbf{Q}Z_1, Z_4)\tau(Z_2). \end{aligned} \tag{35}$$

Putting $Z_2 = Z_4 = v_i$ in the above equation, where $\{v_i\}$ is an orthonormal basis, we get

$$\sum_{i=1}^{2n+1} g((\nabla_{Z_1}\mathbf{Q})\phi v_i, v_i) + \sum_{i=1}^{2n+1} g((\nabla_{Z_1}\mathbf{Q})v_i, \phi v_i) = 0.$$

That is,

$$\sum_{i=1}^{2n+1} g((\nabla_{Z_1}\mathbf{Q})\phi v_i, v_i) = 0. \tag{36}$$

Substituting $Z_1 = Z_4 = v_i$ in (35) yields

$$\sum_{i=1}^{2n+1} g((\nabla_{v_i}\mathbf{Q})Z_2, \phi v_i) = (-4n^2 - r)\tau(Z_2) - div\, \phi\ Z_2 - \frac{1}{2}(\phi Z_2)r. \tag{37}$$

Using (36) and (37) in (32) yields

$$\begin{aligned} \sum_{i=1}^{2n+1} (\nabla_{v_i}\mathbf{C}_0)(v_i, Z_2)\zeta = 2(-4n^2 - r)\tau(Z_2) &- \frac{1}{2}(\phi Z_2 r) \\ &- \frac{1}{4n}[(div\, Dr)\tau(Z_2) - g(\nabla_\zeta Dr, Z_2)]. \end{aligned} \tag{38}$$

Now

$$\begin{aligned} g(\mathbf{Q}v_i, v_j)g(\mathbf{C}(Z_1, v_i)v_j), Z_2) &= -g(\mathbf{C}(Z_1, v_i)Z_2, \mathbf{Q}v_i) \\ &= -g(\mathbf{Q}\mathbf{C}(Z_1, v_i)Z_2, v_i). \end{aligned} \tag{39}$$

Combining (3) and (39), we have

$$\mathcal{B}(Z_1, Z_2) = \frac{1}{2n-1}[\sum_{i=1}^{2n+1} (\nabla_{v_i}\mathbf{C}_0)(v_i, Z_1, Z_2) - \sum_{i=1}^{2n+1} g(\mathbf{Q}\mathbf{C}(Z_1, v_i)Z_2, v_i)] \tag{40}$$

for all smooth vector fields Z_1 and Z_2.

Replacing Z_2 by ζ in (6) and using (20), (38), (40) yields

$$(41)\quad 2(4n-4n^2+r)\tau(Z_1)-\frac{1}{2}(\phi Z_1 r)-\frac{1}{4n}[(divDr)\tau(Z_1)-g(\nabla_\zeta Dr, Z_1)] + \frac{r^2-4n^2}{2n(2n-1)}\tau(Z_1)-[\frac{|\,\mathbf{Q}\,|^2-4n^2}{2n-1}]\tau(Z_1)-\lambda\tau(Z_1)=0.$$

Replacing Z_1 by ϕZ_1 in the above equation implies

$$(42)\quad \nabla_\zeta Dr = 2n\phi Dr.$$

Since ζ is a Killing vector field, so

$$(43)\quad \pounds_\zeta r = 0.$$

Taking exterior derivative d in (43), provides

$$\pounds_\zeta dr = 0,$$

since $\pounds_\zeta$ and d commutes.

From the preceding equation, we have

$$(44)\quad \pounds_\zeta Dr = 0.$$

Using (10) in (44), we have

$$(45)\quad \pounds_\zeta Dr = -\phi Dr.$$

Finally, equations (42) and (45) together reveal $\phi Dr = 0$, that is, $Dr = 0$. Hence the manifold is of constant scalar curvature. Now, since r is constant so from (41), it follows that the Ricci operator of the metric g has a constant norm.

As a result, the following theorem emerges:

Theorem 3.2. *Let (g, ζ, λ) be a Bach almost solitons on a paraSasakian manifold of dimension $(2n+1)$. Then the manifold is of constant scalar curvature and the Ricci operator of the metric g has a constant norm.*

4. Bach almost solitons in 3-dimensional paraSasakian manifolds

In this section we characterize 3-dimensional ps manifolds admitting Bach almost solitons. In a 3-dimensional Riemannian manifold the curvature tensor is given by

$$(46)\quad \mathbf{R}(Z_1,Z_2)Z_3 = g(Z_2,Z_3)\mathbf{Q}Z_1 - g(Z_1,Z_3)\mathbf{Q}Z_2 + \mathbf{S}(Z_2,Z_3)Z_1 - \mathbf{S}(Z_1,Z_3)Z_2 - \frac{r}{2}[g(Z_2,Z_3)Z_1 - g(Z_1,Z_3)Z_2]$$

for all smooth vector fields Z_1, Z_2 and Z_3.

Substituting $Z_1 = Z_3 = \zeta$ in (46) and making use of (12), (13) and (14) implies

$$(47)\quad \mathbf{Q}Z_2 = (-3-\frac{r}{2})\tau(Z_2)\zeta + (1+\frac{r}{2})Z_2.$$

From the foregoing equation, it is quite clear that

$$\mathbf{Q}\phi = \phi\mathbf{Q}. \tag{48}$$

Using (10) and (47), we infer that

$$(\nabla_{Z_1}\mathbf{Q})\zeta = \mathbf{Q}\phi Z_1. \tag{49}$$

From (21) and (49) we have

$$\mathbf{C}_0(Z_1, Z_2)\zeta = -2g(\mathbf{Q}\phi Z_1, Z_2) - \frac{1}{4}[(Z_1 r)\tau(Z_2) - (Z_2 r)\tau(Z_1)]. \tag{50}$$

Using (4), (9), (47) and (50) in (26) yields

$$\begin{aligned}(\nabla_{Z_1}\mathbf{C}_0)(Z_2, Z_3)\zeta = {} & g((\nabla_{Z_2}\mathbf{Q})Z_3, \phi Z_1) - g((\nabla_{Z_3}\mathbf{Q})Z_2, \phi Z_1)\\ & + 2g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_3) + 4g(Z_1, Z_2)\tau(Z_3)\\ & + 2\mathbf{S}(\mathbf{Q}Z_1, Z_3)\tau(Z_2) + \frac{1}{4}[g(Z_2, \phi Z_1)(Z_2 r)\\ & - g(\nabla_{Z_1} Dr, Z_2)\tau(Z_3) - g(\phi Z_1, Z_3)\tau(Z_2)\\ & - g(\nabla_{Z_1} Dr, Z_3)\tau(Z_2)].\end{aligned} \tag{51}$$

Putting $Z_1 = Z_2 = v_i$ in (51), where $\{v_i\}$ is an orthonormal basis, we get

$$\begin{aligned}& (\nabla_{v_i}\mathbf{C}_0)(v_i, Z_3)\zeta\\ & = g((\nabla_{v_i}\mathbf{Q})Z_3, \phi v_i) - g((\nabla_{Z_3}\mathbf{Q})v_i, \phi v_i)\\ & \quad + 2g((\nabla_{v_i}\mathbf{Q})\phi v_i, Z_3) + 12\tau(Z_3) + 2\mathbf{S}(\mathbf{Q}v_i, Z_3)\tau(v_i)\\ & \quad + \frac{1}{4}[g(Z_3, \phi v_i)(v_i r) - g(\nabla_{v_i} Dr, v_i)\tau(Z_3) - g(\phi v_i, Z_3)(v_i)\\ & \quad - g(\nabla_{v_i} Dr, Z_3)\tau(v_i)].\end{aligned} \tag{52}$$

Now from (48), we have

$$\begin{aligned}& g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_3) + g((\nabla_{Z_1}\mathbf{Q})Z_2, \phi Z_3)\\ & = g((\nabla_{Z_1}\phi)\mathbf{Q}Z_2, Z_3) + g(\mathbf{Q}(\nabla_{Z_1}\phi)Z_2, Z_3).\end{aligned} \tag{53}$$

Again using (9) and (48) in the foregoing equation yields

$$\begin{aligned}& g((\nabla_{Z_1}\mathbf{Q})\phi Z_2, Z_3) + g((\nabla_{Z_1}\mathbf{Q})Z_2, \phi Z_3)\\ & = - g(Z_1, \mathbf{Q}Z_2)\tau(Z_3) - 2g(Z_1, Z_3)\tau(Z_2)\\ & \quad + 2g(Z_1, Z_2)\tau(Z_3) + g(\mathbf{Q}Z_1, Z_3)\tau(Z_2).\end{aligned} \tag{54}$$

Taking an orthonormal basis $\{v_i\}$ and replacing Z_2 and Z_3 by v_i, we infer

$$\sum_{i=1}^{3} g((\nabla_{Z_1}\mathbf{Q})\phi v_i, v_i) + \sum_{i=1}^{3} g((\nabla_{Z_1}\mathbf{Q})v_i, \phi v_i) = 0.$$

That is,

$$\sum_{i=1}^{3} g((\nabla_{Z_1}\mathbf{Q})\phi v_i, v_i) = 0. \tag{55}$$

Setting $Z_1 = Z_3 = v_i$ in (46) yields

$$\sum_{i=1}^{3} g((\nabla_{v_i}\mathbf{Q})Z_2, \phi v_i) = (r-2)\eta(Z_2) - \frac{1}{2}(\phi Z_2)r. \tag{56}$$

Making use of (47), (55) and (56) in (52) yields

$$(\nabla_{v_i}\mathbf{C}_0)(v_i, Z_3)\zeta = 3(r+6)\tau(Z_3) - \frac{3}{2}g(\phi Z_3, Dr) + \frac{1}{4}[(div\, Dr)\tau(Z_3) - g(\nabla_\zeta Dr, Z_3)]. \tag{57}$$

Since in a 3-dimensional paraSasakian manifold Weyl curvature tensor vanishes, so equation (5) reduces to

$$\mathcal{B}(Z_1, Z_2) = \sum_{i=1}^{3}[(\nabla_{v_i}\mathbf{C}_0)(v_i, Z_1)Z_2)] \tag{58}$$

for all smooth vector fields Z_1 and Z_2.

Replacing Z_2 by ζ in (6) and using (57) and (58) provides

$$3(r+6)\tau(Z_1) - \frac{3}{2}g(\phi Z_1, Dr) + \frac{1}{4}[(divDr)\tau(Z_1) - g(\nabla_\zeta Dr, X)] - \lambda\tau(Z_1) = 0. \tag{59}$$

Replacing Z_1 by ϕZ_1 in (59) implies

$$\nabla_\zeta Dr = -6(\phi Dr). \tag{60}$$

From (45) and (60), we have $Dr = 0$, that is, r is a constant.

Then from (59), it follows that

$$\lambda = 3(r+6).$$

This leads to the following:

Theorem 4.1. *Let (g, ζ, λ) be a Bach almost solitons on a paraSasakian manifold of dimension 3. Then the manifold is of constant scalar curvature. Moreover, the Bach almost solitons are steady if $r = -6$; shrinking if $r > -6$; expanding if $r < -6$.*

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