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WHEN ALL PERMUTATIONS ARE COMBINATORIAL SIMILARITIES

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Abstract. Let (X, d) be a semimetric space. A permutation Φ of the set X is a combinatorial self similarity of (X, d) if there is a bijective function $f\colon d(X \times X) \to d(X \times X)$ such that

$$d(x, y) = f(d(\Phi(x), \Phi(y)))$$

for all x, $y \in X$. We describe the set of all semimetrics ρ on an arbitrary nonempty set Y for which every permutation of Y is a combinatorial self similarity of (Y, ρ).

1. Introduction

Let us start from the classical notion of metric space introduced by Maurice Fréchet in his thesis [13].

A *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}$ such that for all x, y, $z \in X$:

(i) $d(x, y) \geqslant 0$ with equality if and only if $x = y$, the *positivity property*;
(ii) $d(x, y) = d(y, x)$, the *symmetry property*;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$, the *triangle inequality*.

Here and what follows $X \times X$ is the Cartesian square of X, i.e., $X \times X$ is the set of all ordered pairs (x, y), where x, $y \in X$.

An useful generalization of the notion of metric space is the concept of semimetric space.

Definition 1.1. Let X be a set and let $d\colon X \times X \to \mathbb{R}$ be a symmetric function. The function d is a *semimetric* on X if it satisfies the positivity property.

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If d is a semimetric on X, we say that (X, d) is a *semimetric space*. The semimetric spaces also were first considered by Fréchet in [13], where he called them "classes E". It should be noted that a different terminology is used to denote the class of semimetric spaces: distances spaces [14], spaces endowed with dissimilarities [5], symmetric spaces (for the case of topological spaces with topologies generated by semimetrics) [18, Chapter 10]. We inherit the terminology from Wilson's paper [24], Blumenthal's book [4], and many recent papers [1, 10, 12, 15, 17, 22].

In the next definition we will denote by $d(X \times X)$ the range of d,

$$d(X \times X) := \{d(x, y) : (x, y) \in X \times X\}.$$

Definition 1.2 ([9])**.** Let (X, d) and (Y, ρ) be semimetric spaces. The spaces (X, d) and (Y, ρ) are *combinatorially similar* if there exist bijections $\Psi\colon Y \to X$ and $f\colon d(X \times X) \to \rho(Y \times Y)$ such that

$$\rho(x, y) = f\big(d(\Psi(x), \Psi(y))\big) \tag{1.1}$$

for all x, $y \in Y$. In this case, we will say that $\Psi\colon Y \to X$ is a *combinatorial similarity.*

Remark 1.3. If $Y \xrightarrow{\Psi} X$ is a combinatorial similarity of spaces (X, d) and (Y, ρ), then the inverse mapping $X \xrightarrow{\Psi^{-1}} Y$ is also a combinatorial similarity of these spaces. Moreover, if (Z, δ) is a semimetric space and $X \xrightarrow{\Phi} Z$ is a combinatorial similarity of (Z, δ) and (X, d), then the composition $Y \xrightarrow{\Psi} X \xrightarrow{\Phi} Z$ is also a combinatorial similarity. In particular, for every semimetric space (X, d), the set of all combinatorial self similarities $X \to X$ is a subgroup of the symmetric group $\mathbf{Sym}(X)$ of all permutations on X. In what follows this subgroup will be denoted as $\mathbf{Cs}(X, d)$.

Let us consider some examples of combinatorial similarities.

Example 1.4. Semimetric spaces (X, d) and (Y, ρ) are *isometric* if there is a bijection $\Psi\colon Y \to X$ such that

$$\rho(x, y) = d(\Psi(x), \Psi(y))$$

for all x, $y \in Y$. In this case we will say that Ψ is an *isometry* of (X, d) and (Y, ρ). It is easy to see that all isometrics are combinatorial similarities. Moreover, a combinatorial similarity $\Psi\colon Y \to X$ is an isometry if and only if (1.1) holds for all x, $y \in Y$ with $f(t) = t$ for every $t \in d(X \times X)$.

The group of all self isometries of a semimetric space (X, d) will be denoted as $\mathbf{Iso}(X, d)$.

Example 1.5. Let $\Phi\colon X \to Y$ be a bijection and let $d\colon X \times X \to \mathbb{R}$ and $\rho\colon Y \times Y \to \mathbb{R}$ be some semimetrics. The mapping Φ is a *weak similarity* of (X, d) and (Y, ρ) if and only if the equivalence

$$(d(x, y) \leqslant d(w, z)) \Leftrightarrow (\rho(\Phi(x), \Phi(y)) \leqslant \rho(\Phi(w), \Phi(z))) \tag{1.2}$$

is valid for all $x, y, z, w \in X$.

Equivalence (1.2) evidently implies the validity of

$$(d(x,y) = d(w,z)) \Leftrightarrow (\rho(\Phi(x), \Phi(y)) = \rho(\Phi(w), \Phi(z))).$$

Thus, every weak similarity is a combinatorial similarity (see Lemma 2.2 below). It is interesting to note that, the inverse statement is also valid for the case when (X,d) and (Y,ρ) are ultrametric spaces. Every combinatorial similarity of ultrametric spaces (X,d) and (Y,ρ) is a weak similarity of these spaces (see Theorem 4.7 in [8]). Some questions connected with the weak similarities and combinatorial similarities were studied in [3, 7, 9]. The weak similarities of finite ultrametric and semimetric spaces were also considered by E. Petrov in [21].

The combinatorial similarities are the main morphisms of semimetric spaces which will be studied in the paper. Let us introduce now the subclasses of semimetric spaces that will be important for the future.

Let (X,d) be a metric space. The metric d is said to be *strongly rigid* if, for all x, y, u, $v \in X$, the condition

$$d(x,y) = d(u,v) \neq 0 \tag{1.3}$$

implies

$$(x = u \text{ and } y = v) \text{ or } (x = v \text{ and } y = u). \tag{1.4}$$

(Some properties of strongly rigid metric spaces are described in [2,9,11,16,19].)

The concept of strongly rigid metric can be naturally generalized to the concept of *strongly rigid semimetric*.

Definition 1.6. Let (X,d) be a semimetric space. The semimetric d is *strongly rigid* if (1.3) implies (1.4) for all x, y, u, $v \in X$.

Let us consider an example of strongly rigid semimetric.

Example 1.7. Let $X = \mathbb{N}$ be the set of all positive integer numbers. Let us define a semimetric $d\colon X \times X \to \mathbb{R}$ as

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{\max\{x,y\}} \cdot 3^{\min\{x,y\}} & \text{if } x \neq y. \end{cases}$$

Using the uniqueness of the representation of a natural number by product of prime factors, we see that from $d(x,y) = d(u,v) \neq 0$ it follows that

$$\max\{x,y\} = \max\{u,v\} \quad \text{and} \quad \min\{x,y\} = \min\{u,v\}.$$

The last equalities imply (1.4). Thus, d is strongly rigid.

It is easy to prove that the equality $\mathbf{Cs}(X,d) = \mathbf{Sym}(X)$ holds for every strongly rigid (X,d) (see the proof of Lemma 2.4).

The following generalization of strongly rigid semimetric spaces was recently given in [11].

Definition 1.8. A semimetric space (X, d) is *weakly rigid* if every three-point subspace of (X, d) is strongly rigid.

Example 2.1 from the next section of the paper shows that a semimetric space can be weakly rigid but not strongly rigid.

We say that a semimetric $d\colon X \times X \to \mathbb{R}$ is *discrete* if the inequality

$$|d(X \times X)| \leqslant 2 \tag{1.5}$$

holds, where $|d(X \times X)|$ is the cardinal number of the set $d(X \times X)$. It follows directly from the definitions that a semimetric space (X, d) is discrete if and only if the equality

$$\mathbf{Iso}(X, d) = \mathbf{Sym}(X) \tag{1.6}$$

holds.

Remark 1.9. The standard definition of *discrete metric* can be formulated as: "The metric on X is discrete if the distance from each point of X to every other point of X is one." (See, for example, [23, p. 4].) Thus, a semimetric (X, d) is discrete if there is $k > 0$ such that

$$d(x, y) = k\rho(x, y)$$

for all x, $y \in X$, where ρ is the discrete metric defined on the set X.

The goal of the paper is to describe all possible semimetric spaces (X, d) for which

$$\mathbf{Cs}(X, d) = \mathbf{Sym}(X). \tag{1.7}$$

The paper is organized as follows.

Proposition 2.7 gives us a characterization of discrete $d\colon X \times X \to \mathbb{R}$ in terms of combinatorial self similarities of (X, d).

The main result of the paper, Theorem 2.8, completely characterizes the structure of all semimetric spaces (X, d) satisfying equality (1.7).

In Corollary 2.10 we show that equality (1.7) is equivalent to the equality (1.6) if X is big enough.

2. Permutations and combinatorial self similarities

Let us start from an example of a four-point weakly rigid metric space (Z, d) that is not strongly rigid but satisfies the equality $\mathbf{Cs}(Z, d) = \mathbf{Sym}(Z)$.

Example 2.1. Let $Z = \{z_1, z_2, z_3, z_4\}$ be the four-point subset of the complex plane $\mathbb{C}$,

$$z_1 = 0 + 0i, \quad z_2 = 0 + 3i, \quad z_3 = 4 + 3i, \quad z_4 = 4 + 0i,$$

and let d be the restriction of the usual Euclidean metric on $Z \times Z$. The equalities $d(z_1, z_2) = d(z_3, z_4) = 3$ and $d(z_1, z_4) = d(z_2, z_3) = 4$ imply that (Z, d) is neither strongly rigid nor discrete but it can be proved directly that (Z, d) is weakly rigid and the equality $\mathbf{Cs}(Z, d) = \mathbf{Sym}(Z)$ holds.

Lemma 2.2. *Let (X, d) and (Y, ρ) be semimetric spaces. A bijection $\Phi\colon Y \to X$ is a combinatorial similarity if and only if the equivalence*

$$\big(\rho(x,y) = \rho(u,v)\big) \Leftrightarrow \Big(d\big(\Phi(x), \Phi(y)\big) = d\big(\Phi(u), \Phi(v)\big)\Big) \tag{2.1}$$

is valid for all $x, y, u, v \in Y$.

Proof. If $\Phi\colon Y \to X$ is a combinatorial similarity, then the validity of (2.1) follows directly from Definition 1.2.

Let (2.1) be valid for all x, y, u, $v \in Y$. Using the bijectivity of Φ, we see that the last statement holds if and only if

$$\big(d(x', y') = d(u', v')\big) \Leftrightarrow \Big(\rho\big(\Phi^{-1}(x'), \Phi^{-1}(y')\big) = \rho\big(\Phi^{-1}(u'), \Phi^{-1}(v')\big)\Big) \tag{2.2}$$

is valid for all $x', y', u', v' \in X$, where Φ^{-1} is the inverse mapping of the mapping Φ.

For every $t \in d(X \times X)$ we can find $(x', y') \in X \times X$ such that

$$d(x', y') = t. \tag{2.3}$$

Write

$$f(t) := \rho\big(\Phi^{-1}(x'), \Phi^{-1}(y')\big). \tag{2.4}$$

Equivalence (2.2) shows that

$$d(X \times X) \ni t \mapsto f(t) \in \rho(Y \times Y)$$

is a correctly defined function. This is a surjective function. Indeed, if s is an arbitrary point of $\rho(Y \times Y)$, then

$$s = \rho(p, q) \tag{2.5}$$

holds for some p, $q \in Y$. Write

$$p' := \Phi(p) \text{ and } q' := \Phi(q).$$

The points p' and q' belong to X. Equalities (2.3) and (2.5) with $x' = p'$ and $y' = q'$ give us

$$\begin{aligned} f\big(d\big(\Phi(p), \Phi(q)\big)\big) &= f\big(d\big(p', q'\big)\big) = \rho\big(\Phi^{-1}(p'), \Phi^{-1}(q')\big) \\ &= \rho\big(\Phi^{-1}(\Phi(p)), \Phi^{-1}(\Phi(q))\big) = \rho(p, q) = s. \end{aligned} \tag{2.6}$$

Since for every $(p, q) \in Y \times Y$ there is $s \in \rho(Y \times Y)$ such that (2.5) holds, (2.6) also shows that we have

$$f\big(d\big(\Phi(p), \Phi(q)\big)\big) = \rho(p, q)$$

for all $p, q \in Y$. To complete the proof it suffice to note that (2.2) can be written in the equivalent form

$$\big(d(x', y') \neq d(u', v')\big) \Leftrightarrow \Big(\rho\big(\Phi^{-1}(x'), \Phi^{-1}(y')\big) \neq \rho\big(\Phi^{-1}(u'), \Phi^{-1}(v')\big)\Big)$$

that, together with (2.3) and (2.4), implies the injectivity of f. $\square$

The next proposition characterizes all semimetric spaces which are combinatorially similar to the rectangle from Example 2.1.

Proposition 2.3. *Let* $\rho\colon X \times X \to \mathbb{R}$ *be a semimetric on a set* X *with* $|X| \geqslant 4$. *Then the following conditions are equivalent*:

(i) (X,ρ) *is combinatorially similar to the metric space* (Z,d) *from Example 2.1.*

(ii) (X,ρ) *is weakly rigid and, moreover, all three-point subspaces of* (X,ρ) *are isometric.*

Proof. (i)⇒(ii) The validity of this implication can be proved using Lemma 2.2.

(ii)⇒(i) Let (ii) hold. Then there are different numbers $a, b, c \in (0,\infty)$ such that, for every three-point subspace T of (X,ρ), we have

$$\rho(X \times X) = \rho(T \times T) = \{0, a, b, c\}. \tag{2.7}$$

Using (2.7) it is easy to prove that $|X| = 4$. Indeed, let p be a point of X. If $|X| \geqslant 5$ holds, then we have the inequality

$$|X \setminus \{p\}| \geqslant 4.$$

Consequently, by Pigeonhole principle, there are some different points $x_1, x_2 \in X \setminus \{p\}$ such that

$$\rho(x_1, p) = \rho(x_2, p).$$

Thus, (X,ρ) is not weakly rigid contrary to (ii). It implies the inequality

$$|X| \leqslant 4. \tag{2.8}$$

By condition, we have $|X| \geqslant 4$. The last inequality and (2.8) give us $|X| = 4$.

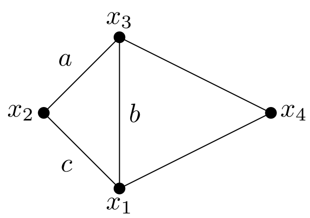

FIGURE 1. Add the point x_4 to the triangle $\{x_1, x_2, x_3\}$.

Let x_1, x_2, x_3 be points of X such that

$$\rho(x_1, x_2) = c, \quad \rho(x_2, x_3) = a \quad \text{and} \quad \rho(x_3, x_1) = b.$$

Since $|X| = 4$ holds, the set $X \setminus \{x_1, x_2, x_3\}$ contains a unique point x_4. Let us consider the triangle $\{x_1, x_3, x_4\}$. From (2.7) with $T = \{x_2, x_3, x_4\}$ and the equality $\rho(x_3, x_1) = b$ it follows that we have either

$$\rho(x_3, x_4) = a \tag{2.9}$$

or $\rho(x_3, x_4) = c$. If (2.9) holds, then we obtain $\rho(x_2, x_3) = \rho(x_3, x_4)$. Hence, (X, ρ) is not weakly rigid contrary to (ii). Thus, we have $\rho(x_3, x_4) = c$. The last equality, the equality $\rho(x_1, x_3) = b$, and (2.7) imply the equality $\rho(x_4, x_1) = a$.

Let us consider now the triangle $\{x_2, x_3, x_4\}$. Then using (2.7) with $T = \{x_2, x_3, x_4\}$ we can prove that $\rho(x_3, x_4) = c$ and $\rho(x_2, x_3) = a$ imply $\rho(x_3, x_1) = b$.

To complete the proof it suffices to note that the bijection

$$F\colon Z \to X, \quad F(z_j) = x_j, \quad j = 1, \dots, 4,$$

is a combinatorial similarity of the semimetric space (X, ρ) and the rectangle (Z, d) (see Figure 2 below). $\square$

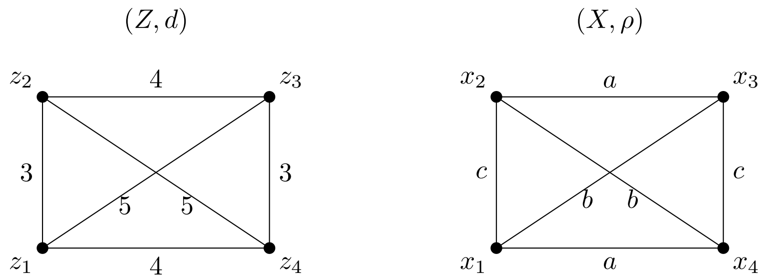

Figure 2. The spaces (X, ρ) and (Z, d) are combinatorially similar.

Lemma 2.4. *Let (X, d) be a three-point metric space. Then the following statements are equivalent*:

(i) *Either (X, d) is strongly rigid or (X, d) is discrete.*
(ii) $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *holds.*

Proof. (i)⇒(ii) Let (i) hold. We must prove that every permutation of X is a combinatorial self similarity of (X, d). Let Φ belong to $\mathbf{Sym}(X)$. By Lemma 2.2, the mapping Φ is a combinatorial similarity if and only if

$$\big(d(x, y) = d(u, v)\big) \Leftrightarrow \big(d(\Phi(x), \Phi(y)) = d(\Phi(u), \Phi(v))\big) \tag{2.10}$$

holds for all $x, y, u, v \in X$. It is clear that (2.10) is valid for $x = y$ or $u = v$. Thus, it suffices to prove the validity of (2.10) for the case when

$$d(x, y) = d(u, v) \neq 0. \tag{2.11}$$

If d is strongly rigid, then (2.11) implies

$$(x, y) = (u, v) \quad \text{or} \quad (x, y) = (v, u) \tag{2.12}$$

(see (1.4)). Since $\Phi\colon X \to X$ is bijective, (2.12) can be written in the following equivalent form

$$(\Phi(x), \Phi(y)) = (\Phi(u), \Phi(v)) \quad \text{or} \quad (\Phi(x), \Phi(y)) = (\Phi(v), \Phi(u))\,.$$

Equivalence (2.10) follows.

If d is discrete, then we have $\mathbf{Sym}(X) = \mathbf{Iso}(X, d)$. That implies (ii) because

$$\mathbf{Iso}(X, d) \subseteq \mathbf{Cs}(X, d) \subseteq \mathbf{Sym}(X).$$

(ii)⇒(i) Let (ii) hold. If (X, d) is neither rigid nor discrete, then we can numbered the points of X and find $a, b \in (0, \infty)$ such that $X = \{x_1, x_2, x_3\}$ and

$$d(x_1, x_2) = d(x_2, x_3) = a \neq b = d(x_3, x_1). \tag{2.13}$$

Let consider the permutation

$$\Phi = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix}. \tag{2.14}$$

Using (2.13) and (2.14) we obtain

$$d(\Phi(x_1), \Phi(x_2)) = d(x_2, x_3) = a$$

and

$$d(\Phi(x_2), \Phi(x_3)) = d(x_3, x_1) = b.$$

Hence, the permutation Φ is not a combinatorial self similarity by Lemma 2.2. The last statement contradicts (ii). □

Lemma 2.5. *Let (X, d) be a semimetric space and let Y be a nonempty subset of X. If $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ holds, then we have the equality $\mathbf{Cs}(Y, d|_{Y \times Y}) = \mathbf{Sym}(Y)$.*

Proof. Each permutation of Y can extended to a permutation of X and, in addition, if $\Psi \colon X \to X$ is a combinatorial self similarity, then $\Phi = \Psi|_Y$ is also a combinatorial self similarity by Lemma 2.2. □

Lemma 2.6. *Let (X, d) and (Y, ρ) be combinatorially similar. Then the equality $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ holds if and only if we have $\mathbf{Cs}(Y, \rho) = \mathbf{Sym}(Y)$.*

Proof. Let $\Phi \colon X \to Y$ be a combinatorial similarity and let

$$\mathbf{Cs}(Y, \rho) = \mathbf{Sym}(Y) \tag{2.15}$$

hold. Let us consider an arbitrary $F \in \mathbf{Sym}(X)$. Then the mapping

$$Y \xrightarrow{\Phi^{-1}} X \xrightarrow{F} X \xrightarrow{\Phi} Y$$

is a permutation of Y. Let us denote this permutation by Ψ. Then Ψ belongs to $\mathbf{Cs}(Y, \rho)$ by (2.15). Since the permutation F of X coincides with the mapping

$$X \xrightarrow{\Phi} Y \xrightarrow{\Phi^{-1}} X \xrightarrow{F} X \xrightarrow{\Phi} Y \xrightarrow{\Phi^{-1}} X,$$

F belongs to $\mathbf{Cs}(X, d)$ by Remark 1.3. Thus, (2.15) implies the equality

$$\mathbf{Cs}(X, d) = \mathbf{Sym}(X). \tag{2.16}$$

Arguing in a similar way, we can prove that (2.15) follows from (2.16). □

Proposition 2.7. *Let* (X, d) *be a semimetric space with* $|X| \geqslant 3$. *Then the following statements are equivalent*:

(i) (X, d) *is discrete.*
(ii) (X, d) *contains an equilateral triangle and the equality* $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *holds.*

Proof. (i)⇒(ii) Let (X, d) be discrete. Then (X, d) contains an equilateral triangle because $|X| \geqslant 3$. Moreover, the equality $\mathbf{Iso}(X, d) = \mathbf{Sym}(X)$ and the inclusions

$$\mathbf{Iso}(X, d) \subseteq \mathbf{Cs}(X, d) \subseteq \mathbf{Sym}(X)$$

imply $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$.

(ii)⇒(i) Let (ii) hold. Then there are some points $x_1, x_2, x_3 \in X$ and $a > 0$ such that

$$d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_1) = a. \tag{2.17}$$

Let p and q be distinct points of (X, d). We must prove the equality

$$d(p, q) = a. \tag{2.18}$$

The last equality trivially holds if $\{p, q\} \subseteq \{x_1, x_2, x_3\}$. Let us consider the case when $p \in \{x_1, x_2, x_3\}$, but $q \notin \{x_1, x_2, x_3\}$. Without loss of generality we may assume that $p = x_1$. Let us define a permutation $F \colon X \to X$ as

$$F(x) := \begin{cases} q & \text{if } x = x_2, \\ x_2 & \text{if } x = q, \\ x & \text{otherwise.} \end{cases} \tag{2.19}$$

Then F is a combinatorial self similarity by statement (ii). It follows from (2.19) that x_1 and x_3 are fixed points of F. Consequently, (2.17) implies $d(F(x_1), F(q)) = a$ by Lemma 2.2. Using (2.19) we can rewrite the last equality as $d(q, x_1) = a$. Since $x_1 = p$, the equality $d(q, x_1) = a$ implies (2.18).

If the sets $\{p, q\}$ and $\{x_1, x_2, x_3\}$ are disjoint, then considering the triangles $\{x_1, x_2, q\}$ and $\{x_1, p, q\}$ we can get (2.18) as above. □

Theorem 2.8. *Let* (X, d) *be a nonempty semimetric space. Then the following statements* (i) *and* (ii) *are equivalent*:

(i) *At least one of the following conditions has been fulfilled*:
  $(\mathrm{i})_1$ (X, d) *is strongly rigid*;
  $(\mathrm{i})_2$ (X, d) *is discrete*;
  $(\mathrm{i})_3$ (X, d) *is weakly rigid and all three-point subspaces of* (X, d) *are isometric.*
(ii) $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *holds.*

Proof. (i)⇒(ii) To prove the validity of (i)⇒(ii) it suffices to show that the implication $(\mathrm{i})_3$⇒(i) is valid.

Let $(\mathrm{i})_3$ hold. Then, by Proposition 2.3, (X, d) is combinatorially similar to the rectangle from Example 2.1. Since every permutation of vertices of this rectangle is a combinatorial self similarity, we obtain the equality $\mathbf{Cs}(X,d) = \mathbf{Sym}(X)$ by Lemma 2.6.

(ii)$\Rightarrow$(i) Let every permutation of X be a combinatorial self similarity of (X,d). We must prove the validity of (i).

Let us consider first the case when $|X| \leqslant 2$. In this case every semimetric d on X is discrete and strongly rigid. Thus, we have the validity of the implications (ii)$\Rightarrow$(i)$_1$ and (ii)$\Rightarrow$(i)$_2$ if $|X| \leqslant 2$.

Let $|X| = 3$ hold. Then exactly one from the implications (ii)$\Rightarrow$(i)$_1$, (ii)$\Rightarrow$(i)$_2$ is true by Lemma 2.4.

Suppose now that $|X| = 4$ holds, but $d\colon X \times X \to \mathbb{R}$ is neither discrete nor strongly rigid. Since (X,d) is not strongly rigid, there are two-point subsets $\{x,y\}$, $\{u,v\}$ of X and $a \in (0,\infty)$ such that

$$\{x,y\} \neq \{u,v\} \tag{2.20}$$

and

$$d(x,y) = d(u,v) = a. \tag{2.21}$$

We claim that (2.20) and (2.21) imply

$$\{x,y\} \cap \{u,v\} = \varnothing. \tag{2.22}$$

Indeed, if (2.22) does not hold, then, using (2.20), we obtain that $Y = \{x,y\} \cup \{u,v\}$ is a triangle. In this case, by Lemma 2.5, from (ii) it follows the equality

$$\mathbf{Cs}(Y, d|_{Y\times Y}) = \mathbf{Sym}(Y). \tag{2.23}$$

Now using Lemma 2.4, we see that the semimetric $d|_{Y\times Y}$ is either strongly rigid or discrete. If $d|_{Y\times Y}$ is strongly rigid, then from $\{x,y\} \neq \{u,v\}$ it follows that $d(x,y) \neq d(u,v)$, contrary to (2.21). Consequently, $d|_{Y\times Y}$ is discrete. By Proposition 2.7, the last statement implies that $d\colon X\times X \to \mathbb{R}$ is discrete contrary to our supposition. Equality (2.22) follows.

From (2.22) and $|X| = 4$ it follows that $X = \{x,y,u,v\}$. Applying Lemma 2.2 to the permutation

$$\begin{pmatrix} x & y & u & v \\ x & u & y & v \end{pmatrix},$$

we obtain the equalities

$$d(x,u) = d(y,v) = b \tag{2.24}$$

for some $b \in (0,\infty)$ and, analogously, for the case of the permutation

$$\begin{pmatrix} x & y & u & v \\ v & x & u & y \end{pmatrix},$$

Lemma 2.2 implies

$$d(v,x) = d(u,y) = c \tag{2.25}$$

for some $c \in (0, \infty)$. To complete the proof of validity of (ii)$\Rightarrow$(i)$_3$ it suffices to show that the numbers a, b, c are pairwise distinct, that can be done in the same way as in the proof of equality (2.22).

Let us consider now the case $|X| \geqslant 5$. We claim that the inequality $|X| \geqslant 5$ and (2.23) imply that d is strongly rigid or discrete. To prove it, suppose contrary that d is neither strongly rigid nor discrete and construct a four-point set $Y \subseteq X$ such that $(Y, d|_{Y\times Y})$ is combinatorially similar to the metric space from Example 2.1. We will construct Y by modification of our proof of validity of (ii)$\Rightarrow$(i) when $|X| = 4$.

Let $\{x, y\}$ and $\{u, v\}$ be two point subsets of X satisfying (2.20) and (2.21) (with some $a > 0$). Write $Y := \{x, y\} \cup \{u, v\}$. As in the case $|X| = 4$, we obtain (2.22) and (2.23). From (2.20) and (2.21) it follows that $d|_{Y\times Y}$ is not strongly rigid. If $d|_{Y\times Y}$ is discrete, then, by Proposition 2.7, equality (2.23) implies that d is also discrete, contrary to our assumption. Moreover, arguing as in the case $|X| = 4$, we can show that equalities (2.21), (2.24) and (2.25) are valid with pairwise distinct a, b, $c \in (0, \infty)$. Thus, we have $|Y| = 4$ and (2.23). It was shown above that (ii)$\Rightarrow$(i)$_3$ is true if $|X| = 4$ and d is neither strongly rigid nor discrete. Consequently, $(Y, d|_{Y\times Y})$ is combinatorially similar to the metric space from Example 2.1.

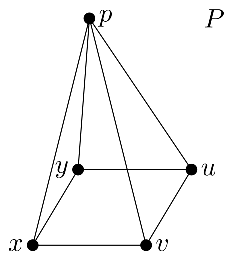

FIGURE 3. From the quadruple $\{x, y, u, v\}$ to the pyramid P.

Since $|X| \geqslant 5$ holds, there is a point $p \in X \setminus Y$. Write

$$P = \{x, y, u, v, p\}$$

(see Figure 3). Then we have the equality

$$\mathbf{Cs}(P, d|_{P\times P}) = \mathbf{Sym}(P) \tag{2.26}$$

by Lemma 2.5. Furthermore, for every four-point set $S \subseteq P$, equality (2.26) implies

$$\mathbf{Cs}(S, d|_{S\times S}) = \mathbf{Sym}(S). \tag{2.27}$$

Now applying (2.27) and using the permutations

$$\begin{pmatrix} x & y & u & v & p \\ p & y & u & v & x \end{pmatrix}, \quad \begin{pmatrix} x & y & u & v & p \\ x & p & u & v & y \end{pmatrix},$$

$$\begin{pmatrix} x & y & u & v & p \\ x & y & p & v & u \end{pmatrix}, \quad \begin{pmatrix} x & y & u & v & p \\ x & y & u & p & v \end{pmatrix}$$

we see that every four-point subspace $(S, d|_{S\times S})$ of $(P, d|_{P\times P})$ is combinatorially similar to $(Y, d|_{Y\times Y})$. In addition, it is easy to see that, for every four-point set $S \subseteq P$, there is a three-point set T such that

$$T \subseteq S \cap Y. \tag{2.28}$$

Since for every three-point T we have the equalities

$$d(T \times T) = d(Y \times Y) = \{0, a, b, c\},$$

(2.27) implies the equality $d(S\times S) = \{0, a, b, c\}$ for every four-point set $S \subseteq P$, because $(S, d|_{S\times S})$ is combinatorially similar to $(Y, d|_{Y\times Y})$.

Let us consider the ribs (p, x), (p, y), (p, u), (p, v) of the pyramid P. Then, by Pigeonhole principle, we can find two different points in Y, say u and v, such that

$$d(p, u) = d(p, v). \tag{2.29}$$

Write $Z := \{p, u, v\}$. Then we have $\mathbf{Cs}(Z, d|_{Z\times Z}) = \mathbf{Sym}(Z)$ by Lemma 2.5. Consequently, (2.29) implies that $(Z, d|_{Z\times Z})$ is discrete by Lemma 2.4. Thus, (X, d) is also discrete by Proposition 2.7. The discreteness of (X, d) contradicts our assumption that (X, d) is neither strongly rigid nor discrete. The proof is completed. □

Corollary 2.9. *The following conditions are equivalent for every nonempty set X:*

(i) $|X| = 4$.
(ii) *There is a semimetric $d\colon X \times X \to \mathbb{R}$ such that d is neither strongly rigid nor discrete but the equality $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ holds.*

Corollary 2.10. *Let (X, d) be a semimetric space such that the inequality*

$$|X| > \mathfrak{c} \tag{2.30}$$

holds, where $\mathfrak{c}$ is the cardinality of the continuum. Then

$$\mathbf{Cs}(X, d) = \mathbf{Sym}(X) \tag{2.31}$$

holds if and only if

$$\mathbf{Iso}(X, d) = \mathbf{Sym}(X). \tag{2.32}$$

Proof. It suffices to show that (2.31) implies (2.32).

Let (2.31) hold. Then, by Corollary 2.9, d is strongly rigid or discrete. Suppose d is strongly rigid and let p be a point of X. Then the mapping

$$X \ni x \mapsto d(x, p) \in \mathbb{R}$$

is injective. Hence, we have the inequality $|X| \leqslant \mathfrak{c}$ contrary to (2.30). Thus, d is discrete, that implies (2.32). □

We conclude the paper by an interesting example of a four-point metric space (Y, ρ) satisfying the equality $\mathbf{Cs}(Y, \rho) = \mathbf{Sym}(Y)$.

Example 2.11. Recall (see [4] for instance) that a four-point metric space (X, ρ) is called a *pseudolinear quadruple* if for a suitable enumeration of the points we have

$$\rho(x_1, x_2) = \rho(x_3, x_4) = s, \qquad \rho(x_2, x_3) = \rho(x_4, x_1) = t,$$
$$\rho(x_2, x_4) = \rho(x_3, x_1) = s + t,$$

with some positive reals s and t. The pseudolinear quadruple (X, ρ) is combinatorially similar to the rectangle from Example 2.1 if and only if $s \neq t$.

The pseudolinear quadruples and their higher-dimensional modifications appeared for the first time in the famous paper [20] of Karl Menger who in particular gave a criterion for the isometric embeddability of metric spaces into $\mathbb{R}^n$. According to Menger, the pseudolinear quadruples are characterized as the metric spaces not isometric to any subset of $\mathbb{R}$ whose every triple of points embeds isometrically into $\mathbb{R}$. There is an elementary proof of this fact in [6].

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