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GENERALIZED m-QUASI-EINSTEIN STRUCTURE IN ALMOST KENMOTSU MANIFOLDS

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Abstract. The goal of this paper is to analyze the generalized m-quasi-Einstein structure in the context of almost Kenmotsu manifolds. Firstly we showed that a complete Kenmotsu manifold admitting a generalized m-quasi-Einstein structure (g, f, m, λ) is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or a warped product $\widetilde{M} \times_\gamma \mathbb{R}$ under certain conditions. Next, we proved that a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting a closed generalized m-quasi-Einstein metric is locally isometric to some warped product spaces. Finally, a generalized m-quasi-Einstein metric (g, f, m, λ) in almost Kenmotsu 3-H-manifold is considered and proved that either it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

1. Introduction

The study of Einstein manifolds and their several generalizations have received a lot of attention in recent years. A Ricci soliton is a Riemannian metric, which satisfies

$$\frac{1}{2}\mathcal{L}_V g + S - \lambda g = 0,$$

where $\mathcal{L}_V$ denotes the Lie-derivative operator along a potential vector field V, S is the Ricci tensor of g and λ is a constant. Clearly, a trivial Ricci soliton is an Einstein metric with V zero or Killing. When $V = df$, i.e., a gradient of smooth function f, it is called a gradient Ricci soliton (see [4]).

Extending the notion of the m-Bakry-Emery Ricci tensor, Case [5] introduced an interesting generalization of gradient Ricci soliton and Einstein manifold. The m-Bakry-Emery Ricci tensor is defined as follows:

$$S_f^m = S + \nabla^2 f - \frac{1}{m} df \otimes df,$$

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where the integer m satisfies $0 < m \leq \infty$, $\nabla^2 f$ denotes the Hessian form of the smooth function f. The m-Bakery-Emery Ricci tensor arises from the warped product $(M \times N, \bar{g})$ of two Riemannian manifolds (M^n, g) and (N^m, h) with the Riemannian metric $\bar{g} = g + e^{-\frac{2f}{m}} h$. We called a quadruple (g, f, m, λ) on a Riemannian manifold (M, g), m-quasi-Einstein structure if it satisfies the equation

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g \tag{1}$$

for some $\lambda \in \mathbb{R}$. Notice that for $m = \infty$, Eq. (1) gives gradient Ricci soliton and for constant f, it becomes Einstein. The m-quasi-Einstein structure has been deeply studied by [5, 6, 17].

Later on Barros-Ribeiro Jr. [1] and Limoncu [22] generalized and studied equation (1) independently, by considering a 1-form $V^\flat$ instead of df, which satisfied

$$S + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^\flat \otimes V^\flat = \lambda g, \tag{2}$$

where $V^\flat$ is the 1-form associated with the potential vector field V. In particular, if the 1-form $V^\flat$ is closed, we called (2) closed m-quasi-Einstein structure. When $V \equiv 0$, the m-quasi-Einstein structure is said to be trivial, and in this case, the metric becomes an Einstein metric. Ghosh [13, 16] studied contact metric manifolds with quasi-Einstein structures (1) and (2). Recently, Chen [8] studied quasi-Einstein structure (g, V, m, λ) in almost cosympletic manifolds and De et al. [10] studied quasi-Einstein metric (g, f, m, λ) in the context of three-dimensional cosympletic manifolds.

Extending the notion of quasi-Einstein structure, Catino [7] introduced and studied the concept of the generalized quasi-Einstein manifold. A particular case of this was proposed by Barros-Ribeiro Jr. [2] which is defined as follows:

A Riemannian manifold (M^n, g) is said to be generalized m-quasi-Einstein (g, f, m, λ) if there exists a function $\lambda : M^n \to \mathbb{R}$ such that

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g. \tag{3}$$

Notice that for $m = \infty$, (3) reduces to gradient Ricci almost soliton. Also when df is replaced by $V^\flat$, then we called (3) generalized m-quasi-Einstein (g, V, m, λ) structure. Moreover if $V \equiv 0$, then it is said to be trivial. Hu et al. [18, 19] studied generalized m-quasi-Einstein metric (g, f, m, λ) with restriction on Ricci curvature and scalar curvature. Ghosh [14] considered generalized m-quasi-Einstein metric (g, f, m, λ) in Sasakian and K-contact manifolds and showed that it is isometric to the unit sphere $\mathbb{S}^{2n+1}$.

In continuation, we studied the generalized m-quasi-Einstein metric in the framework of almost contact manifolds, namely Kenmotsu and almost Kenmotsu manifolds. The paper is organized as follows: After preliminaries, in Section 3 we analyzed generalized m-quasi-Einstein structure in Kenmotsu manifold. Firstly we constructed some examples of Kenmotsu manifold admitting generalized m-quasi-Einstein structure. Next, we showed that if a complete Kenmotsu manifold whose Reeb vector field leaves the scalar curvature invariant, admits a generalized m-quasi-Einstein structure (g, f, m, λ), then it is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or warped product $\widetilde{M} \times_\gamma \mathbb{R}$. Moreover, a generalized m-quasi-Einstein structure whose potential vector field is pointwise collinear with Reeb vector field is studied. Section 4 is devoted to the study of closed generalized m-quasi-Einstein metric in $(\kappa, \mu)'$-almost Kenmotsu manifold. Finally, we looked at 3-dimensional non-Kenmotsu almost Kenmotsu manifold admitting a generalized m-quasi-Einstein structure and showed that it is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1,1)$-tensor field φ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\varphi^2 X = -X + \eta(X)\xi, \ \eta(X) = g(X, \xi) \tag{4}$$

for all vector field X on M. A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for all vector fields X, Y on M. An almost contact manifold $M^{2n+1}(\varphi, \xi, \eta)$ together with a compatible metric g is known as almost contact metric manifold (see Blair [3]).

An almost Kenmotsu manifold is defined as an almost contact metric manifold if it satisfies $d\eta = 0$ and $\Phi = 2\eta \wedge \Phi$, where the fundamental 2-form Φ of the almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \varphi Y)$ for any vector fields X, Y on M (see [28]). On the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J\big(X, f\frac{d}{dt}\big) = \big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1} \times \mathbb{R}$. If J is integrable, then almost contact metric structure on M^{2n+1} is said to be normal. A normal almost Kenmotsu manifold is called a Kenmotsu manifold (see [21]). An almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

for any vector fields X, Y on M^{2n+1}. On a Kenmotsu manifold the following holds [21]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{7}$$

$$Q\xi = -2n\xi \tag{8}$$

for any vector fields X, Y on M^{2n+1}. Here R is the curvature tensor of g and Q the Ricci operator associated with the $(1,2)$ Ricci tensor S given by $S(X,Y) = g(QX,Y)$ for all vector fields X, Y on M^{2n+1}. It is shown that a Kenmotsu manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c and N^{2n} is a Kählerian manifold [21].

On an almost Kenmotsu manifold the following formula is valid [11, 12]:

$$\nabla_X \xi = -\varphi^2 X - \varphi h X \tag{9}$$

for any vector field X on M^{2n+1}. We define two operators h and ℓ by $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $\ell = R(\cdot,\xi)\xi$ on M^{2n+1} satisfying $h\xi = h'\xi = 0$, $Tr.h = Tr.h' = 0$, $h\varphi = -\varphi h$ where $h' = h \cdot \varphi$ and $Tr.$ denotes trace.

3. Normal almost Kenmotsu manifold

In this section, we studied Kenmotsu manifold admitting generalized m-quasi-Einstein structure. Firstly, we construct some examples of Kenmotsu manifold admitting generalized m-quasi-Einstein metric.

Example 3.1. Let (N, J, g_0) be a Kähler manifold of dimension $2n$. Consider the warped product $(M, g) = (\mathbb{R} \times_\sigma N, dt^2 + \sigma^2 g_0)$, where t is the coordinate on $\mathbb{R}$. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R} \times_\sigma N$ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Then the warped product $\mathbb{R} \times_\sigma N$, $\sigma^2 = ce^{2t}$ with the structure (φ, ξ, η, g) is a Kenmotsu manifold [21]. In particular, if we take $N = \mathbb{CH}^{2n}$, then N being Einstein, the Ricci tensor of M becomes $S = -2ng$. Further we define a smooth function $f(t) = ke^t$, $k > 0$. Then it is easy to verify that (M, f, g, λ) is a generalized m-quasi-Einstein structure with $\lambda = \frac{ke^t}{m}(m - ke^t) - 2n$ on $\mathbb{R} \times_\sigma \mathbb{CH}^{2n}$.

Similarly, a large group of examples of generalized m-quasi-Einstein metric on Kenmotsu manifold can be constructed by taking different potential functions on the warped product.

Example 3.2. Consider the warped product $\mathbb{R} \times_\sigma \mathbb{H}^n$ with metric $g = dt^2 + \sigma^2 g_0$, where g_0 is the standard metric on the hyperbolic space $\mathbb{H}^n$ (see [15]). Let $\sigma(t) = \cosh t$. Then the warped product becomes Einstein manifold with Ricci tensor $S = -ng$ and it admits a generalized m-quasi-Einstein structure $(\mathbb{R} \times_\sigma \mathbb{H}^n, f, g, \lambda)$ with $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t - \frac{\cosh^2 t}{m} - n$.

Example 3.3. Let $M^{2n+1} = \mathbb{R} \times_{\cosh t} \mathbb{CH}^{2n}$ with metric $g = dt^2 + (\cosh^2 t)g_0$, where g_0 is the standard metric on the complex hyperbolic space $\mathbb{CH}^{2n}$ (see [15]). Then M^{2n+1} becomes an Einstein manifold with the Ricci tensor $S^M = -2ng$ (see Lemma 1.1 of [27]). Consider a function $f(x,t) = \sinh t$, then $(M^{2n+1}, f, g, \lambda)$ is a generalized m-quasi-Einstein structure if $\lambda = \sinh t - \frac{\cosh^2 t}{m} - 2n$.

Next we state and proved the following result:

Theorem 3.4. *If the metric of a Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ represents a generalized m-quasi-Einstein structure (g, f, m, λ), then it is η-Einstein, provided $1 + \frac{\xi(f)}{m} \neq 0$. Moreover, if M^{2n+1} is complete and Reeb vector field ξ leaves the scalar curvature invariant, then we have*

(a) *If f has a critical point, then M is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*
(b) *If a function f has no critical points, then M is isometric to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0$, $\gamma > 0$.*

Proof. From (3), we have

$$\nabla_X Df = \lambda X + \frac{1}{m} g(X, Df)Df - QX. \tag{10}$$

Taking the covariant derivative of (10) along arbitrary vector field Y, we get

$$\begin{aligned}\nabla_Y \nabla_X Df = (Y\lambda)X + \lambda(\nabla_Y X) + \frac{1}{m}\{g(X, \nabla_Y Df)Df \\ + g(X, Df)(\nabla_Y Df)\} - (\nabla_Y Q)X - Q(\nabla_Y X).\end{aligned} \tag{11}$$

Making use of (10) and (11) in the relation

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$$

we obtain

$$\begin{aligned}R(X,Y)Df = (X\lambda)Y - (Y\lambda)X + (\nabla_Y Q)X - (\nabla_X Q)Y \\ + \frac{\lambda}{m}[g(Y, Df)X - g(X, Df)Y] \\ + \frac{1}{m}[g(X, Df)QY - g(Y, Df)QX].\end{aligned} \tag{12}$$

Taking an inner product of (12) with ξ and using (8) yields

$$\begin{aligned}&g(R(X,Y)Df, \xi) \\ &= (X\lambda)\eta(Y) - (Y\lambda)\eta(X) + g((\nabla_Y Q)\xi, X) \\ &\quad - g((\nabla_X Q)\xi, Y) + \frac{(\lambda + 2n)}{m}[g(Y, Df)\eta(X) - g(X, Df)\eta(Y)].\end{aligned} \tag{13}$$

Taking an inner product of (7) with Df and inserting it in the last equation (13) we obtain

$$\begin{aligned}&(X\lambda)\eta(Y)-(Y\lambda)\eta(X)+g((\nabla_Y Q)\xi,X)-g((\nabla_X Q)\xi,Y)\\ &+\frac{(\lambda+2n+m)}{m}[g(Y,Df)\eta(X)-g(X,Df)\eta(Y)]=0.\end{aligned} \tag{14}$$

Replacing Y by ξ in (14) and making use of the relation $(\nabla_\xi Q)Y=-2QY-4nY$ (see Lemma 2 of [15]) we get

$$\sigma Df-mD\lambda=\{\sigma(\xi f)-m(\xi\lambda)\}\xi, \tag{15}$$

where $\sigma=m+\lambda+2n$. Contracting (12) along arbitrary vector field X gives

$$\frac{(m-1)}{m}S(Y,Df)=\frac{1}{2}(Yr)-2n(Y\lambda)+\frac{1}{m}(2n\lambda-r)g(Y,Df). \tag{16}$$

Replacing Y by ξ and using (8) in (16) we get

$$\frac{1}{m}(2n\sigma-4n^2-r-2n)(\xi f)-2n(\xi\lambda)+\frac{1}{2}(\xi r)=0. \tag{17}$$

Also on the Kenmotsu manifold, we have $\xi r=-2(r+2n(2n+1))$ (Lemma 2 of [15]). Inserting this in the last equation infer

$$\frac{2n}{m}[\sigma(\xi f)-m(\xi\lambda)]=\{r+2n(2n+1)\}\{1+\frac{(\xi f)}{m}\}. \tag{18}$$

Replacing Y by ξ in (12) and using the relation $R(X,\xi)Y=g(X,Y)\xi-\eta(Y)X$, we obtain

$$\begin{aligned}\frac{1}{m}g(X,\sigma Df-mD\lambda)\xi=&\frac{(\sigma-2n)}{m}(\xi f)X-(\xi\lambda)X\\ &-(1+\frac{\xi f}{m})QX-2nX.\end{aligned} \tag{19}$$

Combining (15), (18) and (19) we obtain the following relation

$$(1+\frac{\xi f}{m})(\frac{r}{2n}+2n+1)\eta(X)\xi=(1+\frac{\xi f}{m})\{(\frac{r}{2n}+1)X-QX\}. \tag{20}$$

If possible take $1+\frac{\xi f}{m}\neq 0$, then from the last equation we get

$$QX=(1+\frac{r}{2n})X-(\frac{r}{2n}+2n+1)\eta(X)\xi \tag{21}$$

for any vector field X on M. Therefore, M is η-Einstein.

Suppose that ξ leaves the scalar curvature r invariant, i.e., $\xi r=0$. Consequently, $r=-2n(2n+1)$. By virtue of this in (21) we get $QX=-2nX$, i.e., M is Einstein. Inserting $r=-2n(2n+1)$ in (18) gives $\sigma(\xi f)-m(\xi\lambda)=0$ and hence (15) implies $D\lambda=\frac{\sigma}{m}Df$. Now we consider a function $u=e^{-\frac{f}{m}}$ on M. Then it follows $Du=-\frac{u}{m}Df$. Taking covariant derivative of the forgoing expression along arbitrary vector field X we get

$$\nabla_X Df-\frac{1}{m}g(X,Df)Df=-\frac{m}{u}\nabla_X Du. \tag{22}$$

Using (22) along with the fact that $QX = -2nX$, (10) yields

$$\nabla_X Du = -\frac{(\lambda + 2n)u}{m} X. \tag{23}$$

Also we have $(\lambda + m + 2n)Df = mD\lambda$, simplifying it gives $D(\lambda u) = -(m + 2n)Du$ which implies $\lambda u = -(m + 2n)u + k$, k is a constant. Inserting the forgoing relations in (23) we get

$$\nabla_X Du = (u - \frac{k}{m})X.$$

Applying Kanai's theorem [20], we conclude that if f has a critical point, then M is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or if f is without critical points, then M is isometric to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0, \gamma > 0$. □

Remark 3.5. Suppose $1 + \frac{\xi f}{m} = 0$ in some open set $\mathcal{O}$ of M. Then $\xi f = -m$, since Kenmotsu manifold is locally isometric to the warped product $(-\epsilon, \epsilon) \times_{ce^t} N$, where N is a Kähler manifold of dimension $2n$ and $(-\epsilon, \epsilon)$ is an open interval [21]. Using the local parametrization: $\xi = \frac{\partial}{\partial t}$ then we have $\frac{\partial f}{\partial t} = -m$ hence the potential function is $f = -mt$, $t > 0$.

Theorem 3.6. *If a Kenmotsu manifold admits a non-trivial generalized m-quasi-Einstein structure (g, V, m, λ) whose potential vector field is pointwise collinear with the Reeb vector field ξ, then it is η-Einstein.*

Proof. Suppose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then $V = F\xi$, where F is a smooth function. Differentiating covariantly along arbitrary vector field X of $V = F\xi$ and using (6) we get

$$\nabla_X V = (XF)\xi + F(-\varphi^2 X - \varphi hX). \tag{24}$$

Inserting (24) in (2) gives

$$\begin{aligned} &S(X, Y) + \frac{1}{2}[(XF)\eta(Y) + (YF)\eta(X)] + Fg(h'X, Y) \\ &- (\frac{F^2}{m} + F)\eta(X)\eta(Y) = (\lambda - F)g(X, Y) \end{aligned} \tag{25}$$

for all vector fields X, Y. Replacing X, Y by ξ in (25) and using (8) we get $\xi F = \lambda + 2n + \frac{F^2}{m}$. Further taking Y as ξ and using the last expression in (25) we obtain

$$XF = (\lambda + \frac{F^2}{m} + 2n)\eta(X). \tag{26}$$

Contracting (25) and inserting in the above equation (26), yields

$$r = 2n(\lambda - F - 1). \tag{27}$$

In consequence of (26) and (27), Equation (25) reduces to the following form:

$$QX = (\frac{r}{2n} + 1)X - (\frac{r}{2n} + 2n + 1)\eta(X)\xi \tag{28}$$

for any vector field X. Thus manifold is η-Einstein. This completes the proof. □

Suppose F is constant. Then (26) gives $\lambda = -2n - \frac{F^2}{m}$. This in (27) implies r is constant. Hence $\xi r = 0$ which implies $r = -2n(2n+1)$. Inserting the values of r and λ in (27) gives $F = -m$ which further implies $\lambda = -m - 2n$. Hence we can state the following:

Corollary 3.7. *If a Kenmotsu manifold admits a non-trivial generalized m-quasi-Einstein structure (g, V, m, λ) whose potential vector is a constant multiple of Reeb vector field ξ, then it is Einstein, i.e., $QX = -2nX$ with $\lambda = -m - 2n$.*

4. Non-normal almost Kenmotsu manifold

An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized (κ, μ)-almost Kenmotsu manifold if ξ belongs to the generalized (κ, μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{29}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa, \mu)'$-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y] \tag{30}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M and $h' = h \circ \varphi$ (see [12]). Moreover if both κ and μ are constants in (30), then M is called a $(\kappa, \mu)'$-almost Kenmotsu manifold (see [12, 24, 31]). On generalized (κ, μ) or $(\kappa, \mu)'$-almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$), the following relations hold [12]:

$$h'^2 = (\kappa + 1)\varphi^2, \; h^2 = (\kappa + 1)\varphi^2, \tag{31}$$

$$Q\xi = 2n\kappa\xi. \tag{32}$$

It follows from (31) that $\kappa \leq -1$ and $\nu = \pm\sqrt{-\kappa - 1}$, where ν is an eigenvalue corresponding to eigenvector $X \in \mathcal{D}$ ($\mathcal{D} = \mathrm{Ker}(\eta)$) of h'. The equality holds if and only if $h = 0$ (equivalently, $h' = 0$). Thus $h' \neq 0$ if and only if $\kappa < -1$.

Lemma 4.1 ([31])**.** *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$. For $n > 1$, the Ricci operator Q of M can be expressed as*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - [\mu - 2(n-1)h']X$$

for any vector field X on M. Further, if κ and μ are constants and $n \geq 1$, then $\mu = -2$ and hence

$$QX = -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'X \tag{33}$$

for any vector field X on M. In both cases, the scalar curvature of M is $2n(\kappa - 2n)$.

Proposition 4.2. *There does not exist generalized m-quasi-Einstein structure with $\varphi V = 0$ in $(\kappa,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$.*

Proof. By hypothesis we have $\varphi V = 0$. Operating this with φ gives $V = \eta(V)\xi$, i.e., $V = F\xi$, where F is a smooth function. Taking covariant derivative along arbitrary vector field X of the last equation and inserting it in (2) we obtain

$$\begin{aligned} &S(X,Y) + \frac{1}{2}[(XF)\eta(Y) + (YF)\eta(X)] + Fg(h'X,Y) \\ &-(\frac{F^2}{m} + F)\eta(X)\eta(Y) = (\lambda - F)g(X,Y). \end{aligned} \tag{34}$$

Replacing X by ξ in (34) yields

$$\frac{1}{2}(YF) = [\lambda + \frac{F^2}{m} - 2n\kappa - \frac{1}{2}(\xi F)]\eta(Y) \tag{35}$$

for any vector field X on M. Contracting (32) and using Lemma 4.1, we get

$$\xi F = (2n+1)\lambda - 2n(\kappa - 2n) + \frac{F^2}{m} - 2nF. \tag{36}$$

Replacing Y by ξ in (35) and combining it with (36) gives $F = \lambda + 2n$. Inserting (35) in (34) and using it in Lemma 4.1, we obtain

$$\{2\lambda + \frac{F^2}{m} - F - 2n\kappa + 2n - (\xi F)\}\eta(X)\eta(Y) + \lambda g(h'X,Y) = 0. \tag{37}$$

Replacing X by $h'X$ in (37) implies $\lambda(\kappa+1)g(\varphi X,\varphi Y) = 0$. Since $h' \neq 0$ and $\kappa < -1$, we get $\lambda = 0$ and using it in $F = \lambda + 2n$ gives $F = 2n$. In a consequence of this in (36) we get $\kappa = \frac{2n}{m}$, a contradiction. This completes the proof. □

Now using the above lemmas and proposition we proved the following:

Theorem 4.3. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a $(\kappa,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$. If g admits a closed generalized m-quasi-Einstein metric, then we get one of the following:*

1. *M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*
2. *M^{2n+1} is locally isometric to the warped product*

$$\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n,\ \mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n,$$

where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2m}{n} - \frac{m^2}{n^2}$, $\mathbb{B}^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2m}{n} - \frac{m^2}{n^2}$, $f = ce^{(1-\frac{m}{n})t}$ and $f' = c'e^{(1+\frac{m}{n})t}$, where c, c' are positive constants.

Proof. Since $V^\flat$ is closed, Eq. (2) implies

$$\nabla_X V = \lambda X + \frac{1}{m} g(X,V)V - QX. \tag{38}$$

Making use of the relation $R(X,Y)V = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V$ in (38) we get

$$\begin{aligned} R(X,Y)V &= (X\lambda)Y - (Y\lambda)X + (\nabla_Y Q)X - (\nabla_X Q)Y \\ &\quad + \frac{\lambda}{m}\{g(Y,V)X - g(X,V)Y\} + \frac{1}{m}\{g(X,V)QY - g(Y,V)QX\}. \end{aligned} \tag{39}$$

Taking an inner product of (39) with ξ and using Lemma 4.4 of [25] we obtain

$$\begin{aligned} g(R(X,Y)V,\xi) &= (X\lambda)\eta(Y) - (Y\lambda)\eta(X) \\ &\quad + g(Q\varphi hY, X) - g(Q\varphi hX, Y) \\ &\quad + \frac{(\lambda - 2n\kappa)}{m}\{g(Y,V)\eta(X) - g(X,V)\eta(Y)\}. \end{aligned} \tag{40}$$

Contracting (39) and making use of the fact that scalar curvature is constant yields

$$\frac{(m-1)}{m} S(Y,V) = -2n(Y\lambda) + \frac{1}{m}(2n\lambda - r)g(Y,V). \tag{41}$$

Taking an inner product of (30) with V and inserting it in (40) we get

$$(\xi\lambda)\xi - D\lambda - \frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi^2 V + 2h'V = 0. \tag{42}$$

Operating by φ in (42) yields

$$\frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi V - \varphi D\lambda + 2\varphi h'V = 0. \tag{43}$$

Making use of (33) in (41) and operating the obtained expression by φ we get

$$\{2n\lambda - r + 2n(m-1)\}\varphi V - 2nm\varphi D\lambda + 2n(m-1)\varphi h'V = 0. \tag{44}$$

Combining (43) and (44) we get

$$[2n(\lambda + m - 1) - r - \frac{n(m-1)}{m}\{\lambda - (2n+m)\kappa\}]\varphi V - n(1+m)\varphi D\lambda = 0$$

implies

$$[2n(\lambda + m - 1) - r - \frac{n(m-1)}{m}\{\lambda - (2n+m)\kappa\}]V - n(1+m)D\lambda \in \mathbb{R}\xi.$$

Therefore we can write

$$D\lambda = \alpha V + s\xi, \tag{45}$$

where

$$\alpha = \frac{1}{n(m+1)}[2n(\lambda + m - 1) - r - \frac{n(m-1)}{m}\{\lambda - (2n+m)\kappa\}]$$

and s is a smooth function on M. Inserting (45) in (42) gives

$$(\xi\lambda)\xi - \alpha V - s\xi - \frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi^2 V + 2h'V = 0. \tag{46}$$

Operating (46) by h' we get

$$\frac{1}{m}\{\lambda - (2n+m)\kappa - \alpha m\}h'V + 2(\kappa+1)\varphi^2 V = 0.$$

Inserting the last equation in (46) we obtain

$$\begin{aligned} 4(\kappa+1)\varphi^2 V = &\frac{1}{m}[\lambda - (2n+m)\kappa - \alpha m] \\ &\times [(\xi\lambda)\xi - \alpha V - s\xi - \frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi^2 V], \end{aligned} \tag{47}$$

then operating (47) by φ and using Proposition 4.2, we get

$$[\lambda - (2n+m)\kappa - \alpha m]^2 + 4m^2(\kappa+1) = 0 \tag{48}$$

implies λ is constant. Replacing Y by ξ in (41) and taking λ as constant, gives

$$[\lambda - \frac{r}{2n} - \kappa(m-1)]\eta(V) = 0. \tag{49}$$

So we get either $\eta(V) = 0$ or $\lambda - \frac{r}{2n} - \kappa(m-1) = 0$.

Case-I: Suppose $\eta(V) = 0$. Then taking covariant derivative along ξ and using (38) gives $\lambda = 2n\kappa$. Inserting this in (48) we get $\kappa = -2$. Without loss of generality, we may choose $\nu = 1$. In consequence of this in Theorem 5.1 [24] we get

$$\begin{aligned} &R(X_\nu, Y_\nu)Z_\nu = -4[g(Y_\nu, Z_\nu)X_\nu - g(X_\nu, Z_\nu)Y_\nu], \\ &R(X_{-\nu}, Y_{-\nu})Z_{-\nu} = 0 \end{aligned}$$

for any $X_\nu, Y_\nu, Z_\nu \in [\nu]'$ and $X_{-\nu}, Y_{-\nu}, Z_{-\nu} \in [-\nu]'$. Making use of the fact that $\mu = -2$ it follows from Proposition 4.1 [12] and Proposition 4.3 [12] that $K(X, \xi) = -4$ for any $X \in [\nu]'$ and $K(X, \xi) = 0$ for any $X \in [-\nu]'$. As shown in [12] that the distribution $[\xi] \oplus [\nu]'$ is integrable with totally geodesic leaves and the distribution $[-\nu]'$ is integrable with total umbilical leaves by $H = -(1-\nu)\xi$, where H is the mean curvature vector field for the leaves of $[-\nu]'$ immersed in M^{2n+1}. Taking $\nu = 1$, then the two distribution $[\xi] \oplus [\nu]'$ and $[-\nu]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Hence M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case-II: If $\lambda - \frac{r}{2n} - \kappa(m-1) = 0$, then inserting the value of scalar curvature from Lemma 4.1 gives $\lambda = m\kappa - 2n$. Using this in (48) implies $\kappa = -1 - \frac{m^2}{n^2}$. By applying Dileo-Pastore [12] result we complete the proof. □

Remark 4.4. When $V = Df$, it is clear that $V^\flat$ is closed. Therefore if a non-normal $(\kappa, \mu)'$-almost Kenmotsu manifold admits a generalized m-quasi-Einstein structure (g, f, m, λ), then we get similar results as in Theorem 4.3.

As a particular case of Theorem 4.3, for $m = \infty$ we easily obtain Theorem 3.1 [29].

Let $\mathcal{U}_1$ be the open subset of a 3-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighbourhood of } p\}$. Therefore $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$ we may set $he_1 = \vartheta e_1$ and $he_2 = -\vartheta e_2$, where ϑ is a positive function.

Lemma 4.5 ([9]). *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_\xi e = a\varphi e,\ \nabla_\xi \varphi e = -ae,\\
&\nabla_e \xi = e - \vartheta\varphi e,\ \nabla_e e = -\xi - b\varphi e,\ \nabla_e \varphi e = \vartheta\xi + be,\\
&\nabla_{\varphi e} \xi = -\vartheta e + \varphi e,\ \nabla_{\varphi e} e = \vartheta\xi + c\varphi e,\ \nabla_{\varphi e}\varphi e = -\xi - ce,
\end{aligned}$$

where a, b, c are smooth functions.

From Lemma 4.5, the poisson brackets for $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ are as follows:

$$\begin{cases} [e_3, e_1] = (a + \vartheta)e_2 - e_1,\\ [e_1, e_2] = be_1 - ce_2,\\ [e_2, e_3] = (a - \vartheta)e_1 + e_2. \end{cases} \tag{50}$$

Then the expression for Ricci operator are as follows:

Lemma 4.6. *The Ricci operator Q with respect to the local basis $\{\xi, e, \varphi e\}$ on $\mathcal{U}_1$ can be written as*

$$\begin{aligned}
Q\xi &= -2(\vartheta^2 + 1)\xi - (\varphi e(\vartheta) + 2\vartheta b)e - (e(\vartheta) + 2\vartheta c)\varphi e,\\
Qe &= -(\varphi e(\vartheta) + 2\vartheta b)\xi - (A + 2\vartheta a)e + (\xi(\vartheta) + 2\vartheta)\varphi e,\\
Q\varphi e &= -(e(\vartheta) + 2\vartheta c)\xi + (\xi(\vartheta) + 2\vartheta)e - (A - 2\vartheta a)\varphi e,
\end{aligned}$$

where we set $A = e(c) + b^2 + c^2 + \varphi e(b) + 2$ for simplicity.

Before proceeding to the main result, we recollect a few basic notions of harmonic vector fields. Perrone [26] characterized the harmonicity of an almost Kenmotsu manifold. Let (M^n, g) be a Riemannian manifold and (T^1M, g_s) its unit tangent sphere bundle furnished with the well-known standard Sasakian metric g_s. If M is compact, then the energy $E(V)$ is defined as the energy of the corresponding map V from (M, g) into (T^1M, g_s) by

$$E(V) = \frac{1}{2}\int_M ||dV||^2 dv_g = \frac{m}{2}Vol(M, g) + \frac{1}{2}\int_M ||\nabla V||^2 dv_g,$$

where E indicates the energy function and ∇ being the Levi-Civita connection of g. V, a unit vector field is named harmonic if it is a critical point for E defined on the set of all unit vector fields $\Psi^1(M)$, that is

$$\bar{\Delta}V - ||\nabla V||^2 V = 0,$$

where $\bar{\Delta}$ indicates the rough Laplacian, that is, $\bar{\Delta}V = -tr\nabla^2 V$. The critical point condition still specifies a harmonic vector field even though M is non-compact. A Kenmotsu 3-manifold's Reeb vector field is always harmonic. Now we give the subsequent definition (see [30]).

Definition. An almost Kenmotsu 3-manifold with harmonic Reeb vector field or equivalently, the Reeb vector field is an eigenvector field of the Ricci operator, is called almost Kenmotsu 3-H-manifold.

Now we state and prove the following:

Theorem 4.7. *If a 3-dimensional almost Kenmotsu 3-H-manifold with $h' \neq 0$ admits a generalized m-quasi-Einstein (g, f, m, λ) structure whose potential function is constant along the Reeb vector field, then it is Einstein or is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure.*

Proof. For an almost Kenmotsu 3-H-manifold from Lemma 4.6, we have

$$e(\vartheta) = -2\vartheta c, \ \varphi e(\vartheta) = -\vartheta b. \tag{51}$$

By our assumption, since potential function is constant along the Reeb vector field, we can write

$$Df = f_1 e + f_2 \varphi e \tag{52}$$

for smooth functions $f_1 = f(e)$ and $f_2 = \varphi e(f)$. Substituting $X = \xi$ in (10) and using Lemma 4.5, Lemma 4.6 and (52) gives

$$\begin{cases} \xi f_1 - a f_2 = 0, \\ a f_1 + \xi(f_2) = 0, \\ \lambda = 2(\vartheta^2 + 1). \end{cases} \tag{53}$$

Again, putting $X = e$ in (10) and then using Lemma 4.5, Lemma 4.6 and (52) gives

$$\begin{cases} e(f_1) + b f_2 = \lambda + \frac{f_1^2}{m} - A - 2\vartheta a, \\ \vartheta f_2 - f_1 = 0, \\ e(f_2) - b f_1 = \frac{f_1 f_2}{m} - \xi(\vartheta) - 2\vartheta. \end{cases} \tag{54}$$

Similarly, for $X = \varphi e$, we get

$$\begin{cases} \varphi e(f_1) - c f_2 = \frac{f_1 f_2}{m} - \xi(\vartheta) - 2\vartheta, \\ \varphi e(f_2) + c f_1 = \lambda + \frac{f_2^2}{m} + A - 2a\vartheta, \\ \vartheta f_1 - f_2 = 0. \end{cases} \tag{55}$$

Comparing the second argument of (54) and third argument of (55), we get $(\vartheta^2 - 1) f_2 = 0$. If $f_2 = 0$, then third argument of (55) implies $f_1 = 0$, then (52) gives $Df = 0$, that is, f is constant.

For the case $f_2 \neq 0$, we have $\vartheta = 1$. In consequence, second argument of (54) and third argument of (55) gives $f_1 = f_2$. Moreover, taking $\vartheta = 1$ in (51), we get $b = c = 0$. Also, first and second equation of (53) gives $a = 0$ when $f_1 = f_2$. Inserting the above values in (50), we get

$$[e_3, e_1] = e_2 - e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] = -e_1 + e_2.$$

Using Milnor's result [23], we can conclude that M^3 is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure. This completes the proof. □

In consequence of Theorem 4.7, we can state the following corollary.

Corollary 4.8. *If a 3-dimensional almost Kenmotsu 3-H-manifold admits a non-trivial generalized m-quasi-Einstein (g, f, m, λ) structure whose potential function is constant along the Reeb vector field, then it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. We shall divide the prove into two cases:

Case-I: When $h = 0$, then M is a Kenmotsu manifold. Then we have

$$QX = (\frac{r}{2} + 1)X - (\frac{r}{2} + 3)\eta(X)\xi. \tag{56}$$

By assumption, $\xi f = 0$. Replacing $X = \xi$ in (10) then taking inner product with ξ gives $\lambda = -2n$ under our assumptions. In consequence, (7) gives $\xi r = 0$. Since $\xi r = -2(r + 6)$, we get $r = -6$ which reduces (56) to $QX = -2X$. Clearly, M^3 is conformally flat.

Case-II: When $h \neq 0$, then by Theorem 4.7, we have $a = b = c = 0$. From Lemma 4.6, we see that $r = -2(\vartheta^2 + 1) - 2A$. Making use of the fact that $a = b = c = 0$ implies $r = -8$. It is easy to see that M^3 is conformally flat.

Applying Wang's theorem ([30, Theorem 1.6]), we can conclude that M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. □

Corollary 4.9. *If a 3-dimensional almost Kenmotsu 3-H-manifold admits a non-trivial m-quasi-Einstein (g, f, m, λ) structure whose potential function is constant along the Reeb vector field, then either it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Next, we constructed an example of almost Kenmotsu manifold admitting a generalized m-quasi-Einstein structure.

Example 4.10. Let $(N, J, \bar{g})$ be a strictly almost Kähler Einstein manifold. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R} \times_f N$ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Consider a metric $g = g_0 + \sigma^2 \bar{g}$, where $\sigma^2 = ce^{2t}$, g_0 is the Euclidean metric on $\mathbb{R}$ and c is a positive constant. Then it is easy to verify that the warped product $\mathbb{R} \times_\sigma N$, $\sigma^2 = ce^{2t}$, with the structure (φ, ξ, η, g) is an almost Kenmotsu manifold [11]. Since N is Einstein $S = -2ng$. We define a smooth function $f(x, t) = t^2$.

then it is easy to verify that the warped product $\mathbb{R} \times_\sigma N$, $\sigma^2 = ce^{2t}$ admits a generalized m-quasi-Einstein structure (g, f, m, λ) with $\lambda = \frac{2}{m}(m(1-n)-2t^2)$.

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