Bull. Korean Math. Soc. **60** (2023), No. 3, pp. 705–715
https://doi.org/10.4134/BKMS.b220347
pISSN: 1015-8634 / eISSN: 2234-3016

p-BIHARMONIC HYPERSURFACES IN EINSTEIN SPACE AND CONFORMALLY FLAT SPACE

Ahmed Mohammed Cherif and Khadidja Mouffoki

Abstract. In this paper, we present some new properties for p-biharmonic hypersurfaces in a Riemannian manifold. We also characterize the p-biharmonic submanifolds in an Einstein space. We construct a new example of proper p-biharmonic hypersurfaces. We present some open problems.

1. Introduction

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. The p-energy functional of φ is defined by

$$E_p(\varphi; D) = \frac{1}{p}\int_D |d\varphi|^p v_g, \tag{1}$$

where D is a compact domain in M, $|d\varphi|$ the Hilbert-Schmidt norm of the differential $d\varphi$, v_g the volume element on (M^m, g), and $p \geq 2$.

A smooth map is called p-harmonic if it is a critical point of the p-energy functional (1). We have

$$\frac{d}{dt}E_p(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_p(\varphi), v)v_g,$$

where $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ is a smooth variation of φ supported in D, $v = \frac{\partial\varphi_t}{\partial t}\big|_{t=0}$ the variation vector field of φ, and $\tau_p(\varphi) = \operatorname{div}^M(|d\varphi|^{p-2}d\varphi)$ the p-tension field of φ.

Let ∇^M be the Levi-Civita connection of (M^m, g), and ∇^φ be the pull-back connection on $\varphi^{-1}TN$. Then the map φ is p-harmonic if and only if (see [1,3,5])

$$|d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\operatorname{grad}^M |d\varphi|) = 0,$$

Received May 20, 2022; Accepted September 20, 2022.
2020 *Mathematics Subject Classification.* Primary 53C43, 58E20, 53C25.
Key words and phrases. p-biharmonic maps, p-biharmonic submanifolds, Einstein space.
The authors would like to thank the editor and the reviewers for their useful remarks and suggestions. Partially supported by National Agency Scientific Research of Algeria.

where $\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi$ is the tension field of φ (see [2,4]). The p-bienergy functional of φ is defined by

$$E_{2,p}(\varphi; D) = \frac{1}{2}\int_D |\tau_p(\varphi)|^2 v_g. \tag{2}$$

We say that φ is a p-biharmonic map if it is a critical point of the p-bienergy functional (2), the Euler-Lagrange equation of the p-bienergy functional is given by (see [7])

$$\begin{aligned}\tau_{2,p}(\varphi) = & -|d\varphi|^{p-2}\operatorname{trace}_g R^N(\tau_p(\varphi), d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi |d\varphi|^{p-2}\nabla^\varphi \tau_p(\varphi) \\ & -(p-2)\operatorname{trace}_g \nabla\langle\nabla^\varphi \tau_p(\varphi), d\varphi\rangle |d\varphi|^{p-4} d\varphi = 0,\end{aligned}$$

where R^N is the curvature tensor of (N^n, h) defined by

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z, \quad \forall X, Y, Z \in \Gamma(TN),$$

and ∇^N the Levi-Civita connection of (N^n, h). The p-energy functional (resp. p-bienergy functional) includes as a special case ($p = 2$) the energy functional (resp. bienergy functional), whose critical points are the usual harmonic maps (resp. biharmonic maps [6]).

A submanifold in a Riemannian manifold is called a p-harmonic submanifold (resp. p-biharmonic submanifold) if the isometric immersion defining the submanifold is a p-harmonic map (resp. p-biharmonic map). Will call proper p-biharmonic submanifolds a p-biharmonic submanifols which is non p-harmonic.

2. Main results

Let (M^m, g) be a hypersurface of $(N^{m+1}, \langle,\rangle)$, and $\mathbf{i}: (M^m, g) \hookrightarrow (N^{m+1}, \langle,\rangle)$ the canonical inclusion. We denote by ∇^M (resp. ∇^N) the Levi-Civita connection of (M^m, g) (resp. of $(N^{m+1}, \langle,\rangle)$), grad^M (resp. grad^N) the gradient operator in (M^m, g) (resp. in $(N^{m+1}, \langle,\rangle)$, B the second fundamental form of the hypersurface (M^m, g), A the shape operator with respect to the unit normal vector field η, H the mean curvature of (M^m, g), $\nabla^\perp$ the normal connection of (M^m, g), and by Δ (resp. $\Delta^\perp$) the Laplacian on (M^m, g) (resp. on the normal bundle of (M^m, g) in $(N^{m+1}, \langle,\rangle)$ (see [2,8,10]). Under the notation above we have the following results.

Theorem 2.1. *The hypersurface (M^m, g) with the mean curvature vector $H = f\eta$ is p-bihamronic if and only if*

$$\begin{cases} -\Delta^M(f) + f|A|^2 - f\operatorname{Ric}^N(\eta,\eta) + m(p-2)f^3 = 0; \\ 2A(\operatorname{grad}^M f) - 2f(\operatorname{Ricci}^N \eta)^\top + (p-2+\dfrac{m}{2})\operatorname{grad}^M f^2 = 0, \end{cases} \tag{3}$$

where Ric^N (*resp.* Ricci^N) *is the Ricci curvature* (*resp. Ricci tensor*) *of* $(N^{m+1}, \langle,\rangle)$.

Proof. Choose a normal orthonormal frame $\{e_i\}_{i=1,\dots,m}$ on (M^m, g) at x, so that $\{e_i, \eta\}_{i=1,\dots,m}$ is an orthonormal frame on the ambient space $(N^{m+1}, \langle,\rangle)$. Note that, $d\mathbf{i}(X) = X$, $\nabla^{\mathbf{i}}_X Y = \nabla^N_X Y$, and the p-tension field of $\mathbf{i}$ is given by $\tau_p(\mathbf{i}) = m^{\frac{p}{2}} f\eta$. We compute the p-bitension field of $\mathbf{i}$

$$\begin{aligned}\tau_{2,p}(\mathbf{i}) = &-|d\mathbf{i}|^{p-2}\operatorname{trace}_g R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i}\\ &-(p-2)\operatorname{trace}_g \nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i}\\ &-\operatorname{trace}_g \nabla^{\mathbf{i}}|d\mathbf{i}|^{p-2}\nabla^{\mathbf{i}}\tau_p(\mathbf{i}).\end{aligned} \tag{4}$$

The first term of (4) is given by

$$\begin{aligned}-|d\mathbf{i}|^{p-2}\operatorname{trace}_g R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i} &= -|d\mathbf{i}|^{p-2}\sum_{i=1}^m R^N(\tau_p(\mathbf{i}), d\mathbf{i}(e_i))d\mathbf{i}(e_i)\\ &= -m^{p-1} f\sum_{i=1}^m R^N(\eta, e_i)e_i\\ &= -m^{p-1} f \operatorname{Ricci}^N \eta\\ &= -m^{p-1} f\left[(\operatorname{Ricci}^N \eta)^\perp + (\operatorname{Ricci}^N \eta)^\top\right].\end{aligned}$$

We compute the second term of (4)

$$-(p-2)\operatorname{trace}_g \nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i} = -(p-2)m^{p-2}\sum_{i,j=1}^m \nabla^N_{e_j}\langle\nabla^N_{e_i} f\eta, e_i\rangle e_j,$$

$$\begin{aligned}\sum_{i=1}^m\langle\nabla^N_{e_i} f\eta, e_i\rangle &= \sum_{i=1}^m\left[\langle e_i(f)\eta, e_i\rangle + f\langle\nabla^N_{e_i}\eta, e_i\rangle\right]\\ &= -f\sum_{i=1}^m\langle\eta, B(e_i,e_i)\rangle\\ &= -mf^2.\end{aligned}$$

By the last two equations, we have the following

$$-(p-2)\operatorname{trace}_g \nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i} = m^{p-1}(p-2)\left(\operatorname{grad}^M f^2 + mf^3\eta\right). \tag{5}$$

The third term of (4) is given by

$$\begin{aligned}-\operatorname{trace}_g \nabla^{\mathbf{i}}|d\mathbf{i}|^{p-2}\nabla^{\mathbf{i}}\tau_p(\mathbf{i}) &= -m^{p-1}\sum_{i=1}^m \nabla^N_{e_i}\nabla^N_{e_i} f\eta\\ &= -m^{p-1}\sum_{i=1}^m \nabla^N_{e_i}[e_i(f)\eta + f\nabla^N_{e_i}\eta]\\ &= -m^{p-1}\left[\Delta^M(f)\eta + 2\nabla^N_{\operatorname{grad}^M f}\eta + f\sum_{i=1}^m \nabla^N_{e_i}\nabla^N_{e_i}\eta\right].\end{aligned} \tag{6}$$

Thus, at x, we obtain

$$\begin{aligned}\sum_{i=1}^{m}\nabla^N_{e_i}\nabla^N_{e_i}\eta &= \sum_{i=1}^{m}\nabla^N_{e_i}\left[(\nabla^N_{e_i}\eta)^{\perp}+(\nabla^N_{e_i}\eta)^{\top}\right]\\ &= -\sum_{i=1}^{m}\nabla^N_{e_i}A(e_i)\\ &= -\sum_{i=1}^{m}\nabla^M_{e_i}A(e_i)-\sum_{i=1}^{m}B(e_i,A(e_i)). \end{aligned}\tag{7}$$

Since $\langle A(X),Y\rangle=\langle B(X,Y),\eta\rangle$ for all $X,Y\in\Gamma(TM)$, we get

$$\begin{aligned}\sum_{i=1}^{m}\nabla^M_{e_i}A(e_i) &= \sum_{i,j=1}^{m}\langle\nabla^M_{e_i}A(e_i),e_j\rangle e_j\\ &= \sum_{i,j=1}^{m}\left[e_i\langle A(e_i),e_j\rangle e_j-\langle A(e_i),\nabla^M_{e_i}e_j\rangle e_j\right]\\ &= \sum_{i,j=1}^{m}e_i\langle B(e_i,e_j),\eta\rangle e_j\\ &= \sum_{i,j=1}^{m}e_i\langle\nabla^N_{e_j}e_i,\eta\rangle e_j\\ &= \sum_{i,j=1}^{m}\langle\nabla^N_{e_i}\nabla^N_{e_j}e_i,\eta\rangle e_j. \end{aligned}$$

By using the definition of curvature tensor of $(N^{m+1},\langle,\rangle)$, we conclude

$$\begin{aligned}\sum_{i=1}^{m}\nabla^M_{e_i}A(e_i) &= \sum_{i,j=1}^{m}\Big[\langle R^N(e_i,e_j)e_i,\eta\rangle e_j+\langle\nabla^N_{e_j}\nabla^N_{e_i}e_i,\eta\rangle e_j\Big]\\ &= \sum_{i,j=1}^{m}\Big[-\langle R^N(\eta,e_i)e_i,e_j\rangle e_j+\langle\nabla^N_{e_j}\nabla^N_{e_i}e_i,\eta\rangle e_j\Big]\\ &= -\sum_{j=1}^{m}\langle\operatorname{Ricci}^N\eta,e_j\rangle e_j+\sum_{i,j=1}^{m}e_j\langle\nabla^N_{e_i}e_i,\eta\rangle e_j-\sum_{i,j=1}^{m}\langle\nabla^N_{e_i}e_i,\nabla^N_{e_i}\eta\rangle e_j\\ &= -(\operatorname{Ricci}^N\eta)^{\top}+m\operatorname{grad}^M f. \end{aligned}\tag{8}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{m}B(e_i,A(e_i)) &= \sum_{i=1}^{m}\langle B(e_i,A(e_i)),\eta\rangle\eta\\ &= \sum_{i=1}^{m}\langle A(e_i),A(e_i)\rangle\eta \end{aligned}$$

$$= |A|^2\eta. \tag{9}$$

Substituting (7), (8) and (9) in (6), we obtain

$$\begin{aligned} -\operatorname{trace}_g \nabla^{\mathbf{i}}|d\mathbf{i}|^{p-2}\nabla^{\mathbf{i}}\tau_p(\mathbf{i}) = & -m^{p-1}\big[\Delta^M(f)\eta - 2A(\operatorname{grad}^M f) + f(\operatorname{Ricci}^N \eta)^\top \\ & - \frac{m}{2}\operatorname{grad}^M f^2 - f|A|^2\eta\big]. \end{aligned} \tag{10}$$

Theorem 2.1 follows by (4)-(5), and (10). □

As an immediate consequence of Theorem 2.1 we have:

Corollary 2.2. *A hypersurface (M^m, g) in an Einstein space $(N^{m+1}, \langle,\rangle)$ is p-biharmonic if and only if it's mean curvature function f is a solution of the following PDEs*

$$\begin{cases} -\Delta^M(f) + f|A|^2 + m(p-2)f^3 - \dfrac{S}{m+1}f = 0; \\ 2A(\operatorname{grad}^M f) + (p-2+\dfrac{m}{2})\operatorname{grad}^M f^2 = 0, \end{cases} \tag{11}$$

where S is the scalar curvature of the ambient space.

Proof. It is well known that if $(N^{m+1}, \langle,\rangle)$ is an Einstein manifold, then $\operatorname{Ric}^N(X,Y) = \lambda\langle X,Y\rangle$ for some constant λ, for any $X, Y \in \Gamma(TN)$. So that

$$\begin{aligned} S &= \operatorname{trace}_{\langle,\rangle}\operatorname{Ric}^N \\ &= \sum_{i=1}^m \operatorname{Ric}^N(e_i,e_i) + \operatorname{Ric}^N(\eta,\eta) \\ &= \lambda(m+1), \end{aligned}$$

where $\{e_i\}_{i=1,\ldots,m}$ is a normal orthonormal frame on (M^m, g) at x. Since $\operatorname{Ric}^N(\eta,\eta) = \lambda$, we conclude that

$$\operatorname{Ric}^N(\eta,\eta) = \frac{S}{m+1}.$$

On the other hand, we have

$$\begin{aligned} (\operatorname{Ricci}^N \eta)^\top &= \sum_{i=1}^m \langle \operatorname{Ricci}^N \eta, e_i\rangle e_i \\ &= \sum_{i=1}^m \operatorname{Ric}^N(\eta, e_i)e_i \\ &= \sum_{i=1}^m \lambda\langle \eta, e_i\rangle e_i \\ &= 0. \end{aligned}$$

Corollary 2.2 follows by Theorem 2.1. □

Theorem 2.3. *A totally umbilical hypersurface* (M^m, g) *in an Einstein space* $(N^{m+1}, \langle,\rangle)$ *with non-positive scalar curvature is p-biharmonic if and only if it is minimal.*

Proof. Take an orthonormal frame $\{e_i, \eta\}_{i=1,\dots,m}$ on the ambient space $(N^{m+1}, \langle,\rangle)$ such that $\{e_i\}_{i=1,\dots,m}$ is an orthonormal frame on (M^m, g). We have

$$\begin{aligned} f &= \langle H, \eta\rangle \\ &= \frac{1}{m}\sum_{i=1}^m \langle B(e_i, e_i), \eta\rangle \\ &= \frac{1}{m}\sum_{i=1}^m \langle g(e_i, e_i)\beta\eta, \eta\rangle \\ &= \beta, \end{aligned}$$

where $\beta \in C^\infty(M)$. The p-biharmonic hypersurface equation (11) becomes

$$\begin{cases} -\Delta^M(\beta) + m(p-1)\beta^3 - \dfrac{S}{m+1}\beta = 0; \\ (p-1+\dfrac{m}{2})\beta \operatorname{grad}^M \beta = 0, \end{cases}$$

Solving the last system, we have $\beta = 0$ and hence $f = 0$, or

$$\beta = \pm\sqrt{\frac{S}{m(m+1)(p-1)}},$$

it's constant and this happens only if $S \geq 0$. The proof is complete. □

3. *p*-biharmonic hypersurface in conformally flat space

Let $\mathbf{i} : M^m \hookrightarrow \mathbb{R}^{m+1}$ be a minimal hypersurface with the unit normal vector field η, $\widetilde{\mathbf{i}} : (M^m, \widetilde{g}) \hookrightarrow (\mathbb{R}^{m+1}, \widetilde{h} = e^{2\gamma}h)$, $x \longmapsto \widetilde{\mathbf{i}}(x) = \mathbf{i}(x) = x$, where $\gamma \in C^\infty(\mathbb{R}^{m+1})$, $h = \langle,\rangle_{\mathbb{R}^{m+1}}$, and $\widetilde{g}$ is the induced metric by $\widetilde{h}$, that is

$$\widetilde{g}(X,Y) = e^{2\gamma}g(X,Y) = e^{2\gamma}\langle X, Y\rangle_{\mathbb{R}^{m+1}},$$

where g is the induced metric by h. Let $\{e_i, \eta\}_{i=1,\dots,m}$ be an orthonormal frame adapted to the p-harmonic hypersurface on $(\mathbb{R}^{m+1}, h)$, thus $\{\widetilde{e}_i, \widetilde{\eta}\}_{i=1,\dots,m}$ becomes an orthonormal frame on $(\mathbb{R}^{m+1}, \widetilde{h})$, where $\widetilde{e}_i = e^{-\gamma}e_i$ for all $i = 1, \dots, m$, and $\widetilde{\eta} = e^{-\gamma}\eta$.

Theorem 3.1. *The hypersurface* $(M^m, \widetilde{g})$ *in the conformally flat space* $(\mathbb{R}^{m+1}, \widetilde{h})$ *is p-biharmonic if and only if*

$$(12)\quad \begin{cases} \eta(\gamma)e^{-\gamma}\big[-\Delta^M(\gamma) - m \operatorname{Hess}_\gamma^{\mathbb{R}^{m+1}}(\eta,\eta) + (1-m)|\operatorname{grad}^M\gamma|^2 - |A|^2 \\ +m(1-p)\eta(\gamma)^2\big] + \Delta^M(\eta(\gamma)e^{-\gamma}) + (m-2)(\operatorname{grad}^M\gamma)(\eta(\gamma)e^{-\gamma}) = 0; \\ -2A(\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})) + 2(1-m)\eta(\gamma)e^{-\gamma}A(\operatorname{grad}^M\gamma) \\ +(2p-m)\eta(\gamma)\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}) = 0, \end{cases}$$

where $\mathrm{Hess}_\gamma^{\mathbb{R}^{m+1}}$ *is the Hessian of the smooth function* γ *in* $(\mathbb{R}^{m+1}, h)$.

Proof. By using the Kozul's formula, we have

$$\begin{cases}\widetilde{\nabla}^M_X Y = \nabla^M_X Y + X(\gamma)Y + Y(\gamma)X - g(X,Y)\operatorname{grad}^M \gamma;\\ \widetilde{\nabla}^{\mathbb{R}^{m+1}}_U V = \nabla^{\mathbb{R}^{m+1}}_U V + U(\gamma)V + V(\gamma)U - h(U,V)\operatorname{grad}^{\mathbb{R}^{m+1}} \gamma,\end{cases}$$

for all $X, Y \in \Gamma(TM)$, and $U, V \in \Gamma(T\mathbb{R}^{m+1})$. Consequently

$$\begin{aligned}\nabla^{\widetilde{\mathbf{i}}}_X d\widetilde{\mathbf{i}}(Y) &= \nabla^{\widetilde{\mathbf{i}}}_X Y\\ &= \widetilde{\nabla}^{\mathbb{R}^{m+1}}_{d\mathbf{i}(X)} Y\\ &= \widetilde{\nabla}^{\mathbb{R}^{m+1}}_X Y\\ &= \nabla^{\mathbb{R}^{m+1}}_X Y + X(\gamma)Y + Y(\gamma)X - h(X,Y)\operatorname{grad}^{\mathbb{R}^{m+1}} \gamma, \end{aligned} \tag{13}$$

and the following

$$\begin{aligned}d\widetilde{\mathbf{i}}(\widetilde{\nabla}^M_X Y) &= d\mathbf{i}(\nabla^M_X Y) + X(\gamma)d\mathbf{i}(Y) + Y(\gamma)d\mathbf{i}(X) - g(X,Y)d\mathbf{i}(\operatorname{grad}^M \gamma)\\ &= \nabla^M_X Y + X(\gamma)Y + Y(\gamma)X - g(X,Y)\operatorname{grad}^M \gamma.\end{aligned} \tag{14}$$

From equations (13) and (14), we get

$$\begin{aligned}(\nabla d\widetilde{\mathbf{i}})(X,Y) &= \nabla^{\widetilde{\mathbf{i}}}_X d\widetilde{\mathbf{i}}(Y) - d\widetilde{\mathbf{i}}(\widetilde{\nabla}^M_X Y)\\ &= (\nabla d\mathbf{i})(X,Y) + g(X,Y)[\operatorname{grad}^M \gamma - \operatorname{grad}^{\mathbb{R}^{m+1}} \gamma]\\ &= B(X,Y) - g(X,Y)\eta(\gamma)\eta.\end{aligned} \tag{15}$$

So that, the mean curvature function $\widetilde{f}$ of $(M^m, \widetilde{g})$ in $(\mathbb{R}^{m+1}, \widetilde{h})$ is given by $\widetilde{f} = -\eta(\gamma)e^{-\gamma}$. Indeed, by taking traces in (15), we obtain

$$e^{2\gamma}\widetilde{H} = H - \eta(\gamma)\eta.$$

Since (M^m, g) is minimal in $(\mathbb{R}^{m+1}, h)$, we find that $\widetilde{H} = -e^{-2\gamma}\eta(\gamma)\eta$, that is $\widetilde{H} = -e^{-\gamma}\eta(\gamma)\widetilde{\eta}$.

With the new notations the equation (3) for p-biharmonic hypersurface in the conformally flat space becomes

$$\begin{cases} -\widetilde{\Delta}(\widetilde{f}) + \widetilde{f}|\widetilde{A}|^2_{\widetilde{g}} - \widetilde{f}\,\widetilde{\mathrm{Ric}}^{\mathbb{R}^{m+1}}(\widetilde{\eta},\widetilde{\eta}) + m(p-2)\widetilde{f}^3 = 0;\\ 2\widetilde{A}(\widetilde{\operatorname{grad}}^M \widetilde{f}) - 2\widetilde{f}(\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta})^\top + (p - 2 + \dfrac{m}{2})\widetilde{\operatorname{grad}}^M \widetilde{f}^2 = 0.\end{cases} \tag{16}$$

A straightforward computation yields

$$\begin{aligned}\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\eta = e^{-2\gamma}\big[&\mathrm{Ricci}^{\mathbb{R}^{m+1}}\eta - \Delta^{\mathbb{R}^{m+1}}(\gamma)\eta + (1-m)\nabla^{\mathbb{R}^{m+1}}_\eta \operatorname{grad}^{\mathbb{R}^{m+1}}\gamma\\ &+ (1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2\eta - (1-m)\eta(\gamma)\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma\big];\end{aligned}$$

$$\widetilde{\mathrm{Ric}}^{\mathbb{R}^{m+1}}(\widetilde{\eta},\widetilde{\eta}) = \widetilde{h}(\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta},\widetilde{\eta})$$

$$
\begin{aligned}
&= h(\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\eta,\eta)\\
&= e^{-2\gamma}h\big(\mathrm{Ricci}^{\mathbb{R}^{m+1}}\eta-\Delta^{\mathbb{R}^{m+1}}(\gamma)\eta+(1-m)\nabla^{\mathbb{R}^{m+1}}_{\eta}\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma\\
&\quad+(1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2\eta-(1-m)\eta(\gamma)\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta\big)\\
&= e^{-2\gamma}\big[-\Delta^{\mathbb{R}^{m+1}}(\gamma)+(1-m)\operatorname{Hess}^{\mathbb{R}^{m+1}}_{\gamma}(\eta,\eta)\\
&\qquad+(1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2-(1-m)\eta(\gamma)^2\big]; \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&(\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta})^{\top}\\
&= \sum_{i=1}^{m} h(\widetilde{\mathrm{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta},e_i)e_i\\
&= (1-m)e^{-3\gamma}\sum_{i=1}^{m}\Big[h(\nabla^{\mathbb{R}^{m+1}}_{\eta}\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,e_i)e_i-\eta(\gamma)h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,e_i)e_i\Big]\\
&= (1-m)e^{-3\gamma}\Big[\sum_{i=1}^{m}h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta)e_i-\eta(\gamma)\operatorname{grad}^{M}\gamma\Big]\\
&= (1-m)e^{-3\gamma}\Big[\sum_{i=1}^{m}e_i h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta)e_i-\sum_{i=1}^{m}h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\nabla^{\mathbb{R}^{m+1}}_{e_i}\eta)e_i\\
&\quad-\eta(\gamma)\operatorname{grad}^{M}\gamma\Big]\\
&= (1-m)e^{-3\gamma}\big[\operatorname{grad}^{M}\eta(\gamma)+\sum_{i=1}^{m}h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,Ae_i)e_i-\eta(\gamma)\operatorname{grad}^{M}\gamma\big]\\
&= (1-m)e^{-3\gamma}\big[\operatorname{grad}^{M}\eta(\gamma)+A(\operatorname{grad}^{M}\gamma)-\eta(\gamma)\operatorname{grad}^{M}\gamma\big];
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Delta}(\widetilde{f}) &= e^{-2\gamma}[\Delta(\widetilde{f})+(m-2)d\widetilde{f}(\operatorname{grad}^{M}\gamma)]\\
&= e^{-2\gamma}[-\Delta(\eta(\gamma)e^{-\gamma})-(m-2)(\operatorname{grad}^{M}\gamma)(\eta(\gamma)e^{-\gamma})];
\end{aligned}
$$

$$
\begin{aligned}
|\widetilde{A}|^2_{\widetilde{g}} &= \sum_{i=1}^{m}\widetilde{g}(\widetilde{A}\widetilde{e}_i,\widetilde{A}\widetilde{e}_i)\\
&= \sum_{i=1}^{m}g(\widetilde{A}e_i,\widetilde{A}e_i)\\
&= \sum_{i=1}^{m}h(\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta},\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta})\\
&= \sum_{i=1}^{m}h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}+e_i(\gamma)\widetilde{\eta}+\widetilde{\eta}(\gamma)e_i,\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}+e_i(\gamma)\widetilde{\eta}+\widetilde{\eta}(\gamma)e_i)
\end{aligned}
$$

$$= \sum_{i=1}^{m} \big[h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, \nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}) + 2\widetilde{\eta}(\gamma)h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, e_i) + e_i(\gamma)^2 e^{-2\gamma}$$

$$+ 2e_i(\gamma)h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, \widetilde{\eta})\big] + m\widetilde{\eta}(\gamma)^2. \tag{18}$$

The first term of (18) is given by

$$\begin{aligned}
&\sum_{i=1}^{m} h(\nabla^{\mathbb{R}^{m+1}}_{e_i} e^{-\gamma}\eta, \nabla^{\mathbb{R}^{m+1}}_{e_i} e^{-\gamma}\eta)\\
&= \sum_{i=1}^{m} h(-e^{-\gamma}e_i(\gamma)\eta + e^{-\gamma}\nabla^{\mathbb{R}^{m+1}}_{e_i}\eta, -e^{-\gamma}e_i(\gamma)\eta + e^{-\gamma}\nabla^{\mathbb{R}^{m+1}}_{e_i}\eta)\\
&= \sum_{i=1}^{m} [e^{-2\gamma}e_i(\gamma)^2 + e^{-2\gamma}h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\eta, \nabla^{\mathbb{R}^{m+1}}_{e_i}\eta)]\\
&= e^{-2\gamma}|\operatorname{grad}^M \gamma|^2 + e^{-2\gamma}|A|^2.
\end{aligned}$$

The second term of (18) is given by

$$\begin{aligned}
2\widetilde{\eta}(\gamma)\sum_{i=1}^{m} h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, e_i) &= -2e^{-\gamma}\eta(\gamma)\sum_{i=1}^{m} h(e^{-\gamma}\eta, \nabla^{\mathbb{R}^{m+1}}_{e_i} e_i)\\
&= -2me^{-2\gamma}\eta(\gamma)h(\eta, H)\\
&= 0.
\end{aligned}$$

Here $H = 0$. We have also

$$\begin{aligned}
2\sum_{i=1}^{m} e_i(\gamma)h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, \widetilde{\eta}) &= \sum_{i=1}^{m} e_i(\gamma)e_i h(\widetilde{\eta}, \widetilde{\eta})\\
&= \sum_{i=1}^{m} e_i(\gamma)e_i(e^{-2\gamma})\\
&= -2e^{-2\gamma}\sum_{i=1}^{m} e_i(\gamma)^2\\
&= -2e^{-2\gamma}|\operatorname{grad}^M \gamma|^2.
\end{aligned}$$

Thus

$$|\widetilde{A}|^2_{\widetilde{h}} = e^{-2\gamma}|A|^2 + me^{-2\gamma}\eta(\gamma)^2.$$

We compute

$$\begin{aligned}
\widetilde{\operatorname{grad}}^M \widetilde{f} &= e^{-2\gamma}\sum_{i=1}^{m} e_i(\widetilde{f})e_i\\
&= -e^{-2\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma});
\end{aligned}$$

and the following

$$\widetilde{A}(\widetilde{\operatorname{grad}}^M \widetilde{f}) = -\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}}\widetilde{\eta}$$

$$
\begin{aligned}
&= -\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}} e^{-\gamma}\eta \\
&= e^{-\gamma}(\widetilde{\operatorname{grad}}^M \widetilde{f})(\gamma)\eta - e^{-\gamma}\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}}\eta \\
&= -e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})}\eta \\
&= -e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\eta(\gamma)\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}) \\
&\quad + e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\nabla^{\mathbb{R}^{m+1}}_{\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})}\eta \\
&= e^{-3\gamma}\eta(\gamma)\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}) - e^{-3\gamma}A(\operatorname{grad}^M\eta(\gamma)e^{-\gamma}). \qquad (19)
\end{aligned}
$$

Substituting (17)–(19) in (16), and by simplifying the resulting equation we obtain the system (12). □

Remark 3.2. (1) Using Theorem 3.1, we can construct many examples for proper p-biharmonic hypersurfaces in the conformally flat space (see [9]).

(2) If the functions γ and $\eta(\gamma)$ are non-zero constants on M, then according to Theorem 3.1, the hypersurface $(M^m, \widetilde{g})$ is p-biharmonic in $(\mathbb{R}^{m+1}, \widetilde{h})$ if and only if

$$|A|^2 = m(1-p)\eta(\gamma)^2 - m\eta(\eta(\gamma)).$$

Example 3.3. The hyperplane $\mathbf{i} : \mathbb{R}^m \hookrightarrow (\mathbb{R}^{m+1}, e^{2\gamma(z)}h)$, $x \longmapsto (x, c)$, where $\gamma \in C^\infty(\mathbb{R})$, $h = \sum_{i=1}^m dx_i^2 + dz^2$, and $c \in \mathbb{R}$, is proper p-biharmonic if and only if $(1-p)\gamma'(c)^2 - \gamma''(c) = 0$. Note that, the smooth function

$$\gamma(z) = \frac{\ln\left(c_1(p-1)z + c_2(p-1)\right)}{p-1}, \quad c_1, c_2 \in \mathbb{R},$$

is a solution of the previous differential equation (for all c).

Example 3.4. Let M be a surface of revolution in $\{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$. If M is part of a plane orthogonal to the axis of revolution, so that M is parametrized by

$$(x_1, x_2) \longmapsto (f(x_2)\cos(x_1), f(x_2)\sin(x_1), c)$$

for some constant $c > 0$. Here $f(x_2) > 0$. Then, M is minimal, and according to Theorem 3.1, the surface M is proper p-biharmonic in 3-dimensional hyperbolic space $(\mathbb{H}^3, z^{\frac{2}{p-1}}h)$, where $h = dx^2 + dy^2 + dz^2$.

Open Problems.

(1) If M is a minimal surface of revolution contained in a catenoid, that is M is parametrized by

$$(x_1, x_2) \longmapsto \left(a\cosh\left(\frac{x_2}{a} + b\right)\cos(x_1), a\cosh\left(\frac{x_2}{a} + b\right)\sin(x_1), x_2\right),$$

where $a \neq 0$ and b are constants. Is there $p \geq 2$ and $\gamma \in C^\infty(\mathbb{R}^3)$ such that M is proper p-biharmonic in $\left(\mathbb{R}^3, e^{2\gamma}(dx^2 + dy^2 + dz^2)\right)$?

(2) Is there a proper p-biharmonic submanifolds in Euclidean space $(\mathbb{R}^n, dx_1^2 + \cdots + dx_n^2)$?

References

[1] P. Baird and S. Gudmundsson, *p-harmonic maps and minimal submanifolds*, Math. Ann. **294** (1992), no. 4, 611–624. https://doi.org/10.1007/BF01934344

[2] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, 29, The Clarendon Press, Oxford University Press, Oxford, 2003. https://doi.org/10.1093/acprof:oso/9780198503620.001.0001

[3] B. Bojarski and T. Iwaniec, *p-harmonic equation and quasiregular mappings*, in Partial differential equations (Warsaw, 1984), 25–38, Banach Center Publ., 19, PWN, Warsaw, 1987.

[4] J. Eells, Jr., and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. https://doi.org/10.2307/2373037

[5] A. Fardoun, *On equivariant p-harmonic maps*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **15** (1998), no. 1, 25–72. https://doi.org/10.1016/S0294-1449(99)80020-1

[6] G. Y. Jiang, 2-*harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402.

[7] A. Mohammed Cherif, *On the p-harmonic and p-biharmonic maps*, J. Geom. **109** (2018), no. 3, Paper No. 41, 11 pp. https://doi.org/10.1007/s00022-018-0446-y

[8] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

[9] Y.-L. Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math. **248** (2010), no. 1, 217–232. https://doi.org/10.2140/pjm.2010.248.217

[10] Y. Xin, *Geometry of harmonic maps*, Progress in Nonlinear Differential Equations and their Applications, 23, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4084-6

Ahmed Mohammed Cherif
Department of Mathematics
University of Mustapha Stambouli Mascara
Mascara 29000, Algeria
Email address: a.mohammedcherif@univ-mascara.dz

Khadidja Mouffoki
Department of Mathematics
University of Mustapha Stambouli Mascara
Mascara 29000, Algeria
Email address: khadidja.mouffoki@univ-mascara.dz