Bull. Korean Math. Soc. **60** (2023), No. 3, pp. 687–703
https://doi.org/10.4134/BKMS.b220341
pISSN: 1015-8634 / eISSN: 2234-3016

ON A SPITZER-TYPE LAW OF LARGE NUMBERS FOR PARTIAL SUMS OF INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES UNDER SUB-LINEAR EXPECTATIONS

Miaomiao Wang, Min Wang, and Xuejun Wang

Abstract. In this paper, under some suitable conditions, we study the Spitzer-type law of large numbers for the maximum of partial sums of independent and identically distributed random variables in upper expectation space. Some general results on necessary and sufficient conditions of the Spitzer-type law of large numbers for the maximum of partial sums of independent and identically distributed random variables under sub-linear expectations are established, which extend the corresponding ones in classic probability space to the case of sub-linear expectation space.

1. Introduction

As we all know, the limit theory has been widely studied in probability theory and mathematical statistics. The classical probability limit theory is only valid in the case of model is certainty. In the classical limit theorems, probability and expectation are linear additivity. However, there are uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus. One can refer to Gilboa [8], Peng [18–20], Maccheroni and Marinacci [17], Denis and Martini [6] among others for the details. At this time, nonadditive probabilities and nonadditive expectations are useful tools to study this uncertainty. Recently, motivated by this uncertainty, Peng [19–21] proposed the concept of independent and identically distributed (*i.i.d.*, for short) random variables under sub-linear expectations.

Under the framework of the sub-linear expectation space introduced by Peng [19–21], which is different from the classic linear probability space, many scholars have studied the limit behaviors of some random variables under sub-linear

Received May 16, 2022; Revised December 17, 2022; Accepted January 26, 2023.
2020 *Mathematics Subject Classification.* 60F15.
Key words and phrases. Independent and identically distributed random variables, law of large numbers, complete convergence, sub-linear expectation space, capacity.
Supported by the National Natural Science Foundation of China (11871072, 12201079, 12201600), the Natural Science Foundation of Anhui Province (2108085MA06), and the Scientific Research Project of Graduate Students in Anhui Universities (YJS20210077).

expectations and obtained a lot of interesting results. For example, Denis et al. [5] gave some basic and important properties of several typical Banach spaces of functions of G-Brownian motion paths induced by a sub-linear expectation–G-expectation, and obtained a generalized version of Kolmogorov's criterion for continuous modification of a stochastic process; Chen et al. [3] derived a strong law of large numbers (SLLN, for short) for independent random variables under upper expectations; Chen and Hu [2] extended the Kolmogorov and the Hartman-Wintner laws of the iterated logarithm, and obtained the law of the iterated logarithm for capacities; Hu and Chen [11] gave three laws of large numbers (LLN, for short) without the requirement of identical distribution under some weaker conditions; Zhang [28, 29, 32] obtained the exponential inequalities, Rosenthal's inequalities and some limit theorems for independent or some dependent random variables in sub-linear expectation space; Zhang and Lin [34] gained the Marcinkiewicz's SLLN for non-linear expectations; Wu and Jiang [25] established general SLLN and the Chover's law of the iterated logarithm for a sequence of random variables in a sub-linear expectation space; Feng [7] established law of the logarithm for weighted sums of negatively dependent random variables under sub-linear expectations; Huang and Wu [14] obtained the equivalent relations between Kolmogorov maximal inequality and Hájek-Rényi maximal inequality both in moment and capacity types under sub-linear expectations, and established some SLLN for general random variables; Xu and Zhang [27] established a three series theorem of independent random variables and obtained Marcinkiewicz's SLLN for *i.i.d.* random variables under the sub-linear expectations; Zhang and Lan [33] introduced the notion of asymptotically almost negatively associated (AANA, for short) random variables from the classic probability space to the upper expectation space, and proved some different types of Rosenthal's inequalities under sub-additive expectations; Kuczmaszewska [15] obtained the exponential inequalities, Hoffmann-Jørgensen type inequalities and the complete convergence for widely acceptable random variables; Lin and Feng [16] studied the complete convergence and the SLLN for arrays of rowwice widely negative dependent random variables in upper expectation space; Xu and Cheng [26] established the precise asymptotics in the law of the iterated logarithm for *i.i.d.* random variables under sub-linear expectations; Zhang [30] showed Heyde's theorem under the sub-linear expectations; Guo and Li [9] introduced the notion of pseudo-independence under sub-linear expectations and derived the weak and strong LLN with non-additive probabilities generated by sub-linear expectations, and so on.

The complete convergence is one of the important convergence problems in probability limit theorems. The concept of the complete convergence was first introduced by Hsü and Robbins [10] as follows: A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P\left(|X_n - \theta| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

By the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely ($a.s.$, for short). Hence the complete convergence implies $a.s.$ convergence.

Now let us recall that an important concept of dependent random variables is m-negative association (m-NA, for short), which was introduced by Hu et al. [13] as follows.

Definition 1.1. Let $m \geq 1$ be a fixed integer. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be m-NA if for any $n \geq 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are negatively associated (NA, for short).

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be rowwise m-NA if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is m-NA.

It is obviously seen that if $m = 1$, then the concept of m-NA random variables is equivalent to that of NA random variables. In addition, it is well known that if for any $n \geq 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are independent, then we say that $\{X_n, n \geq 1\}$ are m-dependent. Obviously, NA and m-dependence both imply m-NA. For more details about the m-NA random variables, we refer the readers to Hu et al. [13], Hu et al. [12], Wu et al. [24], Shen et al. [22], Wang and Wang [23], Boukhari [1], and so on.

The following definition of stochastic domination will play an important role in the following theorem.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X, if there exists a constant $C > 0$ such that

$$\sup_{n\geq 1} P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$. We write $\{X_n, n \geq 1\} \prec X$.

Next, we will make the following assumptions for the nonnegative measurable function $\psi(\cdot)$, which are very important in our results.

(H_1) $\psi(\cdot)$ is an increasing unbounded measurable function defined on $(d, +\infty)$ for some $d \geq 0$.

(H_2) (i) There exist $C_1 > 0$ and a positive integer k_0 such that $\psi(x+1) \leq C_1\psi(x)$ for each $x \geq k_0$.
(ii) There exist two constants a and b such that $\psi^2(y)\int_y^{\infty} \frac{dx}{\psi^2(x)} \leq ay+b$ for any $y > d$.

Wang and Wang [23] studied the sufficient part of the Spitzer-type LLN for the maximum of partial sums of m-NA random variables, and obtained the following interesting result.

Theorem 1.1. *Let $\psi(\cdot)$ be as in assumptions (H_1) and (H_2), and let $\{X_n, n \geq 1\}$ be a sequence of m-NA random variables with $EX_n = 0$ for each $n \geq 1$. Assume that $\{X_n, n \geq 1\} \prec X$. If $E[\psi^{-1}(|X|)] < \infty$ and $\frac{n}{\psi(n)} E|X|I(|X| > \psi(n)) \to 0$ as $n \to \infty$, then for any $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq k\leq n} |S_k| \geq \epsilon\psi(n)\right) < \infty, \tag{1.1}$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(|S_n| \geq \epsilon\psi(n)\right) < \infty, \tag{1.2}$$

where, $\psi^{-1}(\cdot)$ is the inverse of a function $\psi(\cdot)$. In addition, if

$$\psi(\sigma x) \leq C_2\psi(x) \tag{1.3}$$

for some $C_2 > 0$ and any $\sigma > 0$ and $x > 0$, then

$$\frac{1}{\psi(n)} \sum_{i=1}^{n} X_k \to 0 \text{ a.s. as } n \to \infty.$$

Boukhari [1] further studied the necessary conditions of the Spitzer-type LLN for the maximum of partial sums of m-NA random variables on the basis of Theorem 1.1, and obtained the following interesting result.

Theorem 1.2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed m-NA random variables with $EX = 0$. Suppose that $\psi(\cdot)$ satisfies the assumptions (H_1), (H_2) and (1.3). If (1.1) is satisfied, then*

$$E[\psi^{-1}(|X|)] < \infty,$$

and

$$\frac{n}{\psi(n)} EXI(|X| > \psi(n)) = o(1).$$

In this paper, under appropriate conditions, we will study the sufficient and necessary conditions of the Spitzer-type LLN for the maximum of partial sums of independent random variables. The results obtained by Wang and Wang [23] and Boukhari [1] from the classic probability space will be extended to the case of sub-linear expectation space.

The structure of this article is as follows. The concept of sub-linear expectations and some preliminary lemmas are stated in Section 2. Main results and their proofs are provided in Section 3.

Throughout this paper, the symbol C represents some positive constant which may be different in various places. Denote $x^+ = xI\ (x \geq 0)$ and $x^- = -xI\ (x \leq 0)$. Let $I(A)$ be the indicator function of the set A, $S_n = \sum_{i=1}^n X_i$, and $S_0 = 0$. Also, let $\lfloor x \rfloor$ be the integer part of x.

2. Preliminaries

In this section, we introduce some basic notations and concepts.

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$, and $\mathcal{M}$ be the set of all probability measures on Ω. Every nonempty subset $\mathcal{P} \subseteq \mathcal{M}$ defines an upper probability

$$\mathbb{V}(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for any } A \in \mathcal{F},$$

and a lower probability

$$v(A) := \inf_{P \in \mathcal{P}} P(A) \text{ for any } A \in \mathcal{F}.$$

Obviously, $\mathbb{V}(\cdot)$ and $v(\cdot)$ are conjugate to each other, that is

$$\mathbb{V}(A) + v(A^c) = 1 \text{ for any } A \in \mathcal{F},$$

where A^c is the complement set of A. It is easy to check that $\mathbb{V}(\cdot)$ satisfies $\mathbb{V}(A \bigcup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{F}$.

Choquet [4] introduced the following definition of capacity.

Definition 2.1. Let $V(\cdot)$ be a set function from $\mathcal{F}$ to $[0,1]$. $V(\cdot)$ is called a capacity if it satisfies the following properties (i) and (ii), and is called a lower or an upper continuous capacity if it further satisfies the following property (iii) or (iv):

(i) $V(\emptyset) = 0,\ V(\Omega) = 1$;

(ii) monotonicity: $V(A) \leq V(B)$, whenever $A \subseteq B$ and $A, B \in \mathcal{F}$;

(iii) lower continuous: $V(A_n) \uparrow V(A)$, if $A_n \uparrow A$, where $A_n, A \in \mathcal{F}$;

(iv) upper continuous: $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

Recently, Denis et al. [5] introduced the following definition of quasi-sure ($q.s.$, for short) convergence.

Definition 2.2. A set D is polar if $\mathbb{V}(D) = 0$ and a property holds "q.s." if it holds outside a polar set.

Now we define the upper expectation $\mathbb{E}[\cdot]$ and the lower expectation $\varepsilon[\cdot]$ on $(\Omega, \mathcal{F})$ generated by $\mathcal{P}$, for each $\mathcal{F}$-measurable real random variables X such that $E_P[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] := \sup_{P \in \mathcal{P}} E_P[X],\ \varepsilon[X] := \inf_{P \in \mathcal{P}} E_P[X].$$

Here and in the sequel, E_P denotes the classic expectation under probability P. $(\Omega, \mathcal{F}, \mathbb{E})$ is called an upper expectation space, and $(\Omega, \mathcal{F}, \varepsilon)$ is called a lower expectation space.

It is easy to check that $\varepsilon[X] = -\mathbb{E}[-X]$, and $\mathbb{E}[\cdot]$ is a sub-linear expectation on $(\Omega, \mathcal{F})$, which means for all $\mathcal{F}$-measurable real random variables X and Y, $\mathbb{E}[\cdot]$ satisfies the following properties:

(i) monotonicity: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(ii) constant preserving: $\mathbb{E}[c] = c$ for $c \in \mathbb{R}$;

(iii) sub-additivity: $\mathbb{E}[X+Y] \leq \mathbb{E}[X]+\mathbb{E}[Y]$;
(iv) positive homogeneity: $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X],\ \lambda \geq 0$.
The corresponding Choquet expectations $(C_{\mathbb{V}}, C_v)$ are defined by

$$C_V(X)=\int_0^{\infty} V(X \geq t)dt+\int_{-\infty}^{0}(V(X \geq t)-1)dt,$$

where V is replaced by $\mathbb{V}$ and v, respectively. Obviously, $\mathbb{E}[X] \leq C_{\mathbb{V}}(X)$.

Now we give the concept of *i.i.d.* random variables in upper expectation space $(\Omega, \mathcal{F}, \mathbb{E})$, which was established by Peng [21].

Definition 2.3. (i) (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors, respectively, defined in sub-linear expectation spaces $(\Omega_1, \mathcal{F}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{E}_2)$. They are called identically distributed, denote by $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2$, if

$$\mathbb{E}_1[\varphi(\mathbf{X}_1)]=\mathbb{E}_2[\varphi(\mathbf{X}_2)] \quad \text{for any } \varphi \in C_{b,Lip}(\mathbb{R}^n),$$

whenever the sub-linear expectations are finite and $C_{b,Lip}(\mathbb{R}^n)$ denotes the space of bounded and Lipschitz continuous functions. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{F}, \mathbb{E})$, a random vector $\mathbf{Y}=(Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent of another random vector $\mathbf{X}=(X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\mathbb{E}$, if for each test function $\varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have

$$\mathbb{E}[\varphi(\mathbf{X}, \mathbf{Y})]=\mathbb{E}[\mathbb{E}[\varphi(x, \mathbf{Y})]|_{x=\mathbf{X}}],$$

whenever $\overline{\varphi}(x):=\mathbb{E}[|\varphi(x, \mathbf{Y})|]<\infty$ for all x and $\mathbb{E}[|\overline{\varphi}(x)|]<\infty$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be independent, if X_{i+1} is independent of $(X_1, \ldots, X_i)$ for each $i \geq 1$.

In the next part of this section, we present the main tools and preliminary lemmas that are needed for the proof of main result. The first one is about basic inequalities in upper expectation space, which has been proved by Chen et al. [3].

Lemma 2.1. *Let X and Y be real measurable random variables in upper expectation space $(\Omega, \mathcal{F}, \mathbb{E})$.*

(i) *Hölder's inequality: For $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\mathbb{E}[|XY|] \leq \left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}}\left(\mathbb{E}\left[|X|^q\right]\right)^{\frac{1}{q}}.$$

(ii) *Chebyshev's inequalities: Let $f(x)>0$ be a nondecreasing function on $\mathbb{R}$. Then for any x,*

$$\mathbb{V}(X \geq x) \leq \frac{\mathbb{E}[f(X)]}{f(x)},\ v(X \geq x) \leq \frac{\varepsilon[f(X)]}{f(x)}.$$

Let $f(x) > 0$ be an even function on $\mathbb{R}$ and nondecreasing on $(0, \infty)$. Then for any $x > 0$,

$$\mathbb{V}(|X| \geq x) \leq \frac{\mathbb{E}[f(X)]}{f(x)}, \ v(|X| \geq x) \leq \frac{\varepsilon[f(X)]}{f(x)}.$$

(iii) *Jensen's inequality: Let $f(\cdot)$ be a convex function on $\mathbb{R}$. Suppose that $\mathbb{E}[X]$ and $\mathbb{E}[f(X)]$ exist. Then*

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

The next one can be referred to Lemma 2.1 in Chen et al. [3].

Lemma 2.2. *$\mathbb{V}(\cdot)$ is a lower continuous capacity and $v(\cdot)$ is an upper continuous capacity.*

Next, we will give the Borel-Cantelli's lemma in upper expectation space.

Lemma 2.3. (i) *Let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$ and $(\mathbb{V}, v)$ be a pair of upper and lower probability generated by $\mathcal{P}$. If $\sum_{n=1}^{\infty} \mathbb{V}(A_n) < \infty$, then*

$$\mathbb{V}(A_n, i.o.) = \mathbb{V}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) = 0.$$

(ii) *Suppose that $\{\xi_n, n \geq 1\}$ is a sequence of independent random variables in $(\Omega, \mathcal{F}, \mathbb{E})$. If $\sum_{n=1}^{\infty} v(\xi_n \geq 2\epsilon) = \infty$ for some $\epsilon > 0$, then*

$$v(\{\xi_n \geq \epsilon\}, i.o.) = v\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{\xi_i \geq \epsilon\}\right) = 1.$$

Proof. The proof is similar to that of Lemma 4.1 in Zhang [31].

(i) By the monotonicity of $\mathbb{V}$ and Lemma 2.2, we have

$$\begin{aligned} 0 \leq \mathbb{V}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) &\leq \mathbb{V}\left(\bigcup_{i=n}^{\infty} A_i\right) \\ &= \lim_{N \to \infty} \mathbb{V}\left(\bigcup_{i=n}^{N} A_i\right) \\ &\leq \sum_{i=n}^{\infty} \mathbb{V}(A_i) \to 0 \text{ as } n \to \infty. \end{aligned}$$

(ii) For given $\epsilon > 0$, denote $A_n = \{\xi_n \geq \epsilon\}$. Let $g(x)$ be a Lipschitz function with $I(x \geq 2\epsilon) \leq g(x) \leq I(x \geq \epsilon)$. By Lemma 2.2 and equation $1 - x \leq e^{-x}$, it is easy to check that

$$\begin{aligned} 0 &\leq 1 - v\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) \\ &= \mathbb{V}\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i^c\right) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathbb{V}\left(\bigcap_{i=n}^{\infty} A_i^c\right) \\
&\le \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{V}\left(\bigcap_{i=n}^{N} A_i^c\right) \\
&\le \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{E}\left[\prod_{i=n}^{N}(1-g(\xi_i))\right] \\
&= \lim_{n\to\infty} \lim_{N\to\infty} \prod_{i=n}^{N} \mathbb{E}\left[(1-g(\xi_i))\right] \\
&= \lim_{n\to\infty} \lim_{N\to\infty} \prod_{i=n}^{N} \left(1-\varepsilon[g(\xi_i)]\right) \\
&\le \lim_{n\to\infty} \exp\left\{-\sum_{i=n}^{\infty} \varepsilon[g(\xi_i)]\right\} \\
&\le \lim_{n\to\infty} \exp\left\{-\sum_{i=n}^{\infty} v\left(\xi_i \ge 2\epsilon\right)\right\} = 0.
\end{aligned}
$$

Hence, we have

$$
v(A_n, i.o.) = v\left(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i\right) = 1.
$$

The proof is completed. □

In the following we give the well-known Kronecker's lemma.

Lemma 2.4. *Suppose that $\{x_n, n \ge 1\}$ and $\{a_n, n \ge 1\}$ are two infinite sequences of real numbers with $0 < a_n \uparrow \infty$. If $\sum_{n=1}^{\infty} \frac{x_n}{a_n}$ converges, then $\frac{\sum_{i=1}^{n} x_i}{a_n} \to 0$ as $n \to \infty$.*

The following is Kolmogorov's inequality in $(\Omega, \mathcal{F}, \mathbb{E})$.

Lemma 2.5. *Let $\{X_1, \ldots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{F}, \mathbb{E})$ with $\mathbb{E}[X_i] = 0,\ i = 1, \ldots, n$. Suppose that X_{i+1} is independent to $(X_1, \ldots, X_i)$ for each $i = 1, \ldots, n-1$. Then*

$$
\mathbb{E}\left[\left(\max_{0\le k<n}(S_n - S_k)\right)^2\right] \le \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right].
$$

In particular,

$$
\mathbb{E}\left[\left(S_n^+\right)^2\right] \le \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right].
$$

Proof. Let

$$T_k = \max(X_k, X_k + X_{k-1}, \ldots, X_k + \cdots + X_1).$$

Obviously, we obtain

$$T_{k+1} = X_{k+1} + T_k^+,$$

and

$$T_{k+1}^2 = X_{k+1}^2 + 2X_{k+1}T_k^+ + \left(T_k^+\right)^2.$$

By sub-additivity of $\mathbb{E}$ and the definition of independence, we have

$$\begin{aligned}\mathbb{E}\left[T_{k+1}^2\right] &\leq \mathbb{E}\left[X_{k+1}^2\right] + 2\mathbb{E}\left[X_{k+1}T_k^+\right] + \mathbb{E}\left[(T_k^+)^2\right] \\ &= \mathbb{E}\left[X_{k+1}^2\right] + 2\mathbb{E}\left[X_{k+1}\right]\mathbb{E}\left[T_k^+\right] + \mathbb{E}\left[(T_k^+)^2\right] \\ &= \mathbb{E}\left[X_{k+1}^2\right] + \mathbb{E}\left[(T_k^+)^2\right] \\ &\leq \mathbb{E}\left[X_{k+1}^2\right] + \mathbb{E}\left[T_k^2\right].\end{aligned}$$

Hence, we have

$$\mathbb{E}\left[\left(\max_{0\leq k<n}(S_n - S_k)\right)^2\right] = \mathbb{E}[T_n^2] \leq \sum_{i=1}^n \mathbb{E}\left[X_i^2\right].$$

The proof is completed. □

Lemma 2.6. *Suppose that $\psi(\cdot)$ is a nonnegative measurable function satisfying* (1.3)*, and X is a real random variable in upper expectation space $(\Omega, \mathcal{F}, \mathbb{E})$. Then for any $c > 0$,*

$$C_V\left(\psi^{-1}(|X|)\right) < \infty \Leftrightarrow \sum_{n=1}^\infty V(|X| \geq c\psi(n)) < \infty,$$

where V is replaced by $\mathbb{V}$ and v.

Proof. Obviously, $C_V\left(\psi^{-1}(|X|)\right) < \infty$ is equivalent to $C_V\left(\psi^{-1}\left(\frac{|X|}{c}\right)\right) < \infty$. Note that

$$\begin{aligned}&C_V\left(\psi^{-1}\left(\frac{|X|}{c}\right)\right) = \int_0^\infty V\left(|X| \geq c\psi(x)\right)dx < \infty \\ \Leftrightarrow &\sum_{n=1}^\infty V\left(|X| \geq c\psi(n)\right) < \infty.\end{aligned}$$

The proof is completed. □

Lemma 2.7. *Suppose that $\{x_n, n \geq 1\}$ is a sequence of real numbers. Then*

$$\left(\max_{1\leq i\leq n} x_i^+\right)^2 \leq \left(\max_{1\leq i\leq n} x_i\right)^2 \text{ and } \left(\max_{1\leq i\leq n} x_i^-\right)^2 \leq \left(\max_{1\leq i\leq n}(-x_i)\right)^2.$$

Proof. For $\max_{1\le i\le n} x_i^+$, we consider the following two cases:

(i) If $\max_{1\le i\le n} x_i^+ > 0$, then $\max_{1\le i\le n} x_i^+ = \max_{1\le i\le n} x_i$. Hence, we have

$$\left(\max_{1\le i\le n} x_i^+\right)^2 = \left(\max_{1\le i\le n} x_i\right)^2.$$

(ii) If $\max_{1\le i\le n} x_i^+ = 0$, then $\max_{1\le i\le n} x_i \le 0$. Hence, we have

$$\left(\max_{1\le i\le n} x_i^+\right)^2 = 0 \le \left(\max_{1\le i\le n} x_i\right)^2.$$

As for $\max_{1\le i\le n} x_i^-$, by $\left(\max_{1\le i\le n} x_i^+\right)^2 \le \left(\max_{1\le i\le n} x_i\right)^2$, we can get

$$\left(\max_{1\le i\le n} x_i^-\right)^2 = \left(\max_{1\le i\le n} (-x_i)^+\right)^2 \le \left(\max_{1\le i\le n} (-x_i)\right)^2.$$

The proof is completed. □

The following lemma comes from Lemma 3.7 of Xu and Zhang [27].

Lemma 2.8. *Suppose that $x, y \in \mathbb{R}$. Then*

$$(x+y)^+ \le x^+ + |y| \text{ and } (x+y)^- \le x^- + |y|.$$

3. Main results and their proofs

In this section, we first give the sufficient conditions of Spitzer's LLN for the maximum of partial sums of independent random variables in $(\Omega, \mathcal{F}, \mathbb{E})$.

Theorem 3.1. *Let $\psi(\cdot)$ be as in (H_1) and $\frac{\psi(n)}{n} \uparrow \infty$, and let $\{X_n, n \ge 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{F}, \mathbb{E})$. Suppose that there exist a random variable X and a constant C_3 satisfying:*

(i) $\mathbb{V}(|X_n| > x) \le C_3 \mathbb{V}(|X| > x)$ *for each $x \ge 0$ and $n \ge 1$;*

(ii) $C_{\mathbb{V}}\left(\psi^{-1}(|X|)\right) < \infty$;

(iii) $\sup_{n\ge 1} \mathbb{E}[|X_n|] < \infty$.

Then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\le k\le n} |S_k| \ge \epsilon\psi(n)\right) < \infty, \tag{3.1}$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(|S_n| \ge \epsilon\psi(n)\right) < \infty. \tag{3.2}$$

In addition, if (1.3) *is satisfied, then*

$$\frac{\max_{1\le k\le n} |S_k|}{\psi(n)} = \frac{\max_{1\le k\le n} \left|\sum_{i=1}^{k} X_i\right|}{\psi(n)} \to 0 \ q.s. \text{ as } n \to \infty. \tag{3.3}$$

Proof. Note that (3.2) follows from (3.1) immediately. So we only need to prove (3.1). For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$X_{ni} = -\psi(n)I(X_i < -\psi(n)) + X_i I(|X_i| \leq \psi(n)) + \psi(n)I(X_i > \psi(n)),$$
$$Y_{ni} = X_i - X_{ni} = (X_i + \psi(n))I(X_i < -\psi(n)) + (X_i - \psi(n))I(X_i > \psi(n)).$$

It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq k\leq n} |S_k| \geq \epsilon\psi(n)\right) \\
\leq &\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} Y_{ni}\right| \geq \frac{\epsilon\psi(n)}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_{ni}\right| \geq \frac{\epsilon\psi(n)}{2}\right) \\
=: &\, I_1 + I_2.
\end{aligned}$$

For I_1, by conditions (i) and (ii), we have

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} Y_{ni}\right| \geq \frac{\epsilon\psi(n)}{2}\right) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq i\leq n} |X_i| > \psi(n)\right) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{V}(|X_i| > \psi(n)) \\
&\leq C \sum_{n=1}^{\infty} \mathbb{V}(|X| > \psi(n)) \\
&\leq C C_{\mathbb{V}}\left(\psi^{-1}(|X|)\right) \\
&< \infty.
\end{aligned}$$

For I_2, it is easy to check that

$$\begin{aligned}
\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_{ni}\right| &\leq \max_{1\leq k\leq n} \left|\sum_{i=1}^{n} X_{ni} - \sum_{i=1}^{k} X_{ni}\right| + \left|\sum_{i=1}^{n} X_{ni}\right| \\
&\leq 2 \max_{0\leq k\leq n-1} \left|\sum_{i=k+1}^{n} X_{ni}\right|.
\end{aligned}$$

Hence, we have by Lemma 2.8 that

$$\begin{aligned}
I_2 &\leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{0\leq k\leq n-1} \left|\sum_{i=k+1}^{n} X_{ni}\right| \geq \frac{\epsilon\psi(n)}{4}\right) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{0\leq k\leq n-1} \left(\sum_{i=k+1}^{n} X_{ni}\right)^{+} \geq \frac{\epsilon\psi(n)}{8}\right)
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}X_{ni}\right)^{-}\ge\frac{\epsilon\psi(n)}{8}\right)$$
$$\le\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}-\mathbb{E}[X_{ni}])\right)^{+}\ge\frac{\epsilon\psi(n)}{8}-\sum_{i=1}^{n}|\mathbb{E}[X_{ni}]|\right)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}+\mathbb{E}[-X_{ni}])\right)^{-}\ge\frac{\epsilon\psi(n)}{8}-\sum_{i=1}^{n}|\mathbb{E}[-X_{ni}]|\right).$$

By conditions (i) and (iii), we obtain that

$$\frac{1}{\psi(n)}\sum_{i=1}^{n}|\mathbb{E}[X_{ni}]|\le\frac{1}{\psi(n)}\sum_{i=1}^{n}\mathbb{E}\left[|X_i|\right]\le C\frac{n}{\psi(n)}\to 0 \text{ as } n\to\infty,$$

and

$$\frac{1}{\psi(n)}\sum_{i=1}^{n}|\mathbb{E}[-X_{ni}]|\le\frac{1}{\psi(n)}\sum_{i=1}^{n}\mathbb{E}[|X_{ni}|]\to 0 \text{ as } n\to\infty.$$

Hence, for all sufficiently large n, we have

$$\sum_{i=1}^{n}|\mathbb{E}[X_{ni}]|\le\frac{\epsilon\psi(n)}{16}\text{ and }\sum_{i=1}^{n}|\mathbb{E}[-X_{ni}]|\le\frac{\epsilon\psi(n)}{16}. \tag{3.4}$$

Note that for fixed $n\ge 1$, $\{X_{ni}-\mathbb{E}[X_{ni}],1\le i\le n\}$ and $\{-X_{ni}-\mathbb{E}[-X_{ni}],1\le i\le n\}$ are still independent random variables by the definition of independence. Hence, by (3.4), Lemma 2.1, Lemma 2.5, Lemma 2.7 and conditions (i), (ii) and (iii), we have

$$I_2\le C\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}-\mathbb{E}[X_{ni}])\right)^{+}\ge\frac{\epsilon\psi(n)}{16}\right)$$
$$+C\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}+\mathbb{E}[-X_{ni}])\right)^{-}\ge\frac{\epsilon\psi(n)}{16}\right)$$
$$\le C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\mathbb{E}\left[\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}-\mathbb{E}[X_{ni}])\right)^{+}\right)^{2}\right]$$
$$+C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\mathbb{E}\left[\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}+\mathbb{E}[-X_{ni}])\right)^{-}\right)^{2}\right]$$
$$\le C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\mathbb{E}\left[\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(X_{ni}-\mathbb{E}[X_{ni}])\right)\right)^{2}\right]$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\mathbb{E}\left[\left(\max_{0\le k\le n-1}\left(\sum_{i=k+1}^{n}(-X_{ni}-\mathbb{E}[-X_{ni}])\right)\right)^2\right]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\sum_{i=1}^{n}\mathbb{E}[X_{ni}^2]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n\psi^2(n)}\sum_{i=1}^{n}\left[\mathbb{E}[X_i^2 I(|X_i|\le\psi(n))]+\psi^2(n)\mathbb{V}(|X_i|>\psi(n))\right]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n\psi(n)}\sum_{i=1}^{n}\mathbb{E}[|X_i|I(|X_i|\le\psi(n))]+C\sum_{n=1}^{\infty}\mathbb{V}(|X|>\psi(n))\\
&\le C\sum_{n=1}^{\infty}\frac{1}{\psi(n)}+CC_{\mathbb{V}}\left(\psi^{-1}(|X|)\right)\\
&<\infty.
\end{aligned}
$$

Hence, we get the desired result (3.1), and thus (3.2) follows.

Next, we prove (3.3). Note that

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\max_{1\le k\le n}|S_k|\ge\epsilon\psi(n)\right)\\
&=\sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}-1}\frac{1}{n}\mathbb{V}\left(\max_{1\le k\le n}|S_k|\ge\epsilon\psi(n)\right)\\
&\ge\sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}-1}\frac{1}{2^{m+1}}\mathbb{V}\left(\max_{1\le k\le 2^m}|S_k|\ge\epsilon\psi\left(2^{m+1}\right)\right)\\
&=\frac{1}{2}\sum_{m=0}^{\infty}\mathbb{V}\left(\max_{1\le k\le 2^m}|S_k|\ge\epsilon\psi\left(2^{m+1}\right)\right),
\end{aligned}
$$

which together with Borel-Cantelli's lemma yields that

$$
\frac{1}{\psi\left(2^{m+1}\right)}\max_{1\le k\le 2^m}\left|\sum_{i=1}^{k}X_i\right|\to 0\ q.s.\text{ as } m\to\infty. \tag{3.5}
$$

Besides, for each $n\ge 1$, there always exists some m such that $2^m\le n<2^{m+1}$. By (3.5) and (1.3), we have

$$
\begin{aligned}
\frac{1}{\psi(n)}\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right| &\le \frac{1}{\psi(2^m)}\max_{1\le k\le 2^{m+1}}\left|\sum_{i=1}^{k}X_i\right|\\
&=\frac{\psi\left(2^{m+2}\right)}{\psi(2^m)}\cdot\frac{1}{\psi\left(2^{m+2}\right)}\max_{1\le k\le 2^{m+1}}\left|\sum_{i=1}^{k}X_i\right|
\end{aligned}
$$

$$\leq \frac{C}{\psi\left(2^{m+2}\right)} \max_{1\leq k\leq 2^{m+1}} \left|\sum_{i=1}^{k} X_i\right|$$
$$\to 0 \ q.s. \text{ as } n\to\infty.$$

Therefore, (3.3) has been proved immediately. This completes the proof of the theorem. □

Remark 3.1. The following is an example of function $\psi(x)$ satisfying assumptions (H_1), (1.3) and $\frac{\psi(n)}{n}\uparrow\infty$. For $x>d\geq 0$, set

$$\psi(x)=x^{\frac{1}{\alpha}},\ 0<\alpha<1.$$

Obviously, $\psi(x)$ is an increasing unbounded measurable function for $x>d\geq 0$,

$$\frac{\psi(\sigma x)}{\psi(x)}=\sigma^{\frac{1}{\alpha}}\leq C_2,$$

and

$$\frac{\psi(n)}{n}=n^{\frac{1}{\alpha}-1}\uparrow\infty.$$

According to Theorem 3.1 and Borel-Cantelli's lemma, we will give the necessary conditions for the Spitzer's LLN for the maximum of partial sums of *i.i.d.* random variables in $(\Omega,\mathcal{F},\mathbb{E})$.

Theorem 3.2. *Let $\{X,X_n,n\geq 1\}$ be a sequence of i.i.d. random variables. Suppose that $\psi(\cdot)$ satisfies the assumption (H_1) and* (1.3). *If* (3.1) *holds, then for any $\epsilon>0$,*

$$\sum_{n=1}^{\infty} v(|X|\geq\epsilon\psi(n))<\infty,$$

and thus

$$C_v\left(\psi^{-1}(|X|)\right)<\infty. \tag{3.6}$$

Proof. Note that

$$|X_n|=|S_n-S_{n-1}|\leq|S_n|+|S_{n-1}|\leq 2\max_{1\leq k\leq n}|S_k|.$$

From the proof of Theorem 3.1, we know that when (3.1) and (1.3) hold, (3.3) also holds. Hence, we have

$$\frac{|X_n|}{\psi(n)}\leq\frac{2\max\limits_{1\leq k\leq n}|S_k|}{\psi(n)}\to 0 \ q.s. \text{ as } n\to\infty. \tag{3.7}$$

Suppose that there exists $\epsilon_0>0$ such that $\sum_{n=1}^{\infty} v(|X|\geq 2\epsilon_0\psi(n))=\infty$. Let $A_n=\{|X|\geq\epsilon_0\psi(n)\}$, $n=1,2,\ldots$. By (ii) of Lemma 2.3, we have

$$\mathbb{V}\left(\frac{|X|}{\psi(n)}\nrightarrow 0\right)\geq\mathbb{V}(A_n,i.o.)\geq v(A_n,i.o.)=1,$$

which implies $\mathbb{V}\left(\frac{|X|}{\psi(n)} \nrightarrow 0\right) = 1$, but it contradicts (3.7). Therefore, we get that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} v(|X| \geq \epsilon\psi(n)) < \infty.$$

Together with Lemma 2.6 it yields (3.6). This completes the proof of the theorem. □

Acknowledgements. The authors are most grateful to the Editor and anonymous referee for carefully reading the manuscript and for valuable suggestions which helped in improving an earlier version of this paper.

References

[1] F. Boukhari, *On a Spitzer-type law of large numbers for partial sums of m-negatively associated random variables*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 4, Paper No. 180, 10 pp. `https://doi.org/10.1007/s13398-021-01128-x`

[2] Z. Chen and F. Hu, *A law of the iterated logarithm under sublinear expectations*, Int. J. Financ. Eng. **1** (2014), no. 2, 1450015, 23 pp. `https://doi.org/10.1142/s2345768614500159`

[3] Z. Chen, P. Y. Wu, and B. M. Li, *A strong law of large numbers for non-additive probabilities*, Internat. J. Approx. Reason. **54** (2013), no. 3, 365–377. `https://doi.org/10.1016/j.ijar.2012.06.002`

[4] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1953/54), 131–295 (1955).

[5] L. Denis, M. Hu, and S. G. Peng, *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths*, Potential Anal. **34** (2011), no. 2, 139–161. `https://doi.org/10.1007/s11118-010-9185-x`

[6] L. Denis and C. Martini, *A theoretical framework for the pricing of contingent claims in the presence of model uncertainty*, Ann. Appl. Probab. **16** (2006), no. 2, 827–852. `https://doi.org/10.1214/105051606000000169`

[7] X. Feng, *Law of the logarithm for weighted sums of negatively dependent random variables under sublinear expectation*, Statist. Probab. Lett. **149** (2019), 132–141. `https://doi.org/10.1016/j.spl.2019.01.033`

[8] I. Gilboa, *Expected utility with purely subjective nonadditive probabilities*, J. Math. Econom. **16** (1987), no. 1, 65–88. `https://doi.org/10.1016/0304-4068(87)90022-X`

[9] X. Guo and X. Li, *On the laws of large numbers for pseudo-independent random variables under sublinear expectation*, Statist. Probab. Lett. **172** (2021), Paper No. 109042, 8 pp. `https://doi.org/10.1016/j.spl.2021.109042`

[10] P. L. Hsü and H. E. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31. `https://doi.org/10.1073/pnas.33.2.25`

[11] F. Hu and Z. Chen, *General laws of large numbers under sublinear expectations*, Comm. Statist. Theory Methods **45** (2016), no. 14, 4215–4229. `https://doi.org/10.1080/03610926.2014.917677`

[12] T. C. Hu, C. Y. Chiang, and R. L. Taylor, *On complete convergence for arrays of rowwise m-negatively associated random variables*, Nonlinear Anal. **71** (2009), no. 12, e1075–e1081. `https://doi.org/10.1016/j.na.2009.01.104`

[13] Y. J. Hu, R. Ming, and W. Q. Yang, *Large deviations and moderate deviations for m-negatively associated random variables*, Acta Math. Sci. Ser. B (Engl. Ed.) **27** (2007), no. 4, 886–896. `https://doi.org/10.1016/S0252-9602(07)60086-1`

[14] W. Huang and P. Y. Wu, *Strong laws of large numbers for general random variables in sublinear expectation spaces*, J. Inequal. Appl. **2019** (2019), Paper No. 143, 18 pp. `https://doi.org/10.1186/s13660-019-2094-7`

[15] A. Kuczmaszewska, *Complete convergence for widely acceptable random variables under the sublinear expectations*, J. Math. Anal. Appl. **484** (2020), no. 1, 123662, 15 pp. `https://doi.org/10.1016/j.jmaa.2019.123662`

[16] Y. Lin and X. Feng, *Complete convergence and strong law of large numbers for arrays of random variables under sublinear expectations*, Comm. Statist. Theory Methods **49** (2020), no. 23, 5866–5882. `https://doi.org/10.1080/03610926.2019.1625924`

[17] F. Maccheroni and M. Marinacci, *A strong law of large numbers for capacities*, Ann. Probab. **33** (2005), no. 3, 1171–1178. `https://doi.org/10.1214/009117904000001062`

[18] S. G. Peng, *Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type*, Probab. Theory Related Fields **113** (1999), no. 4, 473–499. `https://doi.org/10.1007/s004400050214`

[19] S. G. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of itô's type*, Stochastic Anal. Appl. **2** (2006), no. 4, 541–567.

[20] S. G. Peng, *Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation*, Stochastic Process. Appl. **118** (2008), no. 12, 2223–2253. `https://doi.org/10.1016/j.spa.2007.10.015`

[21] S. G. Peng, *Nonlinear expectations and stochastic calculus under uncertainty with robust CLT and G-Brownian motion*, Springer, Berlin, Heidelberg, 2019.

[22] A. Shen, Y. Zhang, B. Xiao, and A. Volodin, *Moment inequalities for m-negatively associated random variables and their applications*, Statist. Papers **58** (2017), no. 3, 911–928. `https://doi.org/10.1007/s00362-015-0731-x`

[23] M. Wang and X. J. Wang, *Some convergence properties for the maximum of partial sums of m-negatively associated random variables*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 2345–2358. `https://doi.org/10.1007/s13398-019-00626-3`

[24] Y. F. Wu, T. C. Hu, and A. I. Volodin, *Complete convergence and complete moment convergence for weighted sums of m-NA random variables*, J. Inequal. Appl. **2015** (2015), 200, 14 pp. `https://doi.org/10.1186/s13660-015-0717-1`

[25] Q. Y. Wu and Y. Jiang, *Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations*, J. Math. Anal. Appl. **460** (2018), no. 1, 252–270. `https://doi.org/10.1016/j.jmaa.2017.11.053`

[26] M. Xu and K. Cheng, *Precise asymptotics in the law of the iterated logarithm under sublinear expectations*, Math. Probl. Eng. **2021** (2021), Art. ID 6691857, 9 pp. `https://doi.org/10.1155/2021/6691857`

[27] J. P. Xu and L. X. Zhang, *Three series theorem for independent random variables under sub-linear expectations with applications*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 2, 172–184. `https://doi.org/10.1007/s10114-018-7508-9`

[28] L. X. Zhang, *Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm*, Sci. China Math. **59** (2016), no. 12, 2503–2526. `https://doi.org/10.1007/s11425-016-0079-1`

[29] L. X. Zhang, *Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications*, Sci. China Math. **59** (2016), no. 4, 751–768. `https://doi.org/10.1007/s11425-015-5105-2`

[30] L. X. Zhang, *Heyde's theorem under the sub-linear expectations*, Statist. Probab. Lett. **170** (2021), Paper No. 108987, 9 pp. `https://doi.org/10.1016/j.spl.2020.108987`

[31] L. X. Zhang, *On the laws of the iterated logarithm under sub-linear expectations*, Probab. Uncertain. Quant. Risk **6** (2021), no. 4, 409–460. `https://doi.org/10.3934/puqr.2021020`

[32] L. X. Zhang, *Strong limit theorems for extended independent random variables and extended negatively dependent random variables under sub-linear expectations*, Acta Math. Sci. Ser. B (Engl. Ed.) **42** (2022), no. 2, 467–490. https://doi.org/10.1007/s10473-022-0203-z

[33] N. Zhang and Y. T. Lan, *Rosenthal's inequalities for asymptotically almost negatively associated random variables under upper expectations*, Chinese Ann. Math. Ser. B **40** (2019), no. 1, 117–130. https://doi.org/10.1007/s11401-018-0122-4

[34] L. X. Zhang and J. Lin, *Marcinkiewicz's strong law of large numbers for nonlinear expectations*, Statist. Probab. Lett. **137** (2018), 269–276. https://doi.org/10.1016/j.spl.2018.01.022

Miaomiao Wang
School of Big Data and Statistics
Anhui University
Hefei 230601, P. R. China

Min Wang
School of Mathematical Sciences
Anhui University
Hefei 230601, P. R. China

Xuejun Wang
School of Big Data and Statistics
Anhui University
Hefei 230601, P. R. China
Email address: wxjahdx@126.com