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THE CRITICAL PODS OF PLANAR QUADRATIC POLYNOMIAL MAPS OF TOPOLOGICAL DEGREE 2

Misong Chang, Sunyang Ko, Chong Gyu Lee, and Sang-Min Lee

Abstract. Let K be an algebraically closed field of characteristic 0 and let f be a non-fibered planar quadratic polynomial map of topological degree 2 defined over K. We assume further that the meromorphic extension of f on the projective plane has the unique indeterminacy point. We define *the critical pod of f* where f sends a critical point to another critical point. By observing the behavior of f at the critical pod, we can determine a good conjugate of f which shows its statue in GIT sense.

1. Introduction

In this article, we introduce the *critical pods* of planar quadratic polynomial maps of topological degree 2. In complex and algebraic dynamics, a polynomial automorphism, which is a map of topological degree 1, is one of popular examples [1, 3–6]. In complex analysis, polynomial maps of small topological degree are studied to generalize the results for polynomial automorphisms [7–9]. So we expect to generalized algebraic and geometric properties of polynomial automorphisms [11, 13].

Let K be an algebraically closed field of characteristic 0 and let f be a planar quadratic polynomial map of topological degree 2 defined over K. When f is a map of topological degree 2, the first observation should be made at the *critical points*. Since topological degree is the number of preimages of a generic point, we have a critical point where preimages overlap. If critical points appear one after another, it is quite interesting.

Definition 1.1. Let f be a planar polynomial map of topological degree 2. We say that $\{P_{-1}, P_0, P_1\}$ is a *critical pod of f* if $P_0 = f(P_{-1})$, $P_1 = f(P_0)$,

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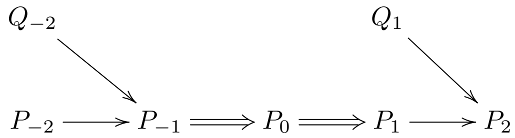
and both P_0 and P_1 have exactly one preimage.



We can describe the critical pod with the dynamical Mordell-Lang conjecture. Xie proves that the dynamical Mordell-Lang conjecture for planar polynomial maps when $K = \overline{\mathbb{Q}}$.

Theorem 1.2 ([16, Theorem 0.1])**.** *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a planar polynomial map defined over $\overline{\mathbb{Q}}$, let C be an irreducible curve in $\mathbb{A}^2$ and let P be a point in $\mathbb{A}^2$. Then the set*

$$D_f(P; C) := \{n \in \mathbb{N} \mid f^n(P) \in C\}$$

is a finite union of arithmetic progressions.

Can we use Xie's result for critical points? If f is a planar quadratic polynomial map of topological degree 2 and $K = \overline{\mathbb{Q}}$, the set of critical points of f is a line L (Proposition 3.1) so that the set $D_f(P; L)$ should be a finite union of arithmetic progressions for any P. We ask further questions - can we find an arithmetic progression with a common difference of 1 in $D_f(P; L)$? If we have one, how long it can be? It is determined by a geometric property of f: we say that f is *fibered* if f sends every line parallel to L to another line parallel to L. We assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point to show that the map f is not fibered if and only if there is an arithmetic progression of length 2, with a common difference of 1, in $D_f(P; L)$ for some P. It guarantees the existence and the uniqueness of the critical pod.

Theorem A (Theorem 3.3)**.** *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 defined over K. Assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point. Then f has the unique critical pod.*

The points where a self map has good dynamical behavior are quite useful in the study of dynamical systems. For example, periodic points of a polynomial automorphism are equidistributed: the probability measures on the set of (saddle) points of period n weakly converges to the invariant measure [2, 12]. The conjugacy classes of quadratic Hénon maps are determined by information at fixed points [11]. The critical pod is also useful to figure out interesting properties of the dynamical system. We define the *determinant and the traces of f* by observing the Jacobian of f at critical pods to find when the critical pod is a fixed point and when algebraic degree of m-th iterate f^m of f is stable.

Definition 1.3. Let f be a non-fibered planar quadratic polynomial map of topological degree 2 defined over K. Assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point. We define

$$T_0 := \operatorname{tr} J_f(P_0), \quad T_1 := \operatorname{tr} J_f(P_1), \quad \text{and} \quad D := \det J_f(P_1),$$

where $\{P_{-1}, P_0, P_1\}$ is the critical pod of f.

Theorem B (Propositions 4.3, 5.3 and 5.4, Corollaries 4.5 and 5.9)**.** *Let f be a non-fibered planar quadratic polynomial map of topological degree* 2 *defined over K. Assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point. Let $\{P_{-1}, P_0, P_1\}$ be the critical pod of f, let f^m be the m-th iterate of f and let* $\deg(f^m)$ *be algebraic degree of f^m. Then the following hold*:

(a) *P_0 is a fixed point of f (i.e., $P_{-1} = P_0 = P_1$) if and only if $D = 0$,*
(b) *P_0 is the unique fixed point if and only if $D = 0$ and $T_0 = 1$,*
(c) $\deg(f^m) \neq 2^m$ *for some $m \in \mathbb{N}$ if and only if* $\operatorname{tr} J_f(P) = T_1$ *for all $P \in \mathbb{A}^2$, and*
(d) *If* $\deg(f^m) = 2^m$ *for all $m \in \mathbb{N}$, the P_0 is a fixed point if and only if $T_0 = T_1$.*

For reader's convenience, we say that f is *algebraically stable* if $\deg(f^m) = 2^m$ for all $m \in \mathbb{N}$ and *algebraically non-stable* otherwise. We will say f is *singular* if $D = 0$. It it not only because such f is determined by $D = 0$, but also we can observe some singularity. When we consider a family f_t of non-fibered planar quadratic maps of topological degree 2, we have three sections $P_{t,-1}, P_{t,0}, P_{t,1}$ which form the 'section' of the critical pod. These three sections meet at a point when f_t is singular.

Also, we use the critical pod to find the conjugacy class of f. By listing coefficients, we can correspond quadratic rational maps to points in the projective space to define the parameter space of quadratic rational maps:

$$\mathcal{R}_2 := \Big\{ [A_{200}; \cdots; B_{200}; \cdots; C_{200}; \cdots] \in \mathbb{P}^{17} \;\Big|\; \Big[\sum A_{ijk} X^i Y^j Z^k; \sum B_{ijk} X^i Y^j Z^k; \sum C_{ijk} X^i Y^j Z^k\Big] : \text{quadratic rational map} \Big\}.$$

When $\sigma \in \mathrm{PGL}_3(K)$, two rational maps g and its conjugate $g^\sigma = \sigma \circ g \circ \sigma^{-1}$ have the same dynamical properties.

$$\begin{array}{ccccccc}
P & \xrightarrow{g} & g(P) & \xrightarrow{g} & g^2(P) & \xrightarrow{g} & \cdots \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \\
\sigma(P) & \xrightarrow[g^\sigma]{} & g^\sigma(\sigma(P)) & \xrightarrow[g^\sigma]{} & (g^\sigma)^2(\sigma(P)) & \xrightarrow[g^\sigma]{} & \cdots
\end{array}$$

So we consider $\mathrm{PGL}_3(K)$-congujacy-action on $\mathcal{R}_2$ and construct the *moduli space* of quadratic rational maps by collecting the conjugacy classes of g. The moduli space of quadratic rational map is well-defined as the categorical quotient. But we want the moduli space to be a good geometric object to study dynamical properties of a family of certain maps. The geometric invariant theory (GIT) shows that which point in the categorical quotient can be distinguished from other point geometrically. We say a point is *stable* if a point is closed, *semistable* if the closure of the point does not contain the zero element, and *unstable* otherwise. We use D, T_0 and T_1 to find a simple representative of f, which shows that f is semistable or unstable. Thought we use the fixed points for Hénon maps in previous work [11], we use the critical pod in this paper because it is easily determined by $L \cap f(L)$ so that we can describe the condition to get the unique critical pod. Moreover, points in the critical pod has natural order while the set of fixed points doesn't have one.

Theorem C (Theorems 4.4, 5.6, and 5.8). *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 defined over K. Assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point. Then*

$$f \sim \begin{cases} \left(\ell(x,y), \frac{1}{2}y^2 + \ell(x,y)\right) & \text{if } f : \text{algebraically stable,} \\ (T_0x - y + D, \frac{1}{2}x^2) & \text{if } f : \text{algebraically non-stable,} \end{cases}$$

where

$$\ell(x,y) = \begin{cases} \frac{D}{T_1 - T_0}x + \left(T_0 - \frac{D}{T_1-T_0}\right)y + (T_1 - T_0) & \text{if } D \neq 0, \\ \frac{\det J_f(P)}{\mathrm{tr} J_f(P) - T_0}x + \left(T_0 - \frac{\det J_f(P)}{\mathrm{tr} J_f(P) - T_0}\right)y + (T_1 - T_0) & \text{if } D = 0 \text{ and } P \text{ any point with } \det J_f(P) \neq 0. \end{cases}$$

Moreover, f is semistable if it is algebraically stable and unstable if it is algebraically non-stable in GIT sense.

The rest of paper is organized as follows. In Section 2, we review some points with geometric properties of f: the indeterminacy points and the infinity fixed points. In Section 3, we prove that the critical pod exists if and only if f is not fibered, and we introduce some properties of the critical pods. In Sections 4 and 5, we use them to find a good representative of each conjugacy class which shows its own status in GIT sense. Unless otherwise stated, we let K be an algebraically closed field of characteristic 0, let λ_f be topological degree of f and let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a planar quadratic polynomial map of topological degree 2 defined over K. Also, we let $\overline{f}$ be the meromorphic extension of f on $\mathbb{P}^2$ with the homogenizing variable Z and let $H = \mathbb{P}^2 \setminus \mathbb{A}^2$ be the hyperplane of infinity defined by the equation $Z = 0$.

2. The indeterminacy points and the infinity fixed points

In this section, we observe two kinds of points with geometric properties; the indeterminacy point and the infinity fixed point. Since the meromorphic extension $\overline{f}$ is of algebraic degree 2 and of topological degree 2, it is not an endomorphism on $\mathbb{P}^2$ and hence $\overline{f}$ has the *indeterminacy locus* $I(\overline{f})$, where we cannot continuously extend f. Since the dimension of the $I(\overline{f})$ is 0 when $\overline{f}$ is a planar rational map, $I(\overline{f})$ consists of finitely many points.

We define the infinity fixed point by observing geometric description of algebraic stability of f with the indeterminacy point.

Lemma 2.1 ([15] Lemma 7.8, planar version)**.** *Let $\phi, \psi : \mathbb{A}^2 \to \mathbb{A}^2$ be polynomial maps. Then*

$$\deg(\phi \circ \psi) < \deg \phi \cdot \deg \psi \quad \text{if and only if} \quad \overline{\psi}(H \setminus I(\overline{\psi})) \subset I(\overline{\phi}),$$

where $\overline{\phi}$, $\overline{\psi}$ are meromorphic extensions of ϕ and ψ on $\mathbb{P}^2$, and $H = \mathbb{P}^2 \setminus \mathbb{A}^2$ is the hyperplane of infinity.

Corollary 2.2. *A planar polynomial map f is algebraically non-stable if and only if $\overline{f}(H \setminus I(\overline{f})) \subset I(\overline{f})$.*

Corollary 2.2 guarantees that $\overline{f}(H \setminus I(\overline{f}))$ should be of dimension 0 when f is algebraically non-stable. Furthermore, we can resolve indeterminacy of $\overline{f}$ [10]: we can find a blowup variety V of $\mathbb{P}^2$ along $I(\overline{f})$ where $\overline{f}$ can be extended to a continuous map $\widetilde{f} : V \to \mathbb{P}^2$. Hence the closure of $H \setminus I(\overline{f})$ should be mapped to a connected set. Therefore, every point on $H \setminus I(\overline{f})$ is mapped to a point in H. We will see that it also happens when f is algebraically stable.

Proposition 2.3. *Let f be a planar quadratic polynomial map of topological degree 2. Then f is of the form*

$$(aQ(x,y) + L_1(x,y),\, bQ(x,y) + L_2(x,y)),$$

where $Q(x,y)$ is a quadratic form, $a, b \in K$ are constants and $L_1, L_2 \in K[x,y]$ are linear polynomials.

Proof. Since f is a planar polynomial map of degree 2, it is of the form

$$(Q_1(x,y) + l_1(x,y),\ Q_2(x,y) + l_2(x,y)),$$

where Q_i's are quadratic forms and l_i's are linear polynomials. Consider the meromorphic extension of f on $\mathbb{P}^2$;

$$\overline{f}[X;Y;Z] = [Q_1(X,Y) + ZL_1(X,Y,Z);\ Q_2(X,Y) + ZL_2(X,Y,Z);\ Z^2].$$

Let (γ, δ) be a generic point in $\mathbb{A}^2$. Then its preimage in $\mathbb{P}^2$ is the intersections of planar curves C_1 and C_2, defined by

$$F_1(X,Y,Z) = Q_1(X,Y) + Z(l_1(X,Y,Z) - \gamma Z) = 0$$

and

$$F_2(X,Y,Z) = Q_2(X,Y) + Z(l_2(X,Y,Z) - \delta Z) = 0,$$

respectively. If F_1 and F_2 have a common factor, then we get infinitely many preimages of a generic point (γ, δ), which contradicts to $\lambda_f = 2$. So, F_1 and F_2 should be coprime. So, by Bézout's theorem, the number of intersection points of C_1 and C_2 is 4. Since we only have two intersection points in $\mathbb{A}^2$, we should have another two intersection points in $H = \mathbb{P}^2 \setminus \mathbb{A}^2$, which means that there are two intersection points with $Z = 0$. So, we get

$$Q_1(X, Y) = a(\beta_1 X - \alpha_1 Y)(\beta_2 X - \alpha_2 Y)$$

and

$$Q_2(X, Y) = b(\beta_1 X - \alpha_1 Y)(\beta_2 X - \alpha_2 Y)$$

for some $a, b \in K$. □

Definition 2.4. Let f be a planar quadratic polynomial map of topological degree 2, of the form in Proposition 2.3. We call

$$\overline{f}(H \setminus I(\overline{f})) = [a; b; 0]$$

the *infinity fixed point of* f.

Note that $\overline{f}$ is not defined at the infinity fixed point if f is algebraically non-stable. But, $\overline{f}$ is constant on $H \setminus I(\overline{f})$ so we can extend $\overline{f}|_{H \setminus I(\overline{f})}$ to a continuous function on H which fixes the infinity fixed point.

By taking a proper conjugate of f, we can locate the infinity fixed point at $[0; 1; 0]$ and hence we may assume that

$$\overline{f}[X; Y; Z] = [Z(a_1 X + b_1 Y + c_1 Z); Q(X, Y) + Z(a_2 X + b_2 Y + c_2 Z); Z^2]$$

and

$$f(x, y) = (a_1 x + b_1 y + c_1, Q(x, y) + a_2 x + b_2 y + c_2), \tag{1}$$

where Q is a quadratic form. We can find that a solution (α, β) of $Q(x, y) = 0$ corresponds to an indeterminacy point $[\alpha; \beta; 0]$. In particular, $Q(x, y)$ becomes a complete square if f has exactly one indeterminacy point.

3. The critical pods

In this section, we study how to find the critical point and the critical pod, and we discuss why we consider 'non-fibered' maps. We start with the observation where the preimages overlap.

Proposition 3.1. *Suppose that f is a planar quadratic polynomial map of topological degree 2. Then the zero locus of the determinant of Jacobian of f is a line.*

Proof. If f is of the form (1);

$$f(x, y) = (a_1 x + b_1 y + c_1, Q(x, y) + a_2 x + b_2 y + c_2),$$

then the determinant of the Jacobian of f is a linear polynomial;

$$\det J_f = \begin{vmatrix} a_1 & b_1 \\ Q_x + a_2 & Q_y + b_2 \end{vmatrix} = a_1 Q_y - b_1 Q_x + (a_1 b_2 - a_2 b_1).$$ □

For convenience, we call the zero locus of $\det J_f$ the *critical line of* f. The map f has interesting geometric behavior on its own critical line.

Proposition 3.2. *Let f be a planar quadratic polynomial map of topological degree 2, and let L be the critical line of f. Then $P \in L$ if and only if P is the only preimage of $f(P)$;*

$$P \in L \ \Leftrightarrow \ f^{-1}(f(\{P\})) = \{P\}.$$

Proof. Let $f = (f_1, f_2)$ be of the form (1), let $P = (x_0, y_0)$ and let $f(P) = (\gamma,\, \delta)$. The preimages of $f(P)$ should satisfy the following system of equations;

$$\begin{cases} a_1x + b_1y + c_1 = \gamma, \\ f_2(x, y) = \delta. \end{cases}$$

We may assume that not both a_1, b_1 are zero: otherwise, $\lambda_f = \infty$. If $a_1 \neq 0$, we can see that y_0 should be the critical point of g if and only if y_0 is the unique solution of a quadratic equation

$$g(y) = f_2\left(\tfrac{b_1y+c_1-\gamma}{-a_1}, y\right) - \delta.$$

Since the equality

$$\frac{dg}{dy} = \frac{\det J_f}{a_1}$$

holds, we can conclude that y_0 is the critical point if and only if $P \in L$. If $a_1 = 0$, then $b_1 \neq 0$ so that we use x instead of y to get the similar result. □

We say that a point P is *wandering* if the orbit $\mathcal{O}_f(P) = \{P, f(P), \ldots\}$ of a P is Zariski dense in $\mathbb{A}^2$. If $P \in L$ is wandering, the probability of $f(P) \in L$ is 0. However, unless f is fibered, we can guarantee the existence and the uniqueness of the critical pod.

Theorem 3.3. *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 defined over K. Assume further that the meromorphic extension $\overline{f}$ of f on $\mathbb{P}^2$ has only one indeterminacy point. Then f has the unique critical pod.*

Proof. Since we assume that $\overline{f}$ has only one indeterminacy point, f should be conjugate with

$$(a_1x + b_1y + c_1,\ (\beta x - \alpha y)^2 + a_2x + b_2y + c_2).$$

Note that $a_1\alpha + b_1\beta \neq 0$; otherwise $\lambda_f = 1$, which is a contradiction. The directional vector of the critical line L of f is (α, β) so that the value of $(\beta x - \alpha y)^2$ should be constant on L. So $f|_L$ forms an affine map and hence $f(L)$ should be a line.

Suppose that L and $f(L)$ are parallel. Since the directional vector of $f(L)$ is

$$(\alpha, \beta)\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

we get

$$(a_1\alpha + b_1\beta)\beta = (a_2\alpha + b_2\beta)\alpha.$$

If L' is any line parallel to L, then $f(L')$ is also parallel to $f(L)$, which contradicts that f is not fibered. Therefore, L and $f(L)$ are not parallel so that there exists the unique intersection point P_0 in $\mathbb{A}^2$ which provides the critical pod $\{P_{-1} = f^{-1}(P_0), P_0, P_1 = f(P_0)\}$. □

Corollary 3.4. *The indeterminacy point $\mathcal{I}$ and two points P_{-1}, P_0 are collinear.*

Proof. We know that both P_0 and P_{-1} are on L. Also, we can see that the directional vector of the critical line L is (α, β) so that L meets with H at $\mathcal{I} = [\alpha; \beta; 0]$. □

We want to describe f with geometric invariants at P_{-1}, P_0 and P_1. If f^σ is a conjugate of f by $\sigma \in \mathrm{GL}_2(K)$, then we get

$$J_{f^\sigma}(\sigma(P)) = J_\sigma(f(P)) \cdot J_f(P) \cdot J_{\sigma^{-1}}(\sigma(P))$$

by the chain rule. So we can say the determinant and trace of the Jacobian of f at a point are geometric information of the dynamical system $(f, \mathbb{A}^2)$ because they only depend on the conjugacy class of f;

$$\det J_f(P) = \det J_{f^\sigma}(\sigma(P)) \quad \text{and} \quad \operatorname{tr} J_f(P) = \operatorname{tr} J_{f^\sigma}(\sigma(P)).$$

If $\{P_{-1}, P_0, P_1\}$ is the critical pod of f, then $\{\sigma(P_{-1}), \sigma(P_0), \sigma(P_1)\}$ is the critical pod of f^σ. Therefore, D, T_0, T_1 are invariants of the conjugacy class of f.

Note that we only have to consider three invariants D, T_0, T_1 of the conjugacy class of f introduced in Definition 1.3. Since both P_{-1} and P_0 are on the critical line, the determinant of J_f may not vanish only at P_1. Moreover, we will show that $\operatorname{tr} J_f(P_{-1}) = T_0$ later.

4. Case I: f is algebraically non-stable

In this section, we will treat non-fibered planar quadratic polynomial maps of topological degree 2 whose meromorphic extension $\overline{f}$ has the unique indeterminacy point, which are algebraically non-stable. By examining T_1 and D, we can find a good conjugate of f which reveals geometric information of f. We start by examining where the indeterminacy point is.

Lemma 4.1. *Let f be a planar quadratic polynomial map of topological degree 2 which is algebraically non-stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$. Then f is conjugate to a non-fibered one*

$$g(x, y) = (ax + by + c,\ x^2 + kax + kby + kc),$$

where $b \in K$ is a non-zero constant.

Proof. Since we assume that $\overline{f}$ has only one indeterminacy point, we may assume that f is conjugate to

$$(a_1x + b_1y + c_1, (\beta x - \alpha y)^2 + a_2x + b_2y + c_2).$$

Since f is algebraically non-stable, Corollary 2.2 says

$$\mathcal{I} = \overline{f}(H \setminus I(\overline{f})) = [0;1;0]$$

and hence

$$f(x,y) = (a_1x + b_1y + c_1,\ x^2 + a_2x + b_2y + c_2). \tag{2}$$

Moreover, $\lambda_f = 2$ guarantees $b_1 \neq 0$: if not,

$$\begin{cases} a_1x + c_1 = \gamma, \\ x^2 + a_2x + b_2y + c_2 = \delta \end{cases}$$

has the unique solution.

Let L be the critical line of f. Since the parametric equation of L is

$$\det J_f = 0 \quad \Leftrightarrow \quad x = \frac{a_1b_2 - a_2b_1}{2b_1} \ \text{ and } \ y = t,$$

the parametric equation of $f(L)$ is

$$\begin{cases} x = a_1\dfrac{a_1b_2 - a_2b_1}{2b_1} + b_1t + c_1, \\ y = \left(\dfrac{a_1b_2 - a_2b_1}{2b_1}\right)^2 + a_2\dfrac{a_1b_2 - a_2b_1}{2b_1} + b_2t + c_2 \end{cases} \tag{3}$$

which satisfies

$$(4b_1b_2)x = (4b_1^2)y + \left\{(a_1b_2 - a_2b_1)^2 - 4b_1(b_1c_2 - b_2c_1)\right\}.$$

Since $b_1 \neq 0$, (3) guarantees that $f(L)$ is not a vertical line and hence f is not fibered: two lines L and $f(L)$ properly intersect at

$$P_0 = \left(\frac{a_1b_2 - a_2b_1}{2b_1}, \frac{2b_2(a_1b_2 - a_2b_1) - (a_1b_2 - a_2b_1)^2 + 4b_1(b_1c_2 - b_2c_1)}{4b_1^2}\right).$$

Note that P_0, $P_1 = f(P_0)$, and $P_{-1} = f^{-1}(P_0)$ form the critical pod of f. We locate P_0 at $[0;0;1]$ to get the relations on the conjugate in (2);

$$a_1b_2 - a_2b_1 = 0, \quad \text{and} \quad b_1c_2 - b_2c_1 = 0$$

so that we may assume

$$f \sim (a_1x + b_1y + c_1,\ x^2 + ka_1x + kb_1y + kc_1)$$

where $k = b_2/b_1$. □

Corollary 4.2. *Three points $\mathcal{I}$, P_0, P_1 are not collinear unless P_0 is a fixed point.*

Proof. In the proof of Lemma 4.1, $P_1 = (c_1, kc_1)$ is on the line $\langle y = kx \rangle$ so three points $\mathcal{I}$, P_0 and P_1 are not on the same line unless $P_0 = P_1$. □

Lemma 4.1 provides a good conjugate of f enough to show the following.

Proposition 4.3. *Let f be a planar quadratic polynomial map of topological degree 2 which is algebraically non-stable. Assume further that $\overline{f}$ has the unique indeterminacy point. Then the following hold*:

(a) $\operatorname{tr} J_f : \mathbb{A}^2 \to K$ *is a constant map, and*
(b) *The mid-point P_0 of the critical pod is a fixed point if and only if $D = 0$.*

Proof. We may assume that f is of the form obtained in Lemma 4.1. The Jacobian of the conjugate f obtained in Lemma 4.1 is

$$J_f(x, y) = \begin{bmatrix} a & b \\ 2x + ka & kb \end{bmatrix},$$

where b is nonzero constant. So we get (a);

$$\operatorname{tr} J_f(x, y) = a + kb.$$

Also, the determinant of the Jacobian of f,

$$\det J_f(x, y) = -2bx$$

guarantees that $D = 0$ if and only if $c_1 = 0$. So, $P_1 = (c_1, kc_1) = P_0$ if and only if $D = 0$. $\square$

To get the good conjugate in Lemma 4.1, we only locate two points at $[0; 1; 0]$ and $[0; 0; 1]$. So we can find a better conjugate by locating P_1 at x-axis while fixing $[0; 1; 0]$ and $[0; 0; 1]$.

Theorem 4.4. *Let f be a planar quadratic polynomial map of topological degree 2 which is algebraically non-stable. Assume further that $\overline{f}$ has the unique indeterminacy point. Then we get*

$$f \sim \left(T_0 x - y + D, \frac{1}{2}x^2\right).$$

Proof. We may assume that f is of the form obtained in Lemma 4.1. We can check that $\overline{\sigma} \in \mathrm{PGL}_3(K)$ will fix $[0; 1; 0]$ and $[0; 0; 1]$ only if $\overline{\sigma} = [rX; sX + tY; Z]$. Using $\sigma = (-2bx, 2b^2(y - kx))$ where b is a nonzero constant, we get

$$f^\sigma \sim \left((a + kb)x - y - 2bc, \frac{1}{2}x^2\right).$$

Since $T_0 = a + kb$, $D = -2bc$, we get the desired result. Note that we get $P_{-1} = (0, D)$ and $P_1 = (D, 0)$. $\square$

Corollary 4.5. *P_0 is the unique fixed point if and only if $D = 0$ and $T_0 = 1$.*

Proof. When $D = 0$, we have two fixed points, P_0 and $(1 - T_0, 2(1 - T_0)^2)$. $\square$

We have another reason why we say the representative in Theorem 4.4 is a good one; we can easily check GIT-stability of f. In such sense, this representative is the best conjugate of f.

Corollary 4.6. *Let f be a planar quadratic polynomial map of topological degree 2 which is algebraically non-stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$. Then f is unstable in GIT sense.*

Proof. We apply the Hilbert-Mumford Criterion [14] to show f is unstable; we will find a 1-parameter subgroup $L_{r,s}(\alpha) = \{[\alpha^r; \alpha^{-r+s}; \alpha^{-s}] \mid r, s \in \mathbb{Z}^{\geq 0}\}$ of $\mathrm{PGL}_3(K)$ which only gives positive weight $\mu(\overline{f},\ L_{r,s})$ to $\overline{f}$. The meromorphic extension of f is of the form

$$[T_0XZ - YZ + DZ^2; \frac{1}{2}X^2; Z^2],$$

we examine Table 1 to find that every exponent of α can be positive. For example, $s = 4, r = 1$ gives

$$\mu(\overline{f},\ L_{1,4}) = \min\{1 + 2 \cdot 4, 2 \cdot 1, 4, -3 \cdot 1 + 4, 4\} = 1 > 0.$$

Therefore, f is unstable. □

Table 1. Exponents of α in algebraically non-stable $\overline{f}^{L_{r,s}}$

	X^2	Y^2	Z^2	XY	YZ	XZ
x-coordinate of $\overline{f}^{L_{r,s}}$	$-$	$-$	$r + 2s$	$-$	$2r$	s
y-coordinate of $\overline{f}^{L_{r,s}}$	$-3r + s$	$-$	$-$	$-$	$-$	$-$
z-coordinate of $\overline{f}^{L_{r,s}}$	$-$	$-$	s	$-$	$-$	$-$

5. Case II: f is algebraically stable

In this section, we will treat a non-fibered planar quadratic polynomial map of topological degree 2 whose meromorphic extension f has the unique indeterminacy point which is algebraically stable and has one indeterminacy point. By examining T_0, T_1 and D, we can find a good conjugate of f which reveals geometric information of f. We start by locating the indeterminacy point at a good place.

Lemma 5.1. *Let f be a planar non-fibered planar quadratic polynomial map of topological degree 2 which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$. Then f is conjugate to*

$$g(x, y) = (ax + by + c,\ y^2 + kax + kby + kc).$$

Proof. Since we assume that f has only one indeterminacy point, we may assume that f is of the form

$$(a_1x + b_1y + c_1, (\beta x - \alpha y)^2 + a_2x + b_2y + c_2).$$

Since f is algebraically stable, $\mathcal{I}$ must be different from $\overline{f}(H \setminus I(\overline{f}))$ by Corollary 2.2. We locate $\mathcal{I}$ and $f(H \setminus I(\overline{f}))$ at $[1;0;0]$ and $[0;1;0]$ respectively to get a conjugate of the form

$$(a_1x + b_1y + c_1,\ y^2 + a_2x + b_2y + c_2). \tag{4}$$

Moreover, $\lambda_f = 2$ and non-fibered condition guarantee $a_1a_2 \neq 0$: $a_1 = 0$ only if $\lambda_f = 1$ and $a_2 = 0$ only if f is fibered.

Let L be the critical line of f. Since the parametric equation of L is

$$\det J_f = 0 \quad \Leftrightarrow \quad x = t,\ y = -\frac{a_1b_2 - a_2b_1}{2a_1},$$

the parametric equation of $f(L)$ is

$$\begin{cases} x = a_1t - b_1\dfrac{a_1b_2 - a_2b_1}{2a_1} + c_1, \\ y = \left(-\dfrac{a_1b_2 - a_2b_1}{2a_1}\right)^2 + a_2t - b_2\dfrac{a_1b_2 - a_2b_1}{2a_1} + c_2 \end{cases}$$

which satisfies

$$(4a_1^2)y = (4a_1a_2)x - \{(a_1b_2 - a_2b_1)^2 - 4a_1(a_1c_2 - a_2c_1)\}.$$

Note that P_0, $P_1 = f(P_0)$, and $P_{-1} = f^{-1}(P_0)$ form the critical pod of f. We locate P_0 at $[0;0;1]$ to get the relations on the conjugate in (4);

$$a_1b_2 - a_2b_1 = 0, \quad \text{and} \quad a_1c_2 - a_2c_1 = 0$$

so that we may assume

$$f \sim (a_1x + b_1y + c_1,\ y^2 + ka_1x + kb_1y + kc_1),$$

where $k = a_2/a_1$. □

Corollary 5.2. *If P_0 is not a fixed point, then $\mathcal{I}, P_0, P_1$ are not colinear. Also, $\overline{f}(H \setminus I(\overline{f})), P_0, P_1$ are not colinear, either.*

Proof. In the proof of Lemma 5.1, $P_1 = (c_1, kc_1)$ is on the line $\langle y = kx\rangle$. Since $k \neq 0$, three points $\mathcal{I}$, P_0 and P_1 are not on the same line unless $P_0 = P_1$. Also, $k \neq \infty$, three points $\overline{f}(H \setminus I(\overline{f}))$, P_0 and P_1 are not on the same line unless $P_0 = P_1$. □

Lemma 5.1 provides a good conjugate of f enough to show the following.

Proposition 5.3. *Let f be a non-fibered planar quadratic polynomial map of topological degree* 2 *which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point. Then the following hold*:

(a) $\operatorname{tr} J_f : \mathbb{A}^2 \to K$ *is not a constant map.*
(b) *The mid-point P_0 of the critical pod is a fixed point if and only if $D = 0$.*
(c) $\det J_f(P) \neq 0$ *if and only if* $\operatorname{tr} J_f(P) \neq T_0$.

Proof. We may assume that f is of the form in Lemma 5.1. The Jacobian of f is

$$J_f = \begin{bmatrix} a & b \\ ka & 2y + kb \end{bmatrix},$$

where a is a nonzero constant. So we get (a);

$$\operatorname{tr} J_f(x, y) = 2y + a + kb.$$

Also, the determinant of the Jocobian of f,

$$\det J_f(x, y) = 2ay$$

guarantees that $D = 0$ if and only if $c_1 = 0$. So, $P_1 = (c_1, kc_1) = P_0$ if and only if $D = 0$. □

Proposition 5.4. *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point. Then the following hold:*

(1) *f is singular if and only if the critical pod consists of a single point.*
(2) *f is singular if and only if $T_0 = T_1$.*

Now we take three points, $\mathcal{I}, \overline{f}(H \setminus I(\overline{f}))$ and P_0. If we can find another point P such that $\mathcal{I}, \overline{f}(H \setminus I(\overline{f}))$, P_0 and P form points in general position, we can find the best conjugate of f by locating them at nice spots. Such point P should satisfy the following property.

Proposition 5.5. *The value*

$$\frac{\det J_f(P)}{\operatorname{tr} J_f(P) - T_0}$$

is a constant for all $P \notin L$.

Proof. We consider the conjugate of f found in Lemma 5.1,

$$g(x, y) = (ax + by + c,\ y^2 + kax + kby + kc).$$

The Jacobian of g is of the form

$$J_g = \begin{bmatrix} a & b \\ ka & 2y + kb \end{bmatrix}$$

so we get

$$\det J_g = 2ay \quad \text{and} \quad \operatorname{tr} J_g = 2y + a + kb.$$

Since y-coordinate of $P \notin L$ is nonzero, we get

$$\frac{\det J_g(P)}{\operatorname{tr} J_g(P) - T_0} = a$$

for all $P \notin L$. Since the determinant and the trace are preserved by the conjugacy action, the above values are same for f. □

5.1. If f is not singular

If $P_0 \neq P_1$, then P_1 can be the point in Proposition 5.5. So, the conjugate we found in Lemma 5.1 has coefficients

$$a = \frac{D}{T_1 - T_0},\ kb = T_0 - \frac{D}{T_1 - T_0}.$$

We will find the best conjugate of f by locating P_1 at a good spot.

Theorem 5.6. *Let f be a non-fibered planar quadratic polynomial map of topological degree* 2 *which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$ and that $D \neq 0$. Then we get a conjugate of f,*

$$h_1(x, y) = \left(px + qy + r,\ \frac{1}{2}y^2 + px + qy + r\right),$$

where

$$p = \frac{D}{T_1 - T_0},\ q = T_0 - p,\ \textit{and}\ \ r = T_1 - T_0.$$

Proof. We consider the conjugate of f found in Lemma 5.1. Using $\overline{\sigma} = [2kX; 2Y; Z]$ which fixes $\overline{f}(H \setminus I(\overline{f})) = [0; 1; 0]$, $\mathcal{I} = [1; 0; 0]$ and $P = [0; 0; 1]$, we get

$$g^{\sigma} = \left(ax + kby + 2kc,\ \frac{1}{2}y^2 + ax + kby + 2kc\right).$$

Since $a = \frac{D}{T_1 - T_0}$, $kb = T_0 - \frac{D}{T_1 - T_0}$ and $T_1 - T_0 = 2kc$, we get the desired result. □

By observing the representative in Theorem 5.6, we can easily check GIT-stability of f. In such sense, this representative is the simplest conjugate of f.

Corollary 5.7. *Let f be a non-fibered planar quadratic polynomial map of topological degree* 2 *which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$ and that $D \neq 0$. Then f is not stable but semistable.*

Proof. Since f is algebraically stable and is not fibered, f is at least semistable [13, Theorem 4.1]. So we only have to apply the Hilbert-Mumford Criterion [14] to show that f is not stable: we will find a 1-parameter subgroup $L_{r,s}(\alpha) = \{[\alpha^r; \alpha^{-r+s}; \alpha^{-s}] \,|\, r, s \in \mathbb{Z}^{\geq 0}\}$ of $\mathrm{PGL}_3(K)$ which only gives non-negative weight to f. Since the meromorphic extension of f is of the form

$$[pXZ + qYZ + rZ^2; \frac{1}{2}2Y^2 + pXZ + qYZ + rZ^2; Z^2],$$

we examine Table 2 to find $\mu(\overline{f},\, L_{r,\,s}) = 0$ when $r = s > 0$. □

TABLE 2. Exponent of α in algebraically stable and nonsingular $\overline{f}^{L_{r,s}}$

	X^2	Y^2	Z^2	XY	YZ	XZ
x-coordinate of $\overline{f}^{L_{r,s}}$	–	–	$r+2s$	–	$2r$	s
y-coordinate of $\overline{f}^{L_{r,s}}$	–	$r-s$	$-r+3s$	–	s	$-2r+2s$
z-coordinate of $\overline{f}^{L_{r,s}}$	–	–	s	–	–	–

5.2. If f is singular

Theorem 5.8. *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$ and that $D = 0$. Then we get a conjugate of f,*

$$h_0(x,y) = (px + qy,\ y^2 + px + qy),$$

where

$$p = \frac{\det J_f(P)}{\operatorname{tr} J_f(P) - T_0},\ q = T_0 - p,$$

and P is any point such that $\det J_f(P) \neq 0$.

Proof. If f is singular, then f should be conjugate to

$$g(x,y) = (ax + kby,\ y^2 + ax + kby).$$

Since $P_1 = (c_1, c_1) = (0,0) = P_0$. In this subcase, we have $D = 0$ and $T_0 = T_1 = a + kb$ so that we do not have enough information. However, if we pick a point P such that $\det J_f(P) \neq 0$. We can calculate the constant

$$\frac{\det J_f(P)}{\operatorname{tr} J_f(P) - T_0}.$$

□

Corollary 5.9. *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$ and that $D = 0$. Then P_0 is the unique fixed point if and only if $T_0 = T_1 = 1$.*

Proof. The y-coordinate of the other fixed point of f is $1 + \frac{q}{p-1}$. So, it should be 0 if and only if $1 - p = q = T_0 - p$. Since f is singular, it is equivalent to $T_0 = T_1 = 1$. □

Corollary 5.10. *Let f be a non-fibered planar quadratic polynomial map of topological degree 2 which is algebraically stable. Assume further that $\overline{f}$ has the unique indeterminacy point $\mathcal{I}$ and that $D = 0$. Then f is not stable but semistable.*

Proof. With the same reason with the nonsingular case, we only have to show that f is not stable using the Hilbert-Mumford Criterion [14]. Since the meromorphic extension of f is of the form

$$[pXZ + qYZ; \frac{1}{2}Y^2 + pXZ + qYZ; Z^2],$$

we use the same 1-parameter subgroup $L_{r,s}(\alpha) = \{[\alpha^r; \alpha^{-r+s}; \alpha^{-s}] \mid r, s \in \mathbb{Z}^{\geq 0}\}$ of $\mathrm{PGL}_3(K)$ to get $\mu(\overline{f}, L_{r,s}) = 0$ when $r = s > 0$. $\square$

Table 3. Exponent of α in algebraically stable and singular $\overline{f}^{L_{r,s}}$

	X^2	Y^2	Z^2	XY	YZ	XZ
x-coordinate of $\overline{f}^{L_{r,s}}$	–	–	–	–	$2r$	s
y-coordinate of $\overline{f}^{L_{r,s}}$	–	$r-s$	–	–	s	$-2r+2s$
z-coordinate of $\overline{f}^{L_{r,s}}$	–	–	s	–	–	–

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Misong Chang
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: rosesong1777@ssu.ac.kr

Sunyang Ko
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: sunyangko@ssu.ac.kr

Chong Gyu Lee
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: cglee@ssu.ac.kr

Sang-Min Lee
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: sangmin@ssu.ac.kr