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CHARACTERIZATION OF WEAKLY COFINITE LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M an arbitrary R-module and X a finite R-module. We prove a characterization for $\mathrm{H}^i_{\mathfrak{a}}(M)$ and $\mathrm{H}^i_{\mathfrak{a}}(X,M)$ to be $\mathfrak{a}$-weakly cofinite for all i, whenever one of the following cases holds: (a) $\operatorname{ara}(\mathfrak{a}) \leq 1$, (b) $\dim R/\mathfrak{a} \leq 1$ or (c) $\dim R \leq 2$. We also prove that, if M is a weakly Laskerian R-module, then $\mathrm{H}^i_{\mathfrak{a}}(X,M)$ is $\mathfrak{a}$-weakly cofinite for all i, whenever $\dim X \leq 2$ or $\dim M \leq 2$ (resp. $(R,\mathfrak{m})$ a local ring and $\dim X \leq 3$ or $\dim M \leq 3$). Let $d = \dim M < \infty$, we prove an equivalent condition for top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$ to be weakly Artinian.

1. Introduction

Throughout this paper R is a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to the ideal $\mathfrak{a}$ is defined as

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

Also, the i^{th} generalized local cohomology module

$$\mathrm{H}^i_{\mathfrak{a}}(X,M) = \varinjlim_n \operatorname{Ext}^i_R(X/\mathfrak{a}^n X, M)$$

for all R-module M and X was introduced by Herzog in [21]. Clearly it is a generalization of the ordinary local cohomology module.

We refer the reader to [10] for more details about the local cohomology.

In [17], Grothendieck conjectured that for any ideal $\mathfrak{a}$ of R and any finite R-module M, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite R-module for all i. Hartshorne [19] provided a counterexample to Grothendieck's conjecture. He defined an R-module M to be $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ are finite for all j and he asked:

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Question. Let M be a finite R-module and I be an ideal of R. When are $\mathrm{H}^i_{\mathfrak{a}}(M)$ $\mathfrak{a}$-cofinite for all $i \geq 0$?

With respect to this question, there are several papers devoted to it; for example see [3, 6–8, 13, 22, 23, 25, 26].

Recall that an R-module M is called *weakly Laskerian* if $\mathrm{Ass}_R(M/N)$ is a finite set for each submodule N of M. The class of weakly Laskerian modules was introduced in [15] by Divaani-Aazar and Mafi. They also, as a generalization of cofinite modules with respect to an ideal, in [16] defined an R-module M to be *weakly cofinite with respect to an ideal* $\mathfrak{a}$ *of* R or $\mathfrak{a}$-weakly cofinite if $\mathrm{Supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geq 0$. Recall also that an R-module M is said to be weakly Artinian if its injective envelope, can be written as $\mathrm{E}_R(M) = \oplus_{i=1}^k \mu^0(\mathfrak{m}_i, M)\, \mathrm{E}_R(R/\mathfrak{m}_i)$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ are maximal ideals of R (see [18, Definition 2.1]).

Recently many authors studied the weakly Laskerianness and weakly cofiniteness of local cohomology modules and answered to the Hartshorne's question in the class of weakly Laskerian modules in some cases (see [2, 3, 5, 9, 15, 16, 18, 27, 28]).

The purpose of this paper is to continue to study weakly cofiniteness of local cohomology and generalized local cohomology modules. In Section 2, we study relationship between weakly cofiniteness of local cohomology and generalized local cohomology modules. We prove that the weakly cofiniteness of these two kinds of local cohomolgy under the following assumptions depends to each other. More precisely, in Corollary 1.9, we show that if $\mathrm{ara}(\mathfrak{a}) \leq 1$, $\dim R/\mathfrak{a} \leq 1$ or $\dim R \leq 2$, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i if and only if $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ is $\mathfrak{a}$-weakly cofinite for every finite R-module X and all i. We also prove the following result as a generalization of [12, Theorem 1.3 and Corollary 5.3]:

Theorem 1.1 (Theorem 1.11 and Corollary 1.12)**.** *Let R be a Noetherian ring (resp. $(R, \mathfrak{m})$ be a Noetherian local ring), X a finite R-module, M a weakly Laskerian R-module and $\mathfrak{a}$ an ideal of R. Assume $\dim X \leq 2$ or $\dim M \leq 2$ (resp. $\dim X \leq 3$ or $\dim M \leq 3$). Then the R-modules $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ are $\mathfrak{a}$-weakly cofinite for all $i \geq 0$.*

In the end, we prove the following result concerning weakly Artinianness of the top local cohomology modules.

Theorem 1.2 (Theorem 1.13)**.** *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be two ideals of a Noetherian ring R and M an R-module with $d = \dim M < \infty$. Then the following conditions are equivalent*:

(i) *The R-module $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} \mathfrak{b}$ is weakly Laskerian*;

(ii) *The R-module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian.*

In particular, if there is an element $x \in \mathfrak{a}$ such that $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} x$ is weakly Laskerian, then the R-module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ will be an ideal of R. Also, for an ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. For any unexplained notation and terminology, we refer the reader to [10], [11] and [24].

1.1. The results

We begin the section with some preliminaries which are needed in the proof of the main results of this section.

Lemma 1.3. *Suppose $x \in \mathfrak{a}$ and $\operatorname{Supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$. If $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite, then M must also be $\mathfrak{a}$-weakly cofinite.*

Proof. The proof is similar to the proof of [25, Corollary 3.4]. □

Lemma 1.4. *Let X be a finite R-module, M be an arbitrary R-module. Then the following statements hold.*

(a) $\Gamma_{\mathfrak{a}}(X, M) \cong \operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(M))$.
(b) *If $\operatorname{Supp}_R(X) \cap \operatorname{Supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, then $\mathrm{H}^i_{\mathfrak{a}}(X, M) \cong \operatorname{Ext}^i_R(X, M)$ for all i.*

Proof. See [27, Lemma 2.5]. □

Lemma 1.5. *Let X be a finite R-module and M an arbitrary R-module. Let t be a non-negative integer such that $\operatorname{Ext}^i_R(X, M)$ is weakly Laskerian for all $0 \leq i \leq t$. Then for any finitely generated R-module L with $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(X)$, $\operatorname{Ext}^i_R(L, M)$ is weakly Laskerian for all integer $0 \leq i \leq t$.*

Proof. Use the method of proof of [13, Proposition 1]. □

The following theorem which is one of our main results is a generalization of [25, Theorem 7.10] to the class of weakly Laskerian and weakly cofinite modules.

Theorem 1.6. *Let R be a Noetherian ring with $\dim R \leq 2$. Let $\mathfrak{a}$ be a proper ideal and M an R-module. The following conditions are equivalent*:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-weakly cofinite for all i.*
(ii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all i.*
(iii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for $i \leq 2$.*

Proof. We need by [25, Proposition 3.9] just to show that (i) follows from (iii). Suppose that M satisfies (iii). If $\mathfrak{a}$ is nilpotent, then by Lemma 1.5, a module is $\mathfrak{a}$-weakly cofinite if and only if it is weakly Laskerian. If $\mathfrak{a}$ is non-nilpotent, take n such that $0 \underset{R}{:} \mathfrak{a}^n = \Gamma_{\mathfrak{a}}(R)$. There is $x \in \mathfrak{a}$ which is regular on $\overline{R} = R/\Gamma_{\mathfrak{a}}(R)$, and therefore $\dim \overline{R}/x\overline{R} \leq 1$. The module $\overline{M} = M/0 \underset{M}{:} \mathfrak{a}^n$ has a natural structure as a module over $\overline{R}$. As by Lemma 1.5 the R-module $0 \underset{M}{:} \mathfrak{a}^n$ is weakly Laskerian, $\overline{M}$ must also satisfy (iii). The exact sequence $0 \longrightarrow 0 \underset{M}{:} \mathfrak{a}^n \longrightarrow$

$M \longrightarrow \overline{M} \longrightarrow 0$ yields the exact sequence $0 \longrightarrow 0 \underset{M}{:} \mathfrak{a}^n \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(\overline{M}) \longrightarrow 0$ and isomorphisms $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(\overline{M})$ for $i \geq 1$. Thus replacing M by $\overline{M}$, we may assume that M is a module over $\overline{R}$. Let $L = \Gamma_{\mathfrak{a}}(N)$, where $N = 0 \underset{M}{:} x \subseteq M$. Since $0 \underset{L}{:} \mathfrak{a} = 0 \underset{M}{:} \mathfrak{a}$ is weakly Laskerian, [20, Proposition 4.5] implies that L is $\mathfrak{a}$-weakly cofinite and therefore satisfies (ii). From the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0$, we get that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ is weakly Laskerian. Hence $\mathrm{Ext}^1_R(R/\mathfrak{a}, N/L)$ is weakly Laskerian. By [25, Lemma 7.9], $\mathrm{Ext}^1_R(R/\mathfrak{a}, N/L) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(N/L))$. Also $\mathrm{H}^1_{\mathfrak{a}}(N) \cong \mathrm{H}^1_{\mathfrak{a}}(N/L)$ and so $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(N)$ is weakly Laskerian. Hence, by [20, Proposition 4.5] the module $\mathrm{H}^1_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-weakly cofinite. Since $\mathrm{H}^i_{\mathfrak{a}}(N) = 0$ for $i > 1$, [25, Proposition 3.9] implies that $N = 0 \underset{M}{:} x$ satisfies (ii). From the exactness of $0 \to N \to M \to xM \to 0$, we therefore get that $\mathrm{Ext}^1_R(R/\mathfrak{a}, xM)$ and $\mathrm{Ext}^2_R(R/\mathfrak{a}, xM)$ are weakly Laskerian. Hence from the exactness of $0 \to xM \to M \to M/xM \to 0$ we get that $\mathrm{Hom}_A(R/\mathfrak{a}, T)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, T)$, where $T = M/xM$, are weakly Laskerian modules. An argument similar to that one, we used to show that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-weakly cofinite for all i shows that $\mathrm{H}^i_{\mathfrak{a}}(T)$ is $\mathfrak{a}$-weakly cofinite for all i.

Consider the homomorphism $f = x1_M$, so $N = \mathrm{Ker}\, f$ and $T = \mathrm{Coker}\, f$. We have shown that $\mathrm{H}^i_{\mathfrak{a}}(\mathrm{Ker}\, f)$ and $\mathrm{H}^i_{\mathfrak{a}}(\mathrm{Coker}\, f)$ are $\mathfrak{a}$-weakly cofinite for each i. By [20, Proposition 4.5], the class of $\mathfrak{a}$-weakly cofinite modules, which are modules over $\bar{R}$ annihilated by x constitute a Serre subcategory of the category of R-modules. Hence it follows from [25, Corollary 3.2] that for all i the modules $\mathrm{Ker}\, \mathrm{H}^i_{\mathfrak{a}}(f)$ and $\mathrm{Coker}\, \mathrm{H}^i_{\mathfrak{a}}(f)$ belong to the same category. Since $x \in \mathfrak{a}$ our Lemma 1.3 implies that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i. □

Theorem 1.7. *Let M be an R-module and suppose one of the following cases holds*:

(a) $\mathrm{ara}(\mathfrak{a}) \leq 1$;
(b) $\dim R/\mathfrak{a} \leq 1$;
(c) $\dim R \leq 2$.

Then, $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-weakly cofinite for all i if and only if* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all i.*

Proof. The case (c) follows by Theorem 1.6.

In the cases (a) and (b), suppose $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i. It follows from [25, Proposition 3.9] that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i.

To prove the converse, we use induction on i. Let $i = 0$. From the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

we obtain that $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ are weakly Laskerian. Now, it follows by [20, Lemma 3.1] in the case (b) and [1, Theorem 2.5] in the case (a) that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite. It follows also that for all i the R-modules $\operatorname{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ are weakly Laskerian. Let $i > 0$ and the case $i-1$ is settled. Consider the exact sequence

$$(\star) \qquad 0 \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow E \longrightarrow N \longrightarrow 0,$$

in which E is an injective $\mathfrak{a}$-torsion free module. It is easy to see that $\mathrm{H}^i_{\mathfrak{a}}(E) = 0 = \operatorname{Ext}^i_R(R/\mathfrak{a}, E)$ for all $i \geq 0$. Now, using the exact sequence $(\star)$, we easily get the isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(N) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$$

and

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, N) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)).$$

Hence the hypothesis is satisfied by N. This completes the inductive step. $\square$

Theorem 1.8. *Let M be an R-module and suppose one of the following cases holds*:

(a) $\operatorname{ara}(\mathfrak{a}) \leq 1$;
(b) $\dim R/\mathfrak{a} \leq 1$;
(c) $\dim R \leq 2$.

Then, for any finite R-module X, $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$ if and only if $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geq 0$.

Proof. First suppose for any finite R-module X, $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$. Let $X = R$. Then it follows by Theorem 1.7, that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$.

To prove the converse we use the induction on n. Let $n = 0$. Then it follows by Lemma 1.4(a) that

$$\mathrm{H}^0_{\mathfrak{a}}(X, M) = \Gamma_{\mathfrak{a}}(X, M) \cong \operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(M)).$$

Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite by Theorem 1.7, so the assertion follows by [1, Corollary 2.11], [9, Corollary 3.7] and [20, Corollary 4.7]. Now, assume that $n > 0$ and that the claim holds for $n-1$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$, the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

yields that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is weakly Laskerian for all $i \geq 0$. Now, by applying the right derived functor of $\Gamma_{\mathfrak{a}}(X, -)$ to the same short exact sequence and using Lemma 1.4(b), we obtain the following long exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Ext}^i_R(X, \Gamma_{\mathfrak{a}}(M)) &\xrightarrow{f_i} \mathrm{H}^i_{\mathfrak{a}}(X, M) \xrightarrow{g_i} \mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M)) \\ &\xrightarrow{h_i} \operatorname{Ext}^{i+1}_R(X, \Gamma_{\mathfrak{a}}(M)) \xrightarrow{f_{i+1}} \mathrm{H}^{i+1}_{\mathfrak{a}}(X, M) \longrightarrow \cdots . \end{aligned}$$

It yields the following short exact sequences

$$0 \longrightarrow \operatorname{Ker} f_i \longrightarrow \operatorname{Ext}^i_R(X, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Im} f_i \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} f_i \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X, M) \longrightarrow \operatorname{Im} g_i \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} g_i \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ker} f_{i+1} \longrightarrow 0.$$

Since $\operatorname{Ext}^i_R(X, \Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$ by [1, Corollary 2.11], [9, Corollary 3.7] and [20, Corollary 4.7], it follows by definition that, $\mathrm{H}^n_{\mathfrak{a}}(X, M)$ is an $\mathfrak{a}$-weakly cofinite R-module if and only if $\mathrm{H}^n_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M))$ is an $\mathfrak{a}$-weakly cofinite R-module. Therefore it suffices to show that $\mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M))$ is an $\mathfrak{a}$-weakly cofinite R-module. To this end consider the exact sequence

$$(\dagger) \qquad 0 \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow E \longrightarrow L \longrightarrow 0,$$

in which E is an injective $\mathfrak{a}$-torsion free module. Since $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0 = \Gamma_{\mathfrak{a}}(E)$, we see that $\operatorname{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) = 0 = \operatorname{Hom}_R(R/\mathfrak{a}, E)$ and $\Gamma_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M)) = 0 = \Gamma_{\mathfrak{a}}(X, E)$ by Lemma 1.4(a). Applying the derived functors of $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ and $\Gamma_{\mathfrak{a}}(X, -)$ to the short exact sequence ($\dagger$) we obtain, for all $i > 0$, the isomorphisms

$$\mathrm{H}^{i-1}_{\mathfrak{a}}(X, L) \cong \mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M))$$

and

$$\operatorname{Ext}^{i-1}_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^i_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)).$$

From what has already been proved, we conclude that $\operatorname{Ext}^{i-1}_R(R/\mathfrak{a}, L)$ is weakly Laskerian for all $i > 0$. Hence $\mathrm{H}^{n-1}_{\mathfrak{a}}(X, L)$ is $\mathfrak{a}$-weakly cofinite by induction hypothesis, which yields that $\mathrm{H}^n_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-weakly cofinite. This completes the inductive step. □

The following corollary is our main result of this paper which is a characterization of weakly cofinite local cohomology and generalized local cohomology modules under the assumptions (a) $\operatorname{ara}(\mathfrak{a}) \leq 1$, (b) $\dim R/\mathfrak{a} \leq 1$ and (c) $\dim R \leq 2$.

Corollary 1.9. *Let M be an R-module and suppose one of the following cases holds*:

(a) $\operatorname{ara}(\mathfrak{a}) \leq 1$;
(b) $\dim R/\mathfrak{a} \leq 1$;
(c) $\dim R \leq 2$.

Then the following conditions are equivalent:

(i) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all i.*
(ii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for $i = 0, 1$ in the cases* (a) *and* (b) (*resp. for $i = 0, 1, 2$ in the case* (c));
(iii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-weakly cofinite for all i;*
(iv) $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ *is $\mathfrak{a}$-weakly cofinite for all i and for every finite R-module X;*
(v) $\operatorname{Ext}^i_R(X, M)$ *is weakly Laskerian for all i and for any finite R-module X with* $\operatorname{Supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$;

(vi) $\operatorname{Ext}^i_R(X,M)$ *is weakly Laskerian for all* i *and for some finite* R*-module* X *with* $\operatorname{Supp}_R(X)=\operatorname{V}(\mathfrak{a})$;

(vii) $\operatorname{Ext}^i_R(X,M)$ *is weakly Laskerian for* $i=0,1$ *in the cases* (a) *and* (b) (*resp. for* $i=0,1,2$ *in the case* (c) *and for any finite* R*-module* X *with* $\operatorname{Supp}_R(X)\subseteq\operatorname{V}(\mathfrak{a})$);

(viii) $\operatorname{Ext}^i_R(X,M)$ *is weakly Laskerian for* $i=0,1$ *in the cases* (a) *and* (b) (*resp. for* $i=0,1,2$ *in the case* (c) *and for some finite* R*-module* X *with* $\operatorname{Supp}_R(X)=\operatorname{V}(\mathfrak{a})$).

Proof. In order to prove (i)⇔(ii), use [1, Theorem 2.5] and [20, Lemma 3.1], in the cases (a) and (b) and use Theorem 1.6, in the case (c).

(i)⇔(iii) It follows by Theorem 1.7.

(i)⇔(iv) It follows by Theorem 1.8.

In order to prove (i)⇔(v) and (ii)⇔(vii) we use [1, Lemma 2.1].

(v)⇒(vi) and (vii)⇒(viii) These are trivial.

In order to prove (vi)⇒(v) and (viii)⇒(vii), we take $X:=R/\mathfrak{a}$ and let L be a finite R-module with $\operatorname{Supp}_R(L)\subseteq\operatorname{V}(\mathfrak{a})$. Then $\operatorname{Supp}_R(L)\subseteq\operatorname{Supp}_R(X)$. Now, the assertion follows by Lemma 1.5. □

Corollary 1.10. *Let* M *be a weakly Laskerian* R*-module,* X *a finite* R*-module and* $\mathfrak{a}$ *an ideal of* R*. Suppose one of the following cases holds*:

(a) $\operatorname{ara}(\mathfrak{a})\leq 1$;

(b) $\dim R/\mathfrak{a}\leq 1$;

(c) $\dim R\leq 2$.

Then $\operatorname{H}^i_{\mathfrak{a}}(M)$ *and* $\operatorname{H}^i_{\mathfrak{a}}(X,M)$ *are* $\mathfrak{a}$*-weakly cofinite for all* i.

Proof. It follows from Corollary 1.9. □

The following two results are generalizations of [12, Theorem 1.3 and Corollary 5.3]. Note that here M is a weakly Laskerian R-module, while in [12, Corollary 5.3], it is a finite R-module.

Theorem 1.11. *Let* R *be a Noetherian ring,* X *a finite module,* M *a weakly Laskerian* R*-module and* $\mathfrak{a}$ *an ideal of* R*. Assume* $\dim X\leq 2$ *or* $\dim M\leq 2$*. Then the* R*-modules* $\operatorname{H}^i_{\mathfrak{a}}(X,M)$ *are* $\mathfrak{a}$*-weakly cofinite for all* $i\geq 0$.

Proof. First suppose $\dim_R X\leq 2$. By applying the derived functor $\Gamma_{\mathfrak{a}}(-,M)$ to the short exact sequence

$$0\longrightarrow\Gamma_{\mathfrak{a}}(X)\longrightarrow X\longrightarrow X/\Gamma_{\mathfrak{a}}(X)\longrightarrow 0,$$

and using Lemma 1.4(b), we obtain the long exact sequence

$$\begin{aligned}\cdots\longrightarrow\operatorname{Ext}^{i-1}_R(\Gamma_{\mathfrak{a}}(X),M)&\xrightarrow{f_i}\operatorname{H}^i_{\mathfrak{a}}(X/\Gamma_{\mathfrak{a}}(X),M)\xrightarrow{g_i}\operatorname{H}^i_{\mathfrak{a}}(X,M)\\&\xrightarrow{h_i}\operatorname{Ext}^i_R(\Gamma_{\mathfrak{a}}(X),M)\longrightarrow\cdots,\end{aligned}$$

which yields the short exact sequences

$$0\longrightarrow\operatorname{Im}f_i\longrightarrow\operatorname{H}^i_{\mathfrak{a}}(X/\Gamma_{\mathfrak{a}}(X),M)\longrightarrow\operatorname{Im}g_i\longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} g_i \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X, M) \longrightarrow \operatorname{Im} h_i \longrightarrow 0.$$

The R-modules $\operatorname{Im} f_i$ and $\operatorname{Im} h_i$ are weakly Laskerian for all $i \geq 0$. So, by the above exact sequences, it follows that $\mathrm{H}^i_{\mathfrak{a}}(X/\Gamma_{\mathfrak{a}}(X), M)$ is $\mathfrak{a}$-weakly cofinite if and only if so is $\mathrm{H}^i_{\mathfrak{a}}(X, M)$. Hence, we may assume that $\Gamma_{\mathfrak{a}}(X) = 0$. Then by [10, Lemma 2.1.1], $\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(X)} \mathfrak{p}$. Thus, there exists an element $r \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(X)} \mathfrak{p}$. Now, the exact sequence $0 \to X \xrightarrow{r} X \to X/rX \to 0$ yields an exact sequence

$$\mathrm{H}^i_{\mathfrak{a}}(X/rX, M) \longrightarrow (0 :_{\mathrm{H}^i_{\mathfrak{a}}(X,M)} r) \longrightarrow 0.$$

Since $\dim_R(X/rX) \leq 1$, it is easy to see that $\dim_R(0 :_{\mathrm{H}^i_{\mathfrak{a}}(X,M)} r) \leq 1$. Now, since $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ is $\mathfrak{a}$-torsion and $r \in \mathfrak{a}$, it follows that

$$\dim_R(\mathrm{H}^i_{\mathfrak{a}}(X, M)) = \dim_R(0 :_{\mathrm{H}^i_{\mathfrak{a}}(X,M)} r) \leq 1$$

for all $i \geq 0$. So, by [4, Corollary 5.5], we get that $\mathrm{H}^i_{\mathfrak{a}}(X, M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$.

In the case $\dim_R M \leq 2$, applying the derived functor $\Gamma_{\mathfrak{a}}(X, -)$ to the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

and using Lemma 1.4(b), we obtain the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Ext}^i_R(X, \Gamma_{\mathfrak{a}}(M)) &\xrightarrow{u_i} \mathrm{H}^i_{\mathfrak{a}}(X, M) \xrightarrow{v_i} \mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M)) \\ &\xrightarrow{w_i} \operatorname{Ext}^{i+1}_R(X, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \cdots. \end{aligned}$$

It yields the short exact sequences

$$0 \longrightarrow \operatorname{Im} u_i \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X, M) \longrightarrow \operatorname{Im} v_i \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} v_i \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Im} w_i \longrightarrow 0.$$

The R-modules $\operatorname{Im} u_i$ and $\operatorname{Im} w_i$ are weakly Laskerian for all $i \geq 0$. So by the above exact sequence, we get that $\mathrm{H}^i_{\mathfrak{a}}(X, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-weakly cofinite if and only if so is $\mathrm{H}^i_{\mathfrak{a}}(X, M)$. Therefore we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Then similar to proof of [10, Lemma 2.1.1], $\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$. Hence, there exists an element $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$. Now, from the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get the exact sequence

$$\mathrm{H}^i_{\mathfrak{a}}(X, M/xM) \longrightarrow (0 :_{\mathrm{H}^{i+1}_{\mathfrak{a}}(X,M)} x) \longrightarrow 0.$$

Since $\dim_R(M/xM) \leq 1$, similar to the previous case we get

$$\dim_R(\mathrm{H}^i_{\mathfrak{a}}(X, M)) = \dim_R(0 :_{\mathrm{H}^i_{\mathfrak{a}}(X,M)} x) \leq 1$$

for all $i \geq 1$. Note that by Lemma 1.4(a),

$$\mathrm{H}^0_{\mathfrak{a}}(X, M) \cong \operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(M)) = \operatorname{Hom}_R(X, 0) = 0.$$

Hence, $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(X,M)) \leq 1$ for all $i \geq 0$. Therefore by [4, Corollary 5.5], it follows that $\mathrm{H}^i_{\mathfrak{a}}(X,M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$. This completes the proof. $\square$

Corollary 1.12. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, X a finite R-module, M a weakly Laskerian R-module and $\mathfrak{a}$ an ideal of R. Assume* $\dim X \leq 3$ *or* $\dim M \leq 3$. *Then the R-modules $\mathrm{H}^i_{\mathfrak{a}}(X,M)$ are $\mathfrak{a}$-weakly cofinite for all $i \geq 0$.*

Proof. Using Lemma [20, Lemma 2.5], the proof is exactly similar to the proof of [12, Corollary 5.3]. $\square$

Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an R-module with $d = \dim M < \infty$. It is proved in [18, Theorem 2.11] that the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian whenever M is a weakly Laskerian R-module. In the following theorem we characterize weakly Artinianness of top local cohomology module.

Theorem 1.13. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be two ideals of a Noetherian ring R and M an R-module with $d = \dim M < \infty$. Then the following conditions are equivalent*:

(i) *The R-module $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} \mathfrak{b}$ is weakly Laskerian*;

(ii) *The R-module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian.*

In particular if there is an element $x \in \mathfrak{a}$ such that $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} x$ is weakly Laskerian, then the R-module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian.

Proof. The conclusion (ii)⇒(i) is obviously true. To prove (i)⇒(ii), since every module is a direct limit of its finite submodules, so $M = \varinjlim_i M_i$, where each M_i is a finite submodule of M. Then $\mathrm{H}^d_{\mathfrak{a}}(M) = \varinjlim_i \mathrm{H}^d_{\mathfrak{a}}(M_i)$. It follows by [25, Proposition 5.1] that $\mathrm{H}^d_{\mathfrak{a}}(M_i)$ is Artinian for each i. Therefore

$$\mathrm{Supp}_R(0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} \mathfrak{b}) \subseteq \mathrm{Supp}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) \subseteq \bigcup_i \mathrm{Supp}_R(\mathrm{H}^d_{\mathfrak{a}}(M_i)) \subseteq \mathrm{Max}\, R.$$

Since $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} \mathfrak{b}$ is weakly Laskerian with support in $\mathrm{Max}\, R$, it follows from Remark [18, Lemma 2.3(b)(v)] that $0 :_{\mathrm{H}^d_{\mathfrak{a}}(M)} \mathfrak{b}$ is weakly Artinian. Since $\mathrm{H}^d_{\mathfrak{a}}(M)$ is $\mathfrak{b}$-torsion, therefore using [18, Lemma 2.8], the R-module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is weakly Artinian. $\square$

It is easy to see that $\mathrm{cd}(\mathfrak{a}, R) \leq \mathrm{ara}\, \mathfrak{a}$ but the equality is not true in general (see [14, Example 2.3]). We close this paper by offering a question and a problem for further research.

Question. Let R be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. Does Corollary 1.9 remain true, when we replace $\mathrm{ara}\, \mathfrak{a} \leq 1$ with $\mathrm{cd}(\mathfrak{a}, R) \leq 1$ in the case (a)?

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