Bull. Korean Math. Soc. **60** (2023), No. 3, pp. 593–610
https://doi.org/10.4134/BKMS.b210773
pISSN: 1015-8634 / eISSN: 2234-3016

SOME RESULTS ON MEROMORPHIC SOLUTIONS OF Q-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract. In view of Nevanlinna theory, we investigate the meromorphic solutions of q-difference differential equations and our results give the estimates about counting function and proximity function of meromorphic solutions to these equations. In addition, some interesting results are obtained for two general equations and a class of system of q-difference differential equations.

1. Introduction and main results

We assume that the readers are familiar with the fundamental results and standard notations of Nevanlinna theory (see [5,10,12,20]). Such as the Nevanlinna characteristic function $T(r,f)$, the proximity function $m(r,f)$ and the counting functions $N(r,f)$, $\overline{N}(r,f)$. A meromorphic function φ is said to be a small function of f if the Nevanlinna characteristic $T(r,\varphi)$ satisfies $T(r,\varphi) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$, $r \to \infty$, possibly outside a set E of r with finite logarithmic measure. In addition,

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)} \quad \text{and} \quad \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

Complex differential equations and complex difference equations play an important role in the complex analysis. In recent decades, the Nevanlinna theory involving q-difference has been developed to study q-difference equations and q-difference polynomials. Many papers have focused on complex difference, giving many difference analogues in value distribution theory of meromorphic functions (see [2,3,7–9,15,16,18,19]).

In 1993, the following theorem was given in [10].

Received October 24, 2021; Revised October 30, 2022; Accepted February 17, 2023.
2010 *Mathematics Subject Classification.* Primary 30D35.
Key words and phrases. q-difference differential equation, meromorphic solution, transcendental, zero order.
This work was partially supported by National Natural Science Foundation of China (NSFC)(NO.12171127).

Theorem A ([10])**.** *Let*

$$\sum a_i(z) f^{i_0}(f')^{i_1}\cdots(f^{(n)})^{i_n} = R(z,f), \tag{1.1}$$

where $R(z,f)$ *is defined as* $R(z,f) = \frac{\sum_{i=0}^{k} a_i(z)f^i(z)}{\sum_{j=0}^{l} b_j(z)f^j(z)}$. *If equation* (1.1) *has a transcendental meromorphic solution, then* $l=0$ *and* $k \le \min\{\Delta, \lambda+\mu(1-\Theta(\infty))\}$*, where*

$$\Delta = \max\{\sum_{\alpha=0}^{n}(1+\alpha)i_\alpha\},\ \lambda = \max\{\sum_{\alpha=0}^{n} i_\alpha\},\ \mu = \max\{\sum_{\alpha=0}^{n}\alpha i_\alpha\}$$

and $\Theta(\infty) = 1 - \limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}$.

In 2010, Zheng and Chen [22] considered the growth of meromorphic solutions of

$$\sum_{j=1}^{n} a_j(z) f(q^j z) = R(z,f) = \frac{P(z,f)}{Q(z,f)},$$

and obtained some results (see, e.g., Theorem 2 in [22]). Later on, Wang et al. [16] investigated the existence of solutions of non-linear q-difference differential equation of the form

$$\sum \alpha_i(z) f^{i_0}(f'(q_1 z))^{i_1}\cdots(f^{(n)}(q_n z))^{i_n} = R(z,f). \tag{1.2}$$

They showed the following Theorem B.

Theorem B ([16])**.** *If equation* (1.2) *has a transcendental meromorphic solution with zero order, then* $l=0$ *and* $k \le \min\{\Delta, \lambda+\mu(1-\Theta(\infty))\}$.

Recently, Laine and Latreuch [11] considered meromorphic solutions of delay-differential equations

$$L(z,f) := \sum_{j=1}^{n} \beta_j(z) f^{(k_j)}(z+c_j) = \frac{P(z,f)}{Q(z,f)} \tag{1.3}$$

and

$$M(z,f) := \prod_{j=1}^{n} f^{(k_j)}(z+c_j) = \frac{P(z,f)}{Q(z,f)}.$$

A result below was given by Laine and Latreuch in [11].

Theorem C ([11])**.** *Suppose* f *is a transcendental meromorphic solution of hyper-order* <1 *to equation* (1.3)*. Then*

$$d = \max\{p,q\} \le 1 + (n-1)(1+\delta(\infty,f)) + K(1-\Theta(\infty,f)),$$

where $K := \sum_{j=1}^{n} k_j$*. Furthermore, the following assertions hold*:

(1) *If* $d \ge 2$ *or* $d = q = 1$*, then* $\lambda_2\left(\frac{1}{f}\right) = \rho_2(f)$.

(2) *If* $\varphi(z)$ *is a small function of* f, *not being a solution of* (1.3), *then* $\lambda_2(f-\varphi)=\rho_2(f)$. *In particular, if* $a_0(z)\not\equiv 0$, *then* $\lambda_2(f)=\rho_2(f)$.

Inspired by Laine and Latreuch [11], Zheng and Chen [22], and the references therein, we will investigate two types of q-difference differential equations

$$\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) = \frac{P(z,f)}{Q(z,f)} \tag{1.4}$$

and

$$\Phi(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) = \frac{P(z,f)}{Q(z,f)}, \tag{1.5}$$

where

$$\frac{P(z,f)}{Q(z,f)} := \frac{\alpha_0(z)+\alpha_1(z)f+\cdots+\alpha_m(z)f^m}{\beta_0(z)+\beta_1(z)f+\cdots+\beta_n(z)f^n}$$

with meromorphic coefficients $\alpha_\mu(z)$ $(\mu=0,1,\ldots,m)$, $\beta_\nu(z)$ $(\nu=0,1,\ldots,n)$, which are small functions of f.

For the convenience of readers, we give some notations below. We set

$$\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) := \sum_{j=1}^{s}\eta_j(z)\prod_{i=1}^{t} f_{q_i}^{\alpha^j_{i,0}}(f'_{q_i})^{\alpha^j_{i,1}}\cdots(f_{q_i}^{(k_i)})^{\alpha^j_{i,k_i}} \tag{1.6}$$

and

$$\Phi(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) := \frac{\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}{\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}, \tag{1.7}$$

where $\eta_j(z)$ are small functions of f, $f_{q_i}\equiv f(q_i z)$ $(i=1,2,\ldots,t)$, $|q_i|\neq 0$ and k_i, $\alpha^j_{i,l}$ $(l=0,1,\ldots,k_i)$ are non-negative integers.

We also need to fix a collection of notations. The quantities

$$\gamma_i(\Omega) := \max_{1\le j\le s}\sum_{l=0}^{k_i}\alpha^j_{i,l},\ \ \Gamma_i(\Omega) := \max_{1\le j\le s}\sum_{l=1}^{k_i} l\alpha^j_{i,l},\ \ \Delta_i(\Omega) := \max_{1\le j\le s}\sum_{l=0}^{k_i}(l+1)\alpha^j_{i,l}$$

are called, respectively, the degree $\gamma_i(\Omega)$, the weight $\Gamma_i(\Omega)$ and the hyper-weight $\Delta_i(\Omega)$ of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ with respect to f_{q_i}. One can observe that

$$\max\{\gamma_i(\Omega), \Gamma_i(\Omega)\} \le \Delta_i(\Omega) \le \gamma_i(\Omega)+\Gamma_i(\Omega).$$

Moreover, the total degree, weight and hyper-weight of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ are defined as

$$\gamma_\Omega := \sum_{i=1}^{t}\gamma_i(\Omega),\ \ \Gamma_\Omega := \sum_{i=1}^{t}\Gamma_i(\Omega),\ \ \Delta_\Omega := \sum_{i=1}^{t}\Delta_i(\Omega), \tag{1.8}$$

and the classical total degree, weight and hyper-weight of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ are usually defined as:

$$\gamma'_\Omega := \max_{1\le j\le s}\{\sum_{i=1}^{t}\sum_{l=0}^{k_i}\alpha_{i,l}^{j}\},$$

$$\Gamma'_\Omega := \max_{1\le j\le s}\{\sum_{i=1}^{t}\sum_{l=1}^{k_i} l\alpha_{i,l}^{j}\},$$

$$\Delta'_\Omega := \max_{1\le j\le s}\{\sum_{i=1}^{t}\sum_{l=0}^{k_i}(l+1)\alpha_{i,l}^{j}\},$$

respectively. Clearly, $\gamma'_\Omega \le \gamma_\Omega$, $\Gamma'_\Omega \le \Gamma_\Omega$ and $\Delta'_\Omega \le \Delta_\Omega$.

Similarly, the total degree, weight and hyper-weight of $\Phi(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ are defined as

$$\gamma_\Phi := \sum_{i=1}^{t}\gamma_i(\Phi),\ \ \Gamma_\Phi := \sum_{i=1}^{t}\Gamma_i(\Phi),\ \ \Delta_\Phi := \sum_{i=1}^{t}\Delta_i(\Phi),$$

where

$$\gamma_i(\Phi) := \max\{\gamma_i(\Omega_1), \gamma_i(\Omega_2)\},$$
$$\Gamma_i(\Phi) := \max\{\Gamma_i(\Omega_1), \Gamma_i(\Omega_2)\},$$
$$\Delta_i(\Phi) := \max\{\Delta_i(\Omega_1), \Delta_i(\Omega_2)\}.$$

We can now state our main results.

Theorem 1.1. *Suppose that f is a transcendental meromorphic solution of zero order of equation* (1.4). *Then*

$$d = \max\{m, n\} \le \gamma'_\Omega + (\gamma_\Omega - \gamma'_\Omega)(1-\delta(\infty, f)) + \Gamma_\Omega(1-\Theta(\infty, f)). \tag{1.9}$$

Furthermore, the following assertions hold:

(1) *If $d \ge \gamma_\Omega$, then f has infinity many poles.*
(2) *If $n \ge m$, then $d = n \le \gamma_\Omega(1-\delta(\infty, f)) + \Gamma_\Omega(1-\Theta(\infty, f))$.*

Example 1.2 shows that the condition of zero order can not be removed. Example 1.3 is given to justify the validity of inequality (1.9).

Example 1.2. The meromorphic function $f(z) = \tan z$ with order 1 solves

$$\Omega(z, f(2z), f(3z)) : = \tan' 2z + \tan 3z = \frac{-f^7 - 6f^6 + 5f^5 - 10f^4 - 7f^3 - 2f^2 + 3f + 2}{-3f^6 + 7f^4 - 5f^2 + 1}.$$

Here, $d = 7$, $\gamma_\Omega = \gamma'_\Omega = 2$, $\Gamma_\Omega = 1$ and $\Theta(\infty, f) = 0$. We observe that the inequality (1.9) fails.

Let $q \in \mathbb{C}$ such that $0 < |q| < 1$. Define

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{1.10}$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$. (1.10) is a meromorphic function with poles at $x = -n \pm 2\pi ik/\log q$, where k and n are non-negative integers [1].

Example 1.3. Let $f(z) := (1-q)^{x-1}\Gamma_q(x)$, $z = q^x$ and $f(0) := (q,q)_\infty$. Then $f(z)$ is a meromorphic function of zero order with no zeros, having its poles at $\{q^{-k}\}_{k=0}^{\infty}$. $f(z)$ is a solution of

$$\frac{f(qz)f(q^2z)}{(1-qz)(1-z)^2} = f^2.$$

Here, $d = 2$, $\gamma_\Omega = \gamma'_\Omega = 2$ and $\Gamma_\Omega = 0$.

Theorem 1.4. *Suppose that f is a transcendental meromorphic solution of zero order of equation* (1.5). *Then*

$$d = \max\{m,n\} \le \min\{\gamma_\Phi + \Gamma_\Phi(1-\Theta(\infty,f)),\ \Delta_\Phi\}. \tag{1.11}$$

Moreover, if $\Omega_1(z,0,0,\ldots,0)\beta_0(z) - \Omega_2(z,0,0,\ldots,0)\alpha_0(z) \not\equiv 0$, then $\delta(0,f) = 0$.

We give an example below to show that the inequality (1.11) can occur.

Example 1.5. Let $f(z) := (1-q)^{x-1}\Gamma_q(x)$, $z = q^x$ and $f(0) := (q,q)_\infty$. Then $f(z)$ is a solution of

$$\frac{f^2(qz) + f(q^2z)}{f(qz) + f^2(q^2z)} = \frac{(1-z)f + (1-qz)}{1 + (1-qz)^2(1-z)f}.$$

Here, $d = 1$, $\gamma_\Phi = 4$, $\Gamma_\Phi(1-\Theta(\infty,f)) = 0$, $\Delta_\Phi = 4$.

Theorem 1.6. *If equation* (1.5) *admits a transcendental meromorphic solution of zero order with $N(r,f) = S(r,f)$, then*

$$m \le \gamma'_{\Omega_1} \quad \text{and} \quad n \le \gamma'_{\Omega_2}.$$

We proceed to consider a more general equation of form

$$\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) + \omega(z)\left(\frac{f'(z)}{f(z)}\right)^{s_1} = \frac{P(z,f)}{Q(z,f)}, \tag{1.12}$$

where s_1 is a positive integer, $\omega(z)$ is a small function of f. We prove the following theorem.

Theorem 1.7. *If equation* (1.12) *admits a transcendental meromorphic solution of zero order, then*

$$d = \max\{m,n\} \le \min\{\gamma_\Omega + \Gamma_\Omega(1-\Theta(\infty,f)), \Delta_\Omega\} + 2s_1 - s_1\Theta(\infty,f) - s_1\Theta(0,f).$$

Furthermore, the following assertions hold:

(1) *If $d = \Delta_\Omega + 2s_1$, then*

$$T(r,f) = \overline{N}(r,f) + S(r,f) = \overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

(2) *If* $Q(z,0)\not\equiv 0$, *then*

$$d\leq \min\{\gamma_\Omega+\Gamma_\Omega(1-\Theta(\infty,f)),\Delta_\Omega\}+s_1(1-\Theta(\infty,f)).$$

For some related results, we refer the reader to [11, 17] and the reference therein. Example 1.8 is given to illustrate the validity of the proposed results in Theorem 1.7.

Example 1.8. Let $f(z):=(1-q)^{x-1}\Gamma_q(x)$, $z=q^x$ and $f(0):=(q,q)_\infty$. Then $f(z)$ is a solution of

$$f(qz)f(q^2z)+\frac{f'(z)}{f(z)}=(1-qz)(1-z)^2f^2-\frac{(z,q)'_\infty}{(q,q)_\infty}f.$$

Here, $d=2$, $s_1=1$, $\gamma_\Omega=\Delta_\Omega=2$, $\Gamma_\Omega=0$, $\Theta(\infty,f)=0$ and $\Theta(0,f)=1$.

Some articles [4,11,14,15] focused on Malmquist-type systems of complex differential, difference and delay-differential equations. We obtain a result about the following system:

$$\begin{cases}\Phi_1(z,g_{q_1},g_{q_2},\ldots,g_{q_t})=R_1(z,f),\\ \Phi_2(z,f_{q_1},f_{q_2},\ldots,f_{q_t})=R_2(z,g),\end{cases}\tag{1.13}$$

where $\Phi_1(z,g_{q_1},g_{q_2},\ldots,g_{q_t})$ and $\Phi_2(z,f_{q_1},f_{q_2},\ldots,f_{q_t})$ are defined as in (1.7). $R_1(z,f):=\frac{a_0(z)+a_1(z)f+\cdots+a_{m_1}(z)f^{m_1}}{b_0(z)+b_1(z)f+\cdots+b_{n_1}(z)f^{n_1}}$ and $R_2(z,g):=\frac{c_0(z)+c_1(z)g+\cdots+c_{m_2}(z)g^{m_2}}{e_0(z)+e_1(z)g+\cdots+e_{n_2}(z)g^{n_2}}$ are irreducible rational functions of degrees $d_1=\max\{m_1,n_1\}$ and $d_2=\max\{m_2,n_2\}$ in f and g, respectively.

Theorem 1.9. *Suppose that* (f,g) *is a transcendental meromorphic solution of zero order of equation* (1.13). *Then*

$$d_1d_2\leq \Delta_{\Phi_1}\Delta_{\Phi_2}.$$

Moreover, if $\overline{N}(r,f)=S(r,f)$, *then*

$$d_1d_2\leq \gamma_{\Phi_1}\gamma_{\Phi_2}.$$

Example 1.10 is given to illustrate the validity of the proposed results in Theorem 1.9.

Example 1.10. As same as Example 1.3, let $f(z):=(1-q)^{x-1}\Gamma_q(x)$, $z=q^x$, $f(0):=(q,q)_\infty$, and $g(z):=(1-z)f(z)$. Then $(f(z),g(z))$ is a solution of

$$\begin{cases}\frac{f^2(qz)+f(q^2z)}{f(qz)+f^2(q^2z)}=\frac{g+(1-qz)}{1+(1-qz)^2g};\\ \frac{g^2(qz)+g(q^2z)}{g^2(qz)-g(q^2z)}=\frac{(1-qz)(1-z)f+(1-q^2z)}{(1-qz)(1-z)f-(1-q^2z)}.\end{cases}$$

Here, $d_1=1$, $d_2=1$, $\Delta_{\Phi_1}=4$, $\Delta_{\Phi_2}=3$.

2. Some lemmas

Lemma 2.1 ([10]). *Let f be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f)=\frac{P(z,f)}{Q(z,f)}:=\frac{a_0(z)+a_1(z)f+\cdots+a_m(z)f^m}{b_0(z)+b_1(z)f+\cdots+b_n(z)f^n}$$

with meromorphic coefficients $a_i(z)$ $(i=0,1,\ldots,m)$, $b_j(z)$ $(j=0,1,\ldots,n)$, such that

$$\begin{cases}T(r,a_i(z))=S(r,f),\ i=0,\ldots,m;\\ T(r,b_j(z))=S(r,f),\ j=0,\ldots,n,\end{cases}$$

the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f))=dT(r,f)+S(r,f),$$

where $d=\max\{m,n\}$.

Lemma 2.2 ([2]). *Let $f(z)$ be a non-constant meromorphic function with zero order and $q\in\mathbb{C}\backslash\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right)=S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.3 ([21]). *Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero constant. Then*

$$T(r,f(qz))=(1+o(1))T(r,f)$$

and

$$N(r,f(qz))=(1+o(1))N(r,f)$$

on a set of logarithmic density 1.

Lemma 2.4. *Let f be a non-constant meromorphic solution of zero order of equation*

$$\Omega(z,f_{q_1},f_{q_2},\ldots,f_{q_t}):=\sum_{j=1}^{s}\eta_j(z)\prod_{i=1}^{t}f_{q_i}^{\alpha^j_{i,0}}(f'_{q_i})^{\alpha^j_{i,1}}\cdots(f_{q_i}^{(k_i)})^{\alpha^j_{i,k_i}}=0,$$

where $|q_i|\neq 0$ are distinct complex numbers, $\alpha^j_{i,l}$ $(l=0,1,\ldots,k_i)$ are non-negative integers and the coefficients $\eta_j(z)$ are small functions of f. If $\Omega(z,a)\not\equiv 0$ for some small meromorphic function $a(z)$, then

$$m\left(r,\frac{1}{f-a}\right)=S(r,f).$$

Proof of Lemma 2.4. Since $g:=f-a$ satisfies a similar q-difference differential equation. Without loss of generality, we assume that $a(z)\equiv 0$. Rewrite $\Omega(z,f_{q_1},f_{q_2},\ldots,f_{q_t})$ as

$$\Omega(z,f_{q_1},f_{q_2},\ldots,f_{q_t})=G(z)+\widetilde{\Omega}(z,f_{q_1},f_{q_2},\ldots,f_{q_t})=0,$$

where $G(z) := \Omega(z, 0, 0, \ldots, 0) \not\equiv 0$ and

$$\widetilde{\Omega}(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) = \sum_{j=1}^{s} \widetilde{\eta_j}(z) \prod_{i=1}^{t} f_{q_i}^{\widetilde{\alpha}_{i,0}^j} (f'_{q_i})^{\widetilde{\alpha}_{i,1}^j} \cdots (f_{q_i}^{(\widetilde{k_i})})^{\widetilde{\alpha}_{i,k_i}^j}$$

with coefficients $\widetilde{\eta_j}(z)$ are small functions of f.

On the one hand, if $|f(z)| > 1$, we have

$$\frac{1}{2\pi} \int_{\{z, |f(z)|<1\} \cap \{z, |z|=r\}} \log^+ \frac{1}{|f|} d\theta = 0.$$

On the other hand, if $|f(z)| \leq 1$, then

$$\frac{1}{|f|} \leq \frac{1}{|f|^{\alpha_{1,0}^j + \cdots + \alpha_{1,k_1}^j + \cdots + \alpha_{t,0}^j + \cdots + \alpha_{t,k_t}^j}}.$$

Hence, we obtain

$$\frac{1}{|f|} \left| \prod_{i=1}^{t} f_{q_i}^{\widetilde{\alpha}_{i,0}^j} (f'_{q_i})^{\widetilde{\alpha}_{i,1}^j} \cdots (f_{q_i}^{(\widetilde{k_i})})^{\widetilde{\alpha}_{i,\widetilde{k_i}}^j} \right| \leq \prod_{i=1}^{t} \left| \frac{f_{q_i}}{f} \right|^{\widetilde{\alpha}_{i,0}^j} \left| \frac{f'_{q_i}}{f} \right|^{\widetilde{\alpha}_{i,1}^j} \cdots \left| \frac{f_{q_i}^{(\widetilde{k_i})}}{f} \right|^{\widetilde{\alpha}_{i,\widetilde{k_i}}^j}.$$

Applying the lemma of logarithmic derivative and Lemma 2.2, we get that $m(r, \frac{\widetilde{\Omega}}{f}) = S(r, f)$. Thus,

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{G}{f} \cdot \frac{1}{G}\right) \leq m\left(r, \frac{G}{f}\right) + m\left(r, \frac{1}{G}\right) \\ &= m\left(r, \frac{\widetilde{\Omega}}{f}\right) + m\left(r, \frac{1}{G}\right) = S(r, f). \end{aligned}$$

We finished the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

In order to give the estimation of the Nevanlinna characteristics function of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$.

Firstly, we estimate the proximity function of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$.

$$\begin{aligned} &|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})| \\ &= \left| \sum_{j=1}^{s} \eta_j(z) \prod_{i=1}^{t} f_{q_i}^{\alpha_{i,0}^j} (f'_{q_i})^{\alpha_{i,1}^j} \cdots (f_{q_i}^{(k_i)})^{\alpha_{i,k_i}^j} \right| \\ &\leq \sum_{j=1}^{s} |\eta_j(z)| \prod_{i=1}^{t} |f(z)|^{\sum_{l=0}^{k_i} \alpha_{i,l}^j} \left| \frac{f_{q_i}}{f} \right|^{\alpha_{i,0}^j} \left| \frac{f'_{q_i}}{f} \right|^{\alpha_{i,1}^j} \cdots \left| \frac{f_{q_i}^{(k_i)}}{f} \right|^{\alpha_{i,k_i}^j}. \end{aligned}$$

Define $D_1 = \{z, |f(z)| < 1\} \cap \{z, |z| = r\}$ and $D_2 = \{z, |z| = r\} \backslash D_1$.

If $z \in D_1$, then we have

$$|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})|$$

$$\leq \sum_{j=1}^{s}|\eta_j(z)|\prod_{i=1}^{t}|f(z)|^{\sum_{l=0}^{k_i}\alpha_{i,l}^j}\left|\frac{f_{q_i}}{f}\right|^{\alpha_{i,0}^j}\left|\frac{f'_{q_i}}{f}\right|^{\alpha_{i,1}^j}\cdots\left|\frac{f_{q_i}^{(k_i)}}{f}\right|^{\alpha_{i,k_i}^j}$$
$$\leq \sum_{j=1}^{s}|\eta_j(z)|\prod_{i=1}^{t}\left|\frac{f_{q_i}}{f}\right|^{\alpha_{i,0}^j}\left|\frac{f'_{q_i}}{f}\right|^{\alpha_{i,1}^j}\cdots\left|\frac{f_{q_i}^{(k_i)}}{f}\right|^{\alpha_{i,k_i}^j}.$$

If $z \in D_2$, then we have

$$\begin{aligned}
&|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})| \\
&\leq \sum_{j=1}^{s}|\eta_j(z)|\prod_{i=1}^{t}|f(z)|^{\sum_{l=0}^{k_i}\alpha_{i,l}^j}\left|\frac{f_{q_i}}{f}\right|^{\alpha_{i,0}^j}\left|\frac{f'_{q_i}}{f}\right|^{\alpha_{i,1}^j}\cdots\left|\frac{f_{q_i}^{(k_i)}}{f}\right|^{\alpha_{i,k_i}^j} \\
&\leq |f(z)|^{\gamma'_\Omega}\sum_{j=1}^{s}|\eta_j(z)|\prod_{i=1}^{t}\left|\frac{f_{q_i}}{f}\right|^{\alpha_{i,0}^j}\left|\frac{f'_{q_i}}{f}\right|^{\alpha_{i,1}^j}\cdots\left|\frac{f_{q_i}^{(k_i)}}{f}\right|^{\alpha_{i,k_i}^j}.
\end{aligned}$$

By using the lemma of logarithmic derivative and Lemma 2.2, we obtain

$$\begin{aligned}
m(r,\Omega) &= \frac{1}{2\pi}\int_{D_1}\log^+|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})|d\theta \\
&\quad+\frac{1}{2\pi}\int_{D_2}\log^+|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})|d\theta \\
&\leq \gamma'_\Omega m(r,f) + S(r,f).
\end{aligned} \tag{3.1}$$

Secondly, to estimate $N(r,\Omega)$, we consider

$$\Omega_{(j)}(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) = \eta_j(z)\prod_{i=1}^{t}f_{q_i}^{\alpha_{i,0}^j}(f'_{q_i})^{\alpha_{i,1}^j}\cdots(f_{q_i}^{(k_i)})^{\alpha_{i,k_i}^j},$$

then we get that

$$\begin{aligned}
n(r,\Omega_{(j)}) &\leq n(r,\eta_j) + \alpha_{i,0}^j n(r,f_{q_i}) + \cdots + \alpha_{i,k_i}^j n(r,f_{q_i}^{(k_i)}) \\
&\leq n(r,\eta_j) + \sum_{l=0}^{k_i}\alpha_{i,l}^j n(r,f_{q_i}) + \cdots + \sum_{l=1}^{k_i} l\alpha_{i,l}^j \overline{n}(r,f_{q_i}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
n(r,\Omega) &\leq \sum_{j=1}^{s}n(r,\eta_j) + \sum_{i=1}^{t}\max_{1\leq j\leq s}\left\{\sum_{l=0}^{k_i}\alpha_{i,l}^j n(r,f_{q_i})\right\} \\
&\quad+\sum_{i=1}^{t}\max_{1\leq j\leq s}\left\{\sum_{l=1}^{k_i}l\alpha_{i,l}^j\overline{n}(r,f_{q_i})\right\} \\
&\leq \sum_{j=1}^{s}n(r,\eta_j) + \gamma_\Omega n(r,f_{q_i}) + \Gamma_\Omega\overline{n}(r,f_{q_i}),
\end{aligned} \tag{3.2}$$

where $n(r,\varphi)$ stands for the order of the poles of φ and $\overline{n}(r,\varphi)$ stands for the order of the poles of φ counted only once. Then, from (3.2) and η_j are small functions of f, applying Lemma 2.3, we obtain

$$N(r,\Omega) \le S(r,f) + \gamma_\Omega N(r,f) + \Gamma_\Omega \overline{N}(r,f). \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$\begin{aligned} T(r,\Omega) &\le \gamma'_\Omega m(r,f) + \gamma_\Omega N(r,f) + \Gamma_\Omega \overline{N}(r,f) + S(r,f) \\ &\le \gamma'_\Omega T(r,f) + (\gamma_\Omega - \gamma'_\Omega)N(r,f) + \Gamma_\Omega \overline{N}(r,f) + S(r,f). \end{aligned} \tag{3.4}$$

Taking the Nevanlinna characteristic function of both sides of equation (1.4) and applying Lemma 2.1, we obtain

$$\begin{aligned} dT(r,f) + S(r,f) = T\left(r, \frac{P(z,f)}{Q(z,f)}\right) & \\ = T(r, \Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})) & \\ \le \gamma'_\Omega T(r,f) + (\gamma_\Omega - \gamma'_\Omega)N(r,f) & \\ + \Gamma_\Omega \overline{N}(r,f) + S(r,f), & \end{aligned} \tag{3.5}$$

which gives that

$$d = \max\{m,n\} \le \gamma'_\Omega + (\gamma_\Omega - \gamma'_\Omega)(1 - \delta(\infty,f)) + \Gamma_\Omega(1 - \Theta(\infty,f)).$$

If $d \ge \gamma'_\Omega$, it follows from (3.5) that

$$(d - \gamma'_\Omega)T(r,f) + S(r,f) \le (\gamma_\Omega - \gamma'_\Omega)N(r,f) + \Gamma_\Omega \overline{N}(r,f) + S(r,f), \tag{3.6}$$

which implies that f has infinity poles.

If $n \ge m$, we may rewrite $Q(z,f)$ as

$$Q(z,f) = \beta_n f^n \left(\frac{\beta_0}{\beta_n f^n} + \frac{\beta_1}{\beta_n f^{n-1}} + \cdots + \frac{\beta_{n-1}}{\beta_n f} + 1\right).$$

Note that $T(r,\beta_\nu) = S(r,f)$, $\nu = 0,1,\ldots,n$. As well as, $T\left(r, \frac{1}{\beta_\nu}\right) = S(r,f)$. Set

$$|\beta(z)| := \max_{1 \le \nu \le n} \left\{1, 2\left|\frac{\beta_{n-\nu}}{\beta_n}\right|^{\frac{1}{\nu}}\right\}. \tag{3.7}$$

We compute it is a proximity function, then we have

$$m(r,\beta) \le \sum_{\nu=0}^{n} m(r,\beta_\nu) + m(r, \frac{1}{\beta_\nu}) + O(1) = S(r,f). \tag{3.8}$$

Divide the circle $|z| = r$ into two parts:

$$E_1 = \{\theta \in [0,2\pi] : |f(re^{i\theta})| \le |\beta(re^{i\theta})|\} \text{ and } E_2 = [0,2\pi] \backslash E_1.$$

Firstly, we consider $z = re^{i\theta}$, here $\theta \in E_1$. We have

$$|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})|$$

$$
\begin{aligned}
&= \left| \sum_{j=1}^{s} \eta_j(z) \prod_{i=1}^{t} f_{q_i}^{\alpha_{i,0}^j} (f_{q_i}^{'})^{\alpha_{i,1}^j} \cdots (f_{q_i}^{(k_i)})^{\alpha_{i,k_i}^j} \right| \\
&\leq \sum_{j=1}^{s} |\eta_j(z)| \prod_{i=1}^{t} |f_{q_i}|^{\alpha_{i,0}^j} |f_{q_i}^{'}|^{\alpha_{i,1}^j} \cdots |f_{q_i}^{(k_i)}|^{\alpha_{i,k_i}^j} \\
&\leq \sum_{j=1}^{s} |\eta_j(z)| \prod_{i=1}^{t} |f|^{\sum_{l=0}^{k_i} \alpha_{i,l}^j} \left| \frac{f_{q_i}}{f} \right|^{\alpha_{i,0}^j} \left| \frac{f_{q_i}^{'}}{f} \right|^{\alpha_{i,1}^j} \cdots \left| \frac{f_{q_i}^{(k_i)}}{f} \right|^{\alpha_{i,k_i}^j} \\
&\leq (2|\beta(z)|)^{\gamma'(\Omega)} \sum_{j=1}^{s} |\eta_j(z)| \prod_{i=1}^{t} \left| \frac{f_{q_i}}{f} \right|^{\alpha_{i,0}^j} \left| \frac{f_{q_i}^{'}}{f} \right|^{\alpha_{i,1}^j} \cdots \left| \frac{f_{q_i}^{(k_i)}}{f} \right|^{\alpha_{i,k_i}^j}.
\end{aligned}
$$

By the lemma of logarithmic derivative, Lemma 2.2, and (3.8), we get that

$$
\frac{1}{2\pi} \int_{E_1} \log^+ |\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})| d\theta = S(r, f). \tag{3.9}
$$

Secondly, we consider $z = re^{i\theta}$, here $\theta \in E_2$. From (3.7), we have

$$
|f(z)| > \beta(z) \geq 2 \left| \frac{\beta_{n-\nu}}{\beta_n} \right|^{\frac{1}{\nu}}.
$$

Therefore, $\left| \frac{\beta_{n-\nu}}{\beta_n} \right| < \frac{|f(z)|^\nu}{2^\nu}$ holds for all $\nu = 1, \ldots, n$. Then, we get that

$$
\begin{aligned}
|Q(z, f)| &= \left| \beta_n f^n \left(\frac{\beta_0}{\beta_n f^n} + \frac{\beta_1}{\beta_n f^{n-1}} + \cdots + \frac{\beta_{n-1}}{\beta_n f} + 1 \right) \right| \\
&\geq |\beta_n||f|^n \left(1 - \sum_{\nu=1}^{n} \frac{|\beta_{n-\nu}|}{|\beta_n||f|^\nu} \right) \\
&\geq |\beta_n||f|^n \left(1 - \sum_{\nu=1}^{n} \frac{1}{2^\nu} \right) = \frac{|\beta_n||f|^n}{2^n}.
\end{aligned} \tag{3.10}
$$

It follows from (1.4) and (3.10) that

$$
\begin{aligned}
|\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})| &= \left| \frac{P(z, f)}{Q(z, f)} \right| \leq \frac{2^n}{|\beta_n||f|^n} \sum_{\mu=0}^{m} |\alpha_\mu(z)||f|^\mu \\
&\leq \frac{2^n}{|\beta_n|} \sum_{\mu=0}^{m} |\alpha_\mu(z)||f|^{\mu-n}.
\end{aligned} \tag{3.11}
$$

From (3.7), note that $|f| > |\beta(z)| \geq 1$. Recall that $n \geq m$. Hence, (3.11) gives that

$$
\frac{1}{2\pi} \int_{E_2} \log^+ |\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})| d\theta = S(r, f). \tag{3.12}
$$

By adding (3.9) to (3.12) together, it yields

$$m(r,\Omega) = S(r,f). \tag{3.13}$$

From (3.3) and (3.13), then (3.5) reduces to

$$\begin{aligned} dT(r,f) + S(r,f) &\le \gamma'_\Omega T(r,f) + (\gamma_\Omega - \gamma'_\Omega)N(r,f) + \Gamma_\Omega \overline{N}(r,f) + S(r,f) \\ &\le \gamma_\Omega N(r,f) + \Gamma_\Omega \overline{N}(r,f) + S(r,f), \end{aligned}$$

which implies that $d = n \le \gamma_\Omega(1-\delta(\infty,f)) + \Gamma_\Omega(1-\Theta(\infty,f))$.

Thus, we finished the proof of Theorem 1.1.

4. Characteristic function estimation of $\Phi(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$

Before proving Theorems 1.4, 1.6, 1.7 and 1.9, we need a sequence of preliminary results. We show that the characteristic function estimation of $\Phi(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$. The following lemma which can be obtained by similar proof of Theorem 1 in [13].

Lemma 4.1. *Let η_j, τ_j be small functions of f, $q \ne 0, 1$ and*

$$\frac{\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}{\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})} := \frac{\displaystyle\sum_{j=1}^{s} \eta_j(z) \prod_{i=1}^{t} f_{q_i}^{\alpha_{i,0}^j} (f'_{q_i})^{\alpha_{i,1}^j} \cdots (f_{q_i}^{(k_i)})^{\alpha_{i,k_i}^j}}{\displaystyle\sum_{j=1}^{s} \tau_j(z) \prod_{i=1}^{t} f_{q_i}^{\alpha_{i,0}^j} (f'_{q_i})^{\alpha_{i,1}^j} \cdots (f_{q_i}^{(l_i)})^{\alpha_{i,l_i}^j}},$$

where $\alpha_{i,l}^j$ $(l = 0, 1, \ldots, k_i)$, $\alpha_{i,t}^j$ $(t = 0, 1, \ldots, l_i)$ are positive integers and $\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ and $\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$ are q-difference differential polynomials of the form (1.6)*. If f has a zero order, then*

$$m(r,\Phi) \le \gamma_\Phi m(r,f) - N(r, \frac{1}{\Omega_2}) + N(r,\Omega_2) + S(r,f).$$

Proof of Lemma 4.1. Let $U(z) = \max\{|\Omega_1|, |\Omega_2|\}$. It follows from (1.7) that

$$\log^+ |\Phi| = \log U(z) - \log |\Omega_2|.$$

By integration, this yields

$$\begin{aligned} m(r,\Phi) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\Phi(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log U(re^{i\theta}) d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log |\Omega_2(re^{i\theta})| d\theta. \end{aligned}$$

Note that (3.1) and $\gamma'_\Omega \le \gamma_\Omega$. Thus, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\log\max\{|\Omega_1|,|\Omega_2|\}d\theta\\ &\le \sum_{i=1}^{t}\gamma_i(\Phi)m(r,f)+S(r,f)\\ &= \gamma_\Phi m(r,f)+S(r,f).\end{aligned}$$

Using Jensen's formula, we get that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|\Omega_2(re^{i\theta})|d\theta = N\left(r,\frac{1}{\Omega_2}\right)-N(r,\Omega_2)+O(1).$$

Hence, from the above equations, we obtain

$$m(r,\Phi)\le \gamma_\Phi m(r,f)-N\left(r,\frac{1}{\Omega_2}\right)+N(r,\Omega_2)+S(r,f).$$

□

Applying Lemma 4.1, we obtain Proposition 4.2 below, as the corresponding proof of (see, e.g., Theorem 3 in [13] and Theorem 4.7 in [6]).

Proposition 4.2. *Let $\Phi(z,f_{q_1},f_{q_2},\ldots,f_{q_t})$ be defined as in (1.7). If f is a meromorphic function with zero order, then*

$$T(r,\Phi)\le \min\left\{\gamma_\Phi T(r,f)+\Gamma_\Phi\overline{N}(r,f),\Delta_\Phi T(r,f)\right\}+S(r,f).$$

Proposition 4.3. *Suppose that f is a transcendental meromorphic solution of (1.5) with zero order and $n\ge m$. Then*

$$m(r,\Phi)=m\left(r,\frac{1}{\Omega_2}\right)+S(r,f).$$

Furthermore, if $N(r,f)=S(r,f)$, we have

$$T(r,\Phi)\le T\left(r,\frac{1}{\Omega_2}\right)+S(r,f).$$

Proof of Proposition 4.3. By the similar method as in the proof of Theorem 1.1.

If $z\in E_1$, then we get that

$$\begin{aligned}&\frac{1}{2\pi}\int_{E_1}\log^+\left|\frac{\Omega_1(z,f_{q_1},f_{q_2},\ldots,f_{q_t})}{\Omega_2(z,f_{q_1},f_{q_2},\ldots,f_{q_t})}\right|d\theta\\ &\le \frac{1}{2\pi}\int_{E_1}\log^+|\Omega_1(z,f_{q_1},f_{q_2},\ldots,f_{q_t})|d\theta\\ &\quad+\frac{1}{2\pi}\int_{E_1}\log^+\left|\frac{1}{\Omega_2(z,f_{q_1},f_{q_2},\ldots,f_{q_t})}\right|d\theta\\ &\le \frac{1}{2\pi}\int_{E_1}\log^+\left|\frac{1}{\Omega_2(z,f_{q_1},f_{q_2},\ldots,f_{q_t})}\right|d\theta+S(r,f).\end{aligned}\tag{4.1}$$

If $z \in E_2$, then it follows from (3.7) that $|f| > 2|\beta(z)| > 1$. Recall that $n \geq m$. Similarly, we get that

$$\left|\frac{\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}{\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}\right| = \left|\frac{P(z,f)}{Q(z,f)}\right| \leq \frac{2^n}{|\beta_n||f|^n} \sum_{\mu=0}^{m} |\alpha_\mu(z)||f|^\mu$$
$$\leq \frac{2^n}{|\beta_n|} \sum_{\mu=0}^{m} |\alpha_\mu(z)||f|^{\mu-n}.$$

Thus

$$\frac{1}{2\pi}\int_{E_2} \log^+ \left|\frac{\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}{\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}\right| d\theta = S(r,f). \tag{4.2}$$

From (4.1) and (4.2), we have

$$m(r,\Phi) = m\left(r, \frac{1}{\Omega_2}\right) + S(r,f). \tag{4.3}$$

Next, suppose that $N(r,f) = S(r,f)$. Recall inequality (3.3), then we have

$$N(r,\Phi) \leq N(r,\Omega_1) + N\left(r, \frac{1}{\Omega_2}\right) = N\left(r, \frac{1}{\Omega_2}\right) + S(r,f). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$T(r,\Phi) \leq T\left(r, \frac{1}{\Omega_2}\right) + S(r,f).$$

This completes the proof of Proposition 4.3. □

5. Proof of Theorem 1.4-1.9

Proof of Theorem 1.4. Taking the Nevanlinna characteristic function of both sides of equation (1.5). Applying Lemma 2.1 and Proposition 4.2, we obtain

$$dT(r,f) \leq \min\left\{\gamma_\Phi T(r, f(z)) + \Gamma_\Phi \overline{N}(r, f(z)), \Delta_\Phi T(r, f(z))\right\} + S(r,f),$$

which implies that

$$d = \max\{p, n\} \leq \min\left\{\gamma_\Phi + \Gamma_\Phi(1 - \Theta(\infty, f)),\ \Delta_\Phi\right\}.$$

Rewrite (1.5) as

$$Y(z,f) := \Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})Q(z,f) - \Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})P(z,f) = 0.$$

Note that $\Omega_1(z, 0, 0, \ldots, 0)\beta_0(z) - \Omega_2(z, 0, 0, \ldots, 0)\alpha_0(z) \not\equiv 0$. That is to say, $Y(z,0) \not\equiv 0$. By using Lemma 2.4, we have $m\left(r, \frac{1}{f}\right) = S(r,f)$. Thus, $\delta(0,f) = 0$.

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.6. Firstly, we rewrite the quotient $\frac{P(z,f)}{Q(z,f)}$ as

$$\frac{P(z,f)}{Q(z,f)} = P_1(z,f) + \frac{P_2(z,f)}{Q(z,f)},$$

where $P_1(z,f), P_2(z,f)$ are polynomials in f such that

$$\deg(P_1(z,f)) = \max\{m-n, 0\},\ \deg(P_2(z,f)) < n.$$

Note that the coefficients of P_1, P_2 are small functions of f. Thus, (1.5) can be written as

$$\frac{\Omega_1(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) - P_1(z,f)\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})}{\Omega_2(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})} = \frac{P_2(z,f)}{Q(z,f)}.$$

Applying Proposition 4.3 and Lemma 2.1 to the above equation, we have

$$nT(r,f) \le T\left(r, \frac{\Omega_1 - P_1\Omega_2}{\Omega_2}\right) + S(r,f) \le T\left(r, \frac{1}{\Omega_2}\right) + S(r,f).$$

Note that $N(r,f) = S(r,f)$. By combining with (3.4), we obtain

$$nT(r,f) \le \gamma'_{\Omega_2} T(r,f) + S(r,f),$$

which implies that $n \le \gamma'_{\Omega_2}$.

Secondly, by using a similar method to $\frac{Q(z,f)}{P(z,f)}$. We can obtain $m \le \gamma'_{\Omega_1}$.

This completes the proof of Theorem 1.6. □

Proof of Theorem 1.7. Taking the Nevanlinna characteristic function of both sides of equation (1.12) and applying Proposition 4.2 and Lemma 2.1, we have

$$\begin{aligned} dT(r,f) \le \min\left\{\gamma_\Omega T(r,f) + \Gamma_\Omega \overline{N}(r,f), \Delta_\Omega T(r,f)\right\} \\ + s_1\overline{N}(r,f) + s_1\overline{N}\left(r, \frac{1}{f}\right) + S(r,f), \end{aligned} \tag{5.1}$$

which implies

$$d = \max\{m,n\} \le \min\{\gamma_\Omega + \Gamma_\Omega(1-\Theta(\infty,f)), \Delta_\Omega\} + 2s_1 - s_1\Theta(\infty,f) - s_1\Theta(0,f).$$

If $d = \Delta_\Omega + 2s_1$, then it follows from (5.1) that

$$(d - \Delta_\Omega)T(r,f) \le s_1\overline{N}(r,f) + s_1\overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

Then, we have

$$(d - \Delta_\Omega - s_1)T(r,f) \le \min\left\{s_1\overline{N}(r,f), s_1\overline{N}\left(r, \frac{1}{f}\right)\right\} + S(r,f),$$

which implies that $T(r,f) = \overline{N}(r,f) + S(r,f) = \overline{N}\left(r, \frac{1}{f}\right) + S(r,f)$.

If $Q(z,0) \not\equiv 0$, note that all zeros of f are not poles of $\frac{P(z,f)}{Q(z,f)}$. From (1.12), we know that all zeros of f are not poles of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) + \omega(z)\left(\frac{f'(z)}{f(z)}\right)^{s_1}$. Hence, the poles of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t}) + \omega(z)\left(\frac{f'(z)}{f(z)}\right)^{s_1}$ appear only in the poles of $\Omega(z, f_{q_1}, f_{q_2}, \ldots, f_{q_t})$, the poles of $\omega(z)$ and the poles of f. Hence, we have

$$\begin{aligned} N\left(r, \Omega + \omega\left(\frac{f'}{f}\right)^{s_1}\right) &\le N(r,\Omega) + N(r,\omega) + s_1\overline{N}(r,f) \\ &\le N(r,\Omega) + s_1\overline{N}(r,f) + S(r,f). \end{aligned}$$

By the same argument as above, we get that

$$dT(r,f) \le \min\{\gamma_\Omega T(r,f) + \Gamma_\Omega \overline{N}(r,f), \Delta_\Omega T(r,f)\} + s_1\overline{N}(r,f) + S(r,f),$$

which yields in

$$d \le \min\{\gamma_\Omega + \Gamma_\Omega(1-\Theta(\infty,f)), \Delta_\Omega\} + s_1(1-\Theta(\infty,f)).$$

This completes the proof of Theorem 1.7. □

Proof of Theorem 1.9. Suppose that (f, g) is a transcendental meromorphic solution of zero order to (1.13). Taking the Nevanlinna characteristic function of both sides of each equation of (1.13) and applying Proposition 4.2 and Lemma 2.1, we get that

$$d_1T(r,f) + S(r,f) = T(r,\Phi_1) \le \Delta_{\Phi_1}T(r,g) + S(r,g),$$

$$d_2T(r,g) + S(r,g) = T(r,\Phi_2) \le \Delta_{\Phi_2}T(r,f) + S(r,f).$$

Then we have

$$(d_1 + o(1))T(r,f) \le (\Delta_{\Phi_1} + o(1))\, T(r,g),$$

$$(d_2 + o(1))T(r,g) \le (\Delta_{\Phi_2} + o(1))\, T(r,f),$$

as $r \to \infty$, outside of a possible set of finite logarithmic measure, we obtain

$$d_1d_2 \le \Delta_{\Phi_1}\Delta_{\Phi_2}.$$

In particular, if $\overline{N}(r,f) = S(r,f)$, then by the same method, we have

$$d_1T(r,f) + S(r,f) = T(r,\Phi_1) \le \gamma_{\Phi_1}T(r,g) + S(r,g),$$

$$d_2T(r,g) + S(r,g) = T(r,\Phi_2) \le \gamma_{\Phi_2}T(r,f) + S(r,f).$$

Similarly, we obtain

$$d_1d_2 \le \gamma_{\Phi_1}\gamma_{\Phi_2}.$$

This completes the proof of Theorem 1.9. □

Acknowledgements. The authors would like to thank the referees for their thorough comments and helpful suggestions.

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