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A NOTE ON TWO NEW CLOSED-FORM EVALUATIONS OF THE GENERALIZED HYPERGEOMETRIC FUNCTION ${}_5F_4$ WITH ARGUMENT $\frac{1}{256}$

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Abstract. The aim of this note is to provide two new and interesting closed-form evaluations of the generalized hypergeometric function ${}_5F_4$ with argument $\frac{1}{256}$. This is achieved by means of separating a generalized hypergeometric function into even and odd components together with the use of two known sums (one each) involving reciprocals of binomial coefficients obtained earlier by Trif and Sprugnoli. In the end, the results are written in terms of two interesting combinatorial identities.

1. Introduction

The generalized hypergeometric function ${}_pF_q(z)$ with p numerator and q denominator parameters is defined by [18]

$$(1.1)\qquad {}_pF_q\left[\begin{array}{c} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the well-known Pochhammer's symbol defined by

$$(a)_n = \begin{cases} a(a+1)\ldots(a+n-1) & ;\ n \in \mathbb{N} \\ \\ 1 & ;\ n = 0. \end{cases}$$

In terms of gamma function, we have

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

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Here, as usual, p and q are non-negative integers and the parameters a_j $(1 \leq j \leq p)$ and b_j $(1 \leq j \leq q)$ can have arbitrary complex values with zero or negative integer values of b_j excluded. The generalized hypergeometric function ${}_pF_q(z)$ converges for $|z| < \infty, (p \leq q)$, $|z| < 1 (p = q+1)$ and $|z| = 1$ $(p = q+1$ and $Re(s) > 0)$, where s is the parametric excess defined by

$$s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering and statistics. For more details about this function, we refer [1, 2, 4, 12, 14, 22].

Further, it is well-known that the process of resolving a generalized hypergeometric function into even and odd components can lead to new results. This composition is facilitated by use of the identities

$$(a)_{2n} = 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a}{2} + \frac{1}{2}\right)_n$$

and

$$(a)_{2n+1} = a2^{2n} \left(\frac{a}{2} + \frac{1}{2}\right)_n \left(\frac{a}{2} + 1\right)_n.$$

Then for the generalized hypergeometric function

$${}_{q+1}F_q \left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ \\ b_1, b_2, \ldots, b_q \end{matrix} ; z \right]$$

it is not difficult to obtain the following two general results:

$$\begin{aligned} &{}_{q+1}F_q \left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix} ; z \right] + {}_{q+1}F_q \left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix} ; -z \right] \\ &= 2 {}_{2q+2}F_{2q+1} \left[\begin{matrix} \frac{a_1}{2}, \frac{a_1}{2} + \frac{1}{2}, \ldots, \frac{a_{q+1}}{2}, \frac{a_{q+1}}{2} + \frac{1}{2} \\ \frac{1}{2}, \frac{b_1}{2}, \frac{b_1}{2} + \frac{1}{2}, \ldots, \frac{b_q}{2}, \frac{b_q}{2} + \frac{1}{2} \end{matrix} ; z^2 \right] \end{aligned} \tag{1.2}$$

and

(1.3)
$$
{}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix} ; z\right] - {}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{matrix} ; -z\right]
$$
$$
= \frac{2za_1a_2\ldots a_{q+1}}{b_1b_2\ldots b_q}\ {}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1}{2}+\frac{1}{2}, \frac{a_1}{2}+1, \ldots, \frac{a_{q+1}}{2}+\frac{1}{2}, \frac{a_{q+1}}{2}+1 \\ \frac{3}{2}, \frac{b_1}{2}+\frac{1}{2}, \frac{b_1}{2}+1, \ldots, \frac{b_{q+1}}{2}+\frac{1}{2}, \frac{b_q}{2}+1 \end{matrix} ; z^2\right]
$$

On the other hand, the binomial coefficients are defined by

$$
\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & ;\ n \geq k \\ 0 & ;\ n < k. \end{cases} \tag{1.4}
$$

for nonnegative integers n and k. The central binomial coefficients are defined by

$$
\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \qquad (n = 0, 1, 2, \ldots). \tag{1.5}
$$

It is well-known that the binomial and reciprocal of binomial coefficients play an important role in many areas of mathematics (including number theory, probability and statistics). Actually the sums containing the central binomial coefficients and reciprocals of the central binomial coefficients have been studied for a long time. A large number of very interesting results can be seen in the research papers by Ji and Hei [6], Ji and Zhang [7], Lehmer [9], Mansour [11], Pla [13], Rockett [20], Sprugnoli [23], Sury [25], Sury et al. [26], Trif [27], Wheelon [28] and Zhao and Wang [29]. Many facts about the central binomial coefficients and the reciprocals of the central binomial coefficients can be found in the book of Koshy [8]. Gould [5] has collected numerous identities involving central binomial coefficients. Riordan [19] is also a good reference. For other results, we refer [3, 10, 15, 16, 17].

In the year 2000, among other combinatorial sums involving reciprocals of binomial coefficients, Trif [27] established the following result asserted in the following theorem.

Theorem. *For m and n to be positive integers with $m > n$*

$$
\sum_{k=0}^{\infty} \binom{mk}{nk}^{-1} = \int_0^1 \frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2} dt \tag{1.6}
$$

and in particular deduced the following elegant sum

$$\sum_{k=0}^{\infty}\binom{4k}{2k}^{-1}=\frac{16}{15}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi\approx 1.18211814\ldots. \tag{1.7}$$

where $\varphi=\dfrac{\sqrt{5}+1}{2}$ *is the Golden ratio.*

Later, in 2006, Sprugnoli [23], among other results, established the following elegant result closely related to (1.7)

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k\binom{4k}{2k}^{-1}\\&=\frac{16}{17}+\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)+\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right)\\&\approx 0.846609430\ldots.\end{aligned}\tag{1.8}$$

Moreover, it is not difficult to see that the result (1.7) and (1.8) can be written in terms of the generalized hypergeometric functions in the following form that will be required in our present investigations. These are

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};\frac{1}{16}\right]=\frac{16}{15}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi, \tag{1.9}$$

and

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{1}{16}\right]\\&=\frac{16}{17}+\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)+\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right).\end{aligned}\tag{1.10}$$

The aim of this note is to provide two new and interesting closed-form evaluations of the generalized hypergeometric function ${}_5F_4$ with argument $\frac{1}{256}$ as well as in terms of combinatorial identities . This is achieved by means of the results (1.2), (1.3), (1.9) and (1.10). The same will be given in the next section.

2. Two New Closed-form Evaluations

In this section, we shall establish the following two new and interesting closed-form evaluations for the generalized hypergeometric function ${}_5F_4$ with argument $\frac{1}{256}$. These are

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4}1,1\\ \frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\end{matrix};\frac{1}{256}\right]\\
&=\frac{1}{2}\left\{\frac{512}{255}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi+\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)\right.\\
&\qquad\left.+\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right)\right\}
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}\frac{3}{4},1,1,\frac{5}{4},\frac{3}{2}\\ \frac{5}{8},\frac{7}{8},\frac{9}{8},\frac{11}{8}\end{matrix};\frac{1}{256}\right]\\
&=3\left\{\frac{32}{255}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi-\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)\right.\\
&\qquad\left.-\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right)\right\}.
\end{aligned}
\tag{2.2}
$$

Proof. The derivations of the results (2.1) and (2.2) are quite straight forward. Thus in order to establish the results (2.1) and (2.2), we proceed as follows. In (1.2) and (1.3) if we set $q = 2$, $a_1 = 1/2, a_2 = a_3 = 1, b_1 = 1/4, b_2 = 3/4, z = 1/16$ and making use of the results (1.9) and (1.10), we easily arrive at the right-hand side of (2.1) and (2.2) respectively. This completes the proof of our main results (2.1) and (2.2). We remark in passing that the results (2.1) and (2.2) can be written in terms of the following two interesting combinatorial identities:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{\binom{8n}{4n}}\\
&=\frac{1}{2}\left\{\frac{512}{255}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi+\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)\right.
\end{aligned}
\tag{2.3}
$$

$$+\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right)\Bigg\}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{\binom{8n+4}{4n+2}}$$

$$(2.4)\quad =\frac{1}{2}\left\{\frac{512}{255}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi+\frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)\right.$$

$$\left.-\frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right)\right\}.$$

□

We conclude this note by remarking that the results (2.1) to (2.2) have been verified using MAPLE.

Further observation: It is not out of place to mention here that the elegant sum (1.7) due to Triff [27] can be obtained in an elementary way as follows:
We have from Lehmer [9]

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}=\frac{1}{3}+\frac{2\sqrt{3}\pi}{27}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\binom{2n}{n}}=-\frac{1}{5}-\frac{4\sqrt{5}}{25}\ln\varphi$$

which can be written as

$$(2.5)\quad \sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}=\frac{4}{3}+\frac{2\sqrt{3}\pi}{27}$$

and

$$(2.6)\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n}{n}}=\frac{4}{5}-\frac{4\sqrt{5}}{25}\ln\varphi.$$

Now adding and subtracting (2) and (2.4), we at once get

$$(2.7)\quad \sum_{n=0}^{\infty}\frac{1}{\binom{4n}{2n}}=\frac{16}{15}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi$$

which is Triff result (1.7) and

$$(2.8)\quad \sum_{n=0}^{\infty}\frac{1}{\binom{4n+2}{2n+1}}=\frac{1}{\binom{4n+2}{2n+1}}=\frac{4}{15}+\frac{\sqrt{3}\pi}{27}+\frac{2\sqrt{5}}{25}\ln\varphi.$$

The result (2.8) appears to be a new result.

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