JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 2, May 2023
http://dx.doi.org/10.14403/jcms.2023.36.2.125

VOLUME PROPERTIES AND A CHARACTERIZATION OF ELLIPTIC PARABOLOIDS

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Abstract. We establish a characterization theorem of elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with extrinsic properties such as the $(n+1)$-dimensional volumes of regions enclosed by the hyperplanes and hypersurfaces, and the n-dimensional areas of projections of the sections of hypersurfaces cut off by hyperplanes.

1. Introduction

Suppose that M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. For a fixed point $p \in M$ and a sufficiently small $t > 0$, let us denote by Φ a hyperplane which intersects M and is parallel to the tangent hyperplane Ψ of M at p with distance t. We aim to characterize elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ by using the n-dimensional areas of projections of the sections cut off by hyperplanes and the $(n+1)$-dimensional volumes of regions enclosed by the hyperplanes and hypersurfaces. In order to do so, we denote by $A_p(t)$ and $V_p(t)$ the n-dimensional area of the section in Φ enclosed by $\Phi \cap M$ and the $(n+1)$-dimensional volume of the region bounded by the hypersurface M and the hyperplane Φ, respectively.

If M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$, for a fixed point $p = (x, f(x)) \in M$ and a real number $k > 0$, we put Φ the hyperplane through $v = (x, f(x) + k)$ which is parallel to the tangent hyperplane Ψ of M at p. We denote by $A_p^*(k)$, $V_p^*(k)$ and $D_p^*(k)$ the n-dimensional area of the section in Φ enclosed by $\Phi \cap M$, the $(n+1)$-dimensional volume of the region of $\mathbb{E}^{n+1}$ bounded by M and

Received January 31, 2023; Accepted May 15, 2023.
2020 Mathematics Subject Classification: 53A07.
Key words and phrases: Elliptic paraboloid; $(n+1)$-dimensional volume; n-dimensional surface area; Gauss-Kronecker curvature.

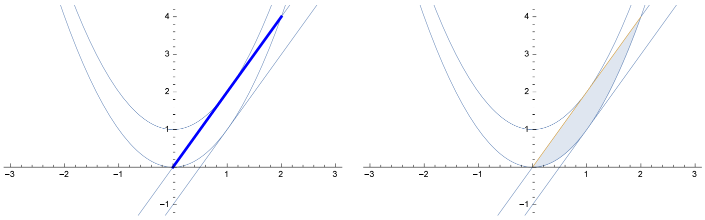

FIGURE 1. $A_p^*(1)$ and $V_p^*(1)$ for $p = (1, 1)$ and $f(x) = x^2$.

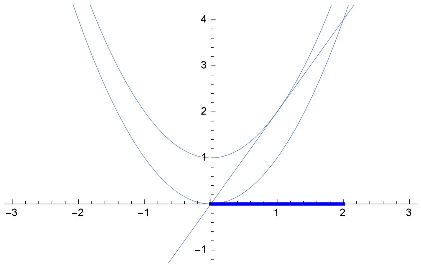

FIGURE 2. $D_p^*(1)$ for $p = (1, 1)$ and $f(x) = x^2$.

the hyperplane Φ and the n-dimensional area of the projection of the section in Φ enclosed by $\Phi \cap M$ onto $\mathbb{R}^n$, respectively. In this case, for a fixed point $p \in M$ and a sufficiently small $t > 0$ we may define $D_p(t)$ as the area of the projection of the section in Φ enclosed by $\Phi \cap M$ onto $\mathbb{R}^n$, where Φ is the hyperplane which intersects M and is parallel to the tangent hyperplane Ψ of M at p with distance t. See Figures 1 and 2.

Let us denote $W(p) = \sqrt{1 + |\nabla f(x)|^2}$, where $p = (x, f(x)) \in M$ and ∇f is the gradient of f. Then we have

$$D_p^*(k) = A_p^*(k)/W(p). \tag{1.1}$$

For details, see Section 2.

For elliptic paraboloids in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, the following characterization theorem has been established ([10, 11]).

Proposition 1. Let M be a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that the Gauss-Kronecker curvature $K(p)$ of M at p with respect to the upward unit normal to M is positive at some point $p \in M$. Then M is an elliptic paraboloid if and only if it satisfies one of the following conditions:

(V^*): $V_p^*(k)$ is a positive function, which depends only on k.
(D^*): $D_p^*(k)$ in (1.1) is a positive function, which depends only on k.

On the other hand, the following characterization theorem of parabolas was established (Theorem 1 of [21]).

Proposition 2. Suppose that $f(x)$ is a differentiable function and for all real numbers a and h with $h > 0$, $l(a, h, x)$ is the secant line determined by the two points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $f(x)$, separated horizontally by h units. Then $f(x)$ is a parabola if and only if the signed area

$$A(a, h) = \int_a^{a+h} l(a, h, x)dx - \int_a^{a+h} f(x)dx$$

between the line $l(a, h, x)$ and the function $f(x)$ over the interval $[a, a+h]$ is a nonzero function of h alone, not dependent on a.

In the convex cases, Proposition 2 can be rewritten as follows:

Proposition 3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a convex differentiable function and M is its graph. Then M is a parabola if and only if it satisfies

$(V^*D) : V_p^*(k)$ is a positive function $\phi(D)$, which depends only on $D = D_p^*(k)$.

Hence, it is quite reasonable to ask whether the above condition (V^*D) also characterize the elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$.

In this paper, in Section 3 we prove the following characterization theorem of elliptic paraboloids, which is an n-dimensional analogue of Proposition 3.

Theorem 4. Let M be a smooth convex hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that the Gauss-Kronecker curvature $K(p)$ of M at p with respect to the upward unit normal to M is positive at some point $p \in M$. Then M is an elliptic paraboloid if and only if it satisfies the following condition:
(V^*D): $V_p^*(k)$ is a positive function $\phi(D)$, which depends only on $D = D_p^*(k)$.

Various properties of conic sections (especially, parabolas) have been proved to be characteristic ones ([1, 2, 4, 7, 8, 12, 14, 15, 17, 19, 21, 24]).

Some characterization theorems for hyperplanes, circular hypercylinders, hyperspheres, ellipsoids, elliptic paraboloids and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$ were established in [3, 4, 6, 9, 10, 11, 13, 18, 22]. For a characterization of hyperbolic space in the Minkowski space $\mathbb{E}_1^{n+1}$, we refer to [16].

Throughout this article, all objects are smooth(C^2) and connected, unless otherwise mentioned.

2. Preliminaries

Suppose that M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. For a fixed point $p \in M$ and a sufficiently small $t > 0$, we make use of notations: $A_p(t)$, $V_p(t)$ and $D_p(t)$ defined in Section 1. We may introduce a coordinate system $(x, z) = (x_1, x_2, \cdots, x_n, z)$ of $\mathbb{E}^{n+1}$ with the origin p, the tangent space of M at p is the hyperplane $z = 0$. Furthermore, we may assume that M is locally the graph of a non-negative convex function $f : \mathbb{R}^n \to \mathbb{R}$.

Then, for a sufficiently small $t > 0$ we have

$$A_p(t) = \iint_{f(x)<t} 1dx$$

and

$$V_p(t) = \iint_{f(x)<t} \{t - f(x)\}dx,$$

where $x = (x_1, x_2, \cdots, x_n)$, $dx = dx_1 dx_2 \cdots dx_n$. Since we also have

$$\begin{aligned} V_p(t) &= \iint_{f(x)<t} \{t - f(x)\}dx \\ &= \int_{z=0}^{t} \{\iint_{f(x)<z} 1dx\}dz, \end{aligned}$$

the fundamental theorem of calculus shows that

$$V_p'(t) = \iint_{f(x)<t} 1dx = A_p(t). \tag{2.1}$$

We have the following ([11]).

Lemma 5. Suppose that the Gauss-Kronecker curvature $K(p)$ of M at p is positive with respect to the upward unit normal to M. Then we have the following:

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^n}A_p(t)=\frac{(\sqrt{2})^n\omega_n}{\sqrt{K(p)}},\tag{2.2}$$

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^{n+2}}V_p(t)=\frac{(\sqrt{2})^{n+2}\omega_n}{(n+2)\sqrt{K(p)}},\tag{2.3}$$

where ω_n denotes the volume of the n-dimensional unit ball.

Now, suppose that M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f:\mathbb{R}^n\to\mathbb{R}$. For a fixed point $p=(x,f(x))\in M$ and a positive number k, we adopt the following notations: $A_p^*(k)$, $V_p^*(k)$, $D_p^*(k)$ and $D_p(t)$ defined in Section 1.

If we denote by θ the angle between the tangent hyperplane Ψ of M at p and $\mathbb{R}^n$, then we have

$$\cos\theta=\frac{1}{W(p)},\tag{2.4}$$

where we put for the gradient ∇f of f

$$W(p)=\sqrt{1+|\nabla f(x)|^2}.$$

For a positive number t with $k=tW(p)$, we have $\cos\theta=1/W(p)=t/k$. Hence we get

$$V_p^*(k)=V_p(t),\quad A_p^*(k)=A_p(t)\quad\text{and}\quad D_p^*(k)=D_p(t).\tag{2.5}$$

Furthermore, it follows from (2.5) that

$$D_p^*(k)=A_p^*(k)/W(p),\tag{2.6}$$

and

$$D_p(t)=A_p(t)/W(p).\tag{2.7}$$

Using $k=tW(p)$, together with (2.5) and (2.6), Lemma 5 implies the following.

Lemma 6. Suppose that M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f:\mathbb{R}^n\to\mathbb{R}$. For a fixed point $p=(x,f(x))\in M$, we suppose that the Gauss-Kronecker curvature $K(p)$ of M at p with respect to the upward unit normal to M is positive. Then we have the following:

$$\lim_{k\to 0}\frac{1}{(\sqrt{k})^n}A_p^*(k)=\frac{(\sqrt{2})^n\omega_n}{\sqrt{K(p)}}\times\frac{1}{(\sqrt{W(p)})^n}, \tag{2.8}$$

$$\lim_{k\to 0}\frac{1}{(\sqrt{k})^{n+2}}V_p^*(k)=\frac{(\sqrt{2})^{n+2}\omega_n}{(n+2)\sqrt{K(p)}}\times\frac{1}{(\sqrt{W(p)})^{n+2}}, \tag{2.9}$$

$$\lim_{k\to 0}\frac{1}{(\sqrt{k})^n}D_p^*(k)=\frac{(\sqrt{2})^n\omega_n}{\sqrt{K(p)}}\times\frac{1}{(\sqrt{W(p)})^{n+2}}, \tag{2.10}$$

where ω_n denotes the volume of the n-dimensional unit ball.

3. Proof of Theorem 4

In this section, we give a proof of Theorem 4 stated in Section 1.

Let us denote by M a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f:\mathbb{R}^n\to\mathbb{R}$.

First, suppose that the hypersurface M satisfies (V^*D). Then $V_p^*(k)$ is a positive function $\phi(D)$, which depends only on $D=D_p^*(k)$. Hence we have

$$\frac{V_p^*(k)}{(\sqrt{k})^{n+2}}=\frac{\phi(D)}{D^{(n+2)/n}}\times\frac{(D_p^*(k))^{(n+2)/n}}{(\sqrt{k})^{n+2}}. \tag{3.1}$$

It follows from (2.10) in Lemma 6 that

$$\lim_{k\to 0}\frac{(D_p^*(k))^{(n+2)/n}}{(\sqrt{k})^{n+2}}=\lim_{k\to 0}\{\frac{D_p^*(k)}{(\sqrt{k})^n}\}^{(n+2)/n}=\frac{(\sqrt{2})^{n+2}\omega_n^{(n+2)/n}}{(\sqrt{K(p)})^{(n+2)/n}}\times\frac{1}{(\sqrt{W(p)})^{(n+2)^2/n}}. \tag{3.2}$$

By the assumption, we have

$$\lim_{D\to 0}\frac{\phi(D)}{D^{(n+2)/n}}=\delta,$$

where δ is a constant independent of p. Hence together with (2.9),(3.1), and (3.2) we obtain

$$\frac{(\sqrt{2})^{n+2}\omega_n}{(n+2)\sqrt{K(p)}}\times\frac{1}{(\sqrt{W(p)})^{n+2}}=\delta\times\frac{(\sqrt{2})^{n+2}\omega_n^{(n+2)/n}}{(\sqrt{K(p)})^{(n+2)/n}}\times\frac{1}{(\sqrt{W(p)})^{(n+2)^2/n}},$$

from which we get

$$K(p) = (n+2)^n \delta^n \omega_n^2 \frac{1}{W(p)^{n+2}}. \tag{3.3}$$

Note that the Gauss-Kronecker curvature $K(p)$ of M at p is given by ([23], p.93)

$$K(p) = \frac{\det D^2 f(x)}{W(p)^{n+2}}. \tag{3.4}$$

It follows from (3.3) and (3.4) that the determinant $\det D^2 f(x)$ of the Hessian of the function f is a positive constant. Thus $f(x)$ is a globally defined quadratic polynomial ([5, 20]), and hence M is an elliptic paraboloid. This completes the proof of the if part of Theorem 4.

Conversely, let us consider an elliptic paraboloid defined by

$$M : z = f(x) = \Sigma_{i=1}^n a_i^2 x_i^2, a_i > 0,$$

a tangent hyperplane Ψ to M at a fixed point $p = (x, z) \in M$ and a hyperplane Φ through $v = (x, z+k), k > 0$ which is parallel to the tangent hyperplane Ψ to M at p. Then the proof of Theorem 5 of [11] shows that

$$V_p^*(k) = \alpha_n k^{(n+2)/2}, \quad \alpha_n = \frac{2\sigma_{n-1}}{n(n+2)a_1 a_2 \cdots a_n}, \tag{3.5}$$

where σ_{n-1} denotes the surface area of the $(n-1)$-dimensional unit sphere. Hence, from (2.5) with $k = tW(p)$ we have

$$V_p(t) = \alpha_n W(p)^{(n+2)/2} t^{(n+2)/2}.$$

It follows from (2.1) that

$$A_p(t) = \beta_n W(p)^{(n+2)/2} t^{n/2}, \quad \beta_n = \frac{n+2}{2}\alpha_n$$

and hence we get from $k = tW(p)$

$$A_p^*(k) = \beta_n W(p) k^{n/2}.$$

Using (2.6), we have

$$D_p^*(k) = \frac{1}{W(p)} A_p^*(k) = \beta_n k^{n/2}. \tag{3.6}$$

Together with (3.6), (3.5) implies

$$V_p^*(k) = \gamma_n D_p^*(k)^{(n+2)/n}, \quad \gamma_n = \frac{\alpha_n}{(\beta_n)^{(n+2)/n}}.$$

This completes the proof of the only if part of Theorem 4.

References

[1] Á. Bényi, P. Szeptycki and F. Van Vleck, *Archimedean properties of parabolas*, Amer. Math. Monthly, **107** (2000), 945–949.

[2] Á. Bényi, P. Szeptycki and F. Van Vleck, *A generalized Archimedean property*, Real Anal. Exchange, **29** (2003/04), 881–889.

[3] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitatstheorie, Band I, Elementare Differentialgeometrie, (German) 3rd ed.*, Dover Publications, New York, N. Y., 1945.

[4] O. Ciaurri, E. Fernandez and L. Roncal, *Revisiting floating bodies*, Expo. Math., **34** (2016), no. 4, 396–422.

[5] K. Jörgens, *Über die Losungen der Differentialgleichung* $rt - s^2 = 1$, Math. Ann., **127**(1954), 130–134.

[6] D.-S. Kim, *Ellipsoids and elliptic hyperboloids in the Euclidean space* $\mathbb{E}^{n+1}$, Linear Algebra Appl., **471** (2015), 28–45.

[7] D.-S. Kim and S. H. Kang, *A characterization of conic sections*, Honam Math. J., **33** (2011), 335–340.

[8] D.-S. Kim and Y. H. Kim, *A characterization of ellipses*, Amer. Math. Monthly, **114** (2007), 66–70.

[9] D.-S. Kim and Y. H. Kim, *New characterizations of spheres, cylinders and W -curves*, Linear Algebra Appl., **432** (2010), 3002–3006.

[10] D.-S. Kim and Y. H. Kim, *Some characterizations of spheres and elliptic paraboloids*, Linear Algebra Appl., **437** (2012), no. 1, 113–120.

[11] D.-S. Kim and Y. H. Kim, *Some characterizations of spheres and elliptic paraboloids II*, Linear Algebra Appl., **438** (2013) no.3, 1356–1364.

[12] D.-S. Kim and Y. H. Kim, *On the Archimedean characterization of parabolas*, Bull. Korean Math. Soc., **50** (2013), no. 6, 2103–2114.

[13] D.-S. Kim and Y. H. Kim, *A characterization of concentric hyperspheres in* $\mathbb{R}^n$, Bull. Korean Math. Soc., **51** (2014), 531–538.

[14] D.-S. Kim, Y. H. Kim and J. H. Park, *Some characterizations of parabolas*, Kyungpook Math. J., **53** (2013), 99–104.

[15] D.-S. Kim, S. Park and Y. H. Kim, *Center of gravity and a characterization of parabolas*, Kyungpook Math. J., **55** (2015), no. 2, 473–484.

[16] D.-S. Kim, Y. H. Kim and D. W. Yoon, *On standard imbeddings of hyperbolic spaces in the Minkowski space*, C. R. Math. Acad. Sci. Paris, **352** (2014), 1033–1038.

[17] D.-S. Kim and K.-C. Shim, *Area of triangles associated with a curve*, Bull. Korean Math. Soc., **51** (2014), 901–909.

[18] D.-S. Kim and B. Song, *A characterization of elliptic hyperboloids*, Honam Math. J., **35** (2013), 37–49.

[19] J. Krawczyk, *On areas associated with a curve*, Zesz. Nauk. Uniw. Opol. Mat., **29** (1995), 97–101.

[20] A. V. Pogorelov, *On the improper convex affine hyperspheres*, Geom. Dedicata, **1** (1972), no. 1, 33–46.

[21] B. Richmond and T. Richmond, *How to recognize a parabola*, Amer. Math. Monthly, **116** (2009), 910–922.

[22] O. Stamm, *Umkehrung eines Satzes von Archimedes über die Kugel,* Abh. Math. Sem. Univ. Hamburg, **17** (1951), 112–132.

[23] J. A. Thorpe, *Elementary topics in differential geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1979.

[24] Y. Yu and H. Liu, *A characterization of parabola*, Bull. Korean Math. Soc., **45** (2008), 631–634.

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