JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 2, May 2023
http://dx.doi.org/10.14403/jcms.2023.36.2.93

CONTINUUM-WISE EXPANSIVE DIFFEOMORPHISMS ON TWO DIMENSIONAL MANIFOLD

Manseob Lee

Abstract. Let $f : M \to M$ be a diffeomorphism of two dimensional manifold M. If f has a homoclinic tangency associated to a hyperbolic periodic point p then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not continuum-wise expansive.

1. Introduction and proof of main Theorem

Assume that (M, d) is compact smooth manifold with Riemannian metric d. For $x \in M$ and any $\delta > 0$, we define $\Gamma(x, \delta) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta, \ \forall i \in \mathbb{Z}\}$. A diffeomorphism $f : M \to M$ is said to be *expansive* if there is $\delta > 0$ such that $\Gamma(x, \delta) = \{x\}$. We say that a set $C \subset M$ is *continuum* that it contains more than one point, that is, called *nondegenerate*. A subset C of a continuum M is called a *subcontinuum* of M. A diffeomorphism $f : M \to M$ is said to be *continuum-wise expansive*(simply, Cw-expansive) if there is $\delta > 0$ such that for a subcontinuum $C \subset \Gamma(x, \delta)$, if $\mathrm{diam} f^n(C) \leq \delta$ for all $n \in \mathbb{Z}$ then C is an one point set. Clearly, if a diffeomorphism f is expansive then f is Cw-expansive. But the converse is not true(see [4]). For Cw-expansive diffeomorphisms, many results are published in [1, 5, 7, 9, 15]. Among that Lee [7] proved that C^1 generically, if a diffeomorphism f is Cw-expansive then it is Axiom A. About the results, we consider the relationship with a horseshoe

Received April 17, 2023; Accepted May 26, 2023.
2020 Mathematics Subject Classification: 37A50, 37B45, 37C29.
Key words and phrases: expansive, continuum-wise expansive, horseshoe, homoclinic tangency.

and Cw-expansivity in the paper. For a horseshoe, we see that a remarkable results from Bowen [2], that is, there is a horseshoe such that it has positive Lebesgue measure. Also, Robinson and Young [14] showed that there is a horseshoe such that it is invariant and has positive Lebesgue measure.

A horseshoe is very close to homoclinic tangency, that is, for a hyperbolic periodic point p, we say that $x \in M$ is *homoclinic tangency* associated to p if $x \in W^s(p) \cap W^u(p) \setminus W^s(p) \pitchfork W^u(p)$, where $W^s(p) = \{x \in M : d(f^{ki}(x), p) \to 0, i \to \infty\}$ and $W^u(p) = \{x \in M : d(f^{ki}(x), p) \to 0, i \to -\infty\}(f^k(p) = p)$. We say that a diffeomorphism $f : M \to M$ is *measure expansive* if there is $\delta > 0$, such that for any $x \in M$, $\mu(\Gamma(\delta, x)) = 0$, where μ is a non-atomic probability measure on M. And a diffeomorphism $f : M \to M$ is *asymptotic measure expansive* if there is $\delta > 0$ such that for any $x \in M$, $\mu(f^n(\Gamma(\delta, x))) = 0$ as $n \to \infty$, where μ is a non-atomic probability measure on M.

According to [3], if a diffeomorphism f is measure expansive then f is asymptotic measure expansive. But the converse is not true (see [3, Example 1.1]). The notion of homoclinic tangency can constructs a small horseshoe which has positive Lebesgue measure. In [13], the authors proved that if a diffeomorphism f has a homoclinic tangency associated to a hyperbolic periodic point p then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not measure expansive. Also, in [3] proved that if if a diffeomorphism f has a homoclinic tangency associated to a hyperbolic periodic point p then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not asymptotic measure expansive. For measure and asymptotic measure expansivities of C^1 perturbation properties, we can see in [1, 8, 10–12, 16]. Moreover, Lee proved in [6] that if a diffeomorphism f has a homoclinic tangency associated to a hyperbolic periodic point p then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not weak measure expansive.

About the results, we consider that the relationship with Cw-expansivity and a horseshoe which is associated to a homoclinic tangency.

We say that $f : M \to M$ is a *flat tangency* if a small arc $\mathcal{J}_x$ contained in $W^s(p)$ and $W^u(p)$, that is, $\mathcal{J}_x \subset W^s(p) \cap W^u(p)$.

LEMMA 1.1. [13, Lemma 4.2] *Let p be a hyperbolic periodic point of f. For a diffeomorphism $f : M \to M$, if $x \in W^s(p) \cap W^u(p)$ then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is a flat tangency.*

The following is in [13, Lemma 4.3] and [3, Proposition 3.2.].

LEMMA 1.2. *Let $g : M \to M$ be a diffeomorphism such that g has a flat homoclinic tangency. Then there is a diffeomorphism $h : M \to M$ with C^1 close to g such that h has a sequence of horseshoes $\mathcal{H}_n$ with the following properties:*

(a) *for all $k \in \mathbb{Z}$, $\mathrm{diam}(h^k(\mathcal{H}_n)) < r_n$ with $r_n \to 0$ as $n \to \infty$.*
(b) *$h^j(\mathcal{H}_n) = \mathcal{H}_n (n \in \mathbb{N})$, for some $j > 0$.*
(c) *$\mu_L(\mathcal{H}_n) > 0$, where $\mu_L(A)$ is a Lebesgue measure of A.*

For a diffeomorphism f, we say that a Borel probability measure μ of M is *continuum-wise expansive*(Shin [17]) if there is $\delta > 0$ such that every subcontinuum $C \subset M$ with $\mu(C) > 0$, then $\mathrm{diam} f^n(C) > \delta$ for some $n \in \mathbb{Z}$.

Shin proved in [17] that a homeomorphism f of a separable metric space is Cw expansive if and only if every nonatomic Borel probability measure is Cw-expansive.

LEMMA 1.3. *For a diffeomorphism $f : M \to M$ of a two dimensional manifold M, if f has a homoclinic tangency associated to $p \in Per_h(f)$, then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that a nonatomic Borel probability measure μ is not Cw-expansive with respect to g, where $Per_h(f)$ is the set of all hyperbolic periodic points of f.*

Proof. Suppose that a diffeomorphism $f : M \to M$ has a homoclinic tangency associated to a hyperbolic periodic point p. As in Lemma 1.1 and Lemma 1.2, there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g has a sequence of horseshoes $\mathcal{H}_n$ with the following properties:

(a) for all $k \in \mathbb{Z}$, $\mathrm{diam}(g^k(\mathcal{H}_n)) < r_n$ with $r_n \to 0$ as $n \to \infty$.

(b) $g^j(\mathcal{H}_n) = \mathcal{H}_n(n \in \mathbb{N})$, for some $j > 0$, and

(c) $\mu(\mathcal{H}_n) > 0$, for all $n \in \mathbb{N}$, where μ is the Lebesgue measure on M.

As in the above items (a), (b) and (c), there is a hyperbolic periodic point $p_n \in \mathcal{H}_n$ such that $\mathcal{H}_n \subset \Gamma_{2r_n}(p_n, g)$ and $\mathrm{diam}(g^k(\mathcal{H}_n)) < r_n(\forall k \in \mathbb{Z})$ with $r_n \to 0$ as $n \to \infty$ and $\mu(\mathcal{H}_n) > 0$. Thus if a diffeomrphism $f : M \to M$ has a homoclinic tangency associated to $p \in Per_h(f)$ then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that μ is not Cw-expansive. $\square$

Theorem 1.4. For a diffeomorphism $f : M \to M$ of a two dimensional manifold M, if f has a homoclinic tangency associated to $p \in Per_h(f)$, then there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not Cw-expansive.

Proof. As the result of Shin [17], a nonatomic Borel probability measure μ on M is Cw-expansive if and only if f is Cw-expansive. So, according to Lemma 1.3, we have that there is a diffeomorphism $g : M \to M$ with C^1 close to f such that g is not Cw-expansive. $\square$

References

[1] A. Artigue and D. Carrasco-Olivera, *A note on measure-expansive diffeomorphisms*, J. Math. Anal. Appl., **428** (2015), 713-716.

[2] R. Bowen, *A horseshoe with positive measure*, Invent. Math., **29**(20175), 203-204.

[3] A. Fakhari, C. A. Morales and K. Tajbakhsh, *Asymptotic measure expansive diffeomorphisms*, J. Math. Anal. Appl., **435**(2016), 1682-1687.

[4] H. Kato, Continuum-wise expansive homeomorphisms, Can. J. Math., **45** (1993), 576-598.

[5] M. Lee, *Continuum-wise expansive diffeomorphisms and conservative systems*, J. Inequal. Appl., 2014, Article number: 379 (2014).

[6] M. Lee, *Weak measure expansiveness for partially hyperbolic diffeomorphisms*, Chaos, Solitons & Fractals, **103** (2017), 256-260.

[7] M. Lee, *Continuum-wise expansiveness for generic diffeomorphisms*, Nonlearity, **31**(2018), 2982-2988.

[8] M. Lee, *Measure expansiveness for generic diffeomorphisms*, Dynamic Syst. Appl., **27** (2018), 629-635.

[9] M. Lee, *Continuum-wise expansiveness for discrete dynamical systems*, Revist. Real Acad. Cie. Exactas, Fisicas y Natur. Serie A. Matema., **115**, Article number: 113 (2021).

[10] M. Lee, *Asymptotic measure-expansiveness for generic diffeomorphisms*, Open Math., **19** (2021), 470-476.

[11] M. Lee amd J. Park, *Measure expansive symplectic diffeomorphisms and Hamiltonian systems*, International J. Math., **27**, 1650077 (2016).

[12] C. A. Morales and V. F. Sirvent, *Expansive measures*, In: 29 Col. Brasil. Matem., 2013.

[13] M. J. Pacifico and J. L. Vieitez, *On measure expansive diffeomorphisms*, Pro. Amer. Math. Soc., **143**(2015), 811-819.

[14] C. Robinson and L. S. Young, *Nonabsolutely continuous folations for an Anosov diffeomorphisms*, Invent. Math., **61**(1980), 159-176.

[15] K. Sakai, *Continuum-wise expansive diffeomorphisms*, Publ. Mat., **41** (1997), 375-382.

[16] K. Sakai, N. Sumi, and K. Yamamoto, *Measure-expansive diffeomorphisms*, J. Math. Anal. Appl., **414** (2014), 546-552.

[17] B. Shin, *Continuum-wise expansive measures*, J. Math. Anal. Appl., **506** (2022), 125551.

Manseob Lee
Department of Marketing Big Data and Mathematics
Mokwon National University
Daejeon 302-729, Korea
E-mail: `lmsds@mokwon.ac.kr`