JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 2, May 2023
http://dx.doi.org/10.14403/jcms.2023.36.2.87

FINITE DIMENSIONAL SUBSPACES OF A BESOV SPACE

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Abstract. It is concerned with the size of bounded functions in Besov spaces. It reports that every closed subspace consisted with bounded functions in Besov spaces $B^s_{p,p}(\mathbb{T}^d)$ $(s<0)$ is finite dimensional.

1. Main results and notations

An early version of Besov spaces was first studied by Taibleson as a generalization of Lipschitz spaces. These spaces were named after Besov, who obtained a trace theorem and embeddings of them [1, 2]. Then the special (modern) Besov space $B^s_{2,2}$ was considered by Hörmander [4], and the general cases of Besov spaces $B^s_{p,q}$ were introduced by Peetre in the connection with modern Littlewood-Paley theory [5].

Littlewood-Paley theory gives a unified perspective to the theory of function spaces. That is to say, well-known spaces such as Lebesgue, Hardy, Sobolev, Lipschitz spaces, and the space of BMO are all special cases of either Besov spaces $B^s_{p,q}$ or Triebel-Lizorkin spaces $F^s_{p,q}$.

One of the main merits of Besov spaces $B^s_{p,q}$ is the effectiveness of measuring regularity of functions. Based on the Sobolev borderline, the super critical Besov spaces contain some unbounded functions. We are interested in the *size* of unbounded functions (and so the size of bounded functions) in super critical Besov spaces $B^s_{p,q}$. As an our first report in this direction, we present that every closed subspace consisted of bounded functions in Besov spaces $B^s_{p,p}(\mathbb{T}^d)$ $(s<0)$ is finite dimensional. We state our results as follows.

Theorem 1.1. *Let $1 \le p \le \infty$ and $s<0$. Let V be a subspace of a Besov space $B^s_{p,p}(\mathbb{T}^d)$. Suppose that there is a positive constant K such*

Received April 15, 2023; Accepted May 16, 2023.
2020 Mathematics Subject Classification: 42B35, 30H25.
Key words and phrases: Besov space, Littlewood-Paley decomposition, finite dimensional subspace.

that for every $f \in V$

$$\|f\|_{L^\infty} \leq K \|f\|_{B^s_{p,p}}. \tag{1.1}$$

Then the dimension of V *is finite.*

Theorem 1.1 illustrates:

Corollary 1.2. *Let* $1 \leq p \leq \infty$ *and* $s < 0$. *Let* V *be a subspace of the space* $L^\infty(\mathbb{T}^d) \cap B^s_{p,p}(\mathbb{T}^d)$. *If* V *is a closed subspace of* $B^s_{p,p}(\mathbb{T}^d)$, *then the dimension of* V *is finite.*

Our results are a Besov space version of the theorem of Grothendieck for the Lebesgue spaces.

We present some notations used in this report. We choose a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$, and $\chi = 1$ for $|\xi| \leq \frac{3}{4}$. Set

$$h_j(\xi) := \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi), \quad j \geq 0$$

and $h_{-1} = \chi$.

The d-torus $\mathbb{T}^d$ is the cube $[0,1]^d$ with opposite sides identified. This means that the points $(x_1, \cdots, 0, \cdots, x_d)$ and $(x_1, \cdots, 1, \cdots, x_d)$ are identified whenever 0 and 1 appear in the same coordinate. More precisely, for $x, y \in \mathbb{R}^d$, we say that $x \equiv y$ if $x - y \in \mathbb{Z}^d$. Here $\mathbb{Z}^d$ is the additive subgroup of all points in $\mathbb{R}^d$ with integer coordinates. The d-torus $\mathbb{T}^d$ is then defined as the set $\mathbb{R}^d/\mathbb{Z}^d$ of all such equivalence classes.

For a tempered distribution f on the torus $\mathbb{T}^d$, we consider the Littlewood-Paley projections

$$\Delta_j f = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) h_j(k) e^{2\pi i k \cdot x}, \quad j \geq -1,$$

where the k-th Fourier coefficient $\hat{f}(k)$ of f is defined by

$$\hat{f}(k) = \int_{\mathbb{Z}^d} f(x) e^{-2\pi i k \cdot x} dx.$$

We define $\Delta_j f = 0$ for $j \leq -2$ for simplicity. Then we have an analog of a partition of unity:

$$f = \sum_{j=-1}^{\infty} \Delta_j f$$

in the sense of distributions. We also note that

$$\Delta_j f = \Delta_j \sum_{k=-1}^{\infty} \Delta_k f = \Delta_j \Delta_{j-1} f + \Delta_j \Delta_j f + \Delta_j \Delta_{j+1} f. \tag{1.2}$$

For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the (nonhomogeneous) Besov space $B^s_{p,q}(\mathbb{T}^d)$ is the space of all tempered distributions f on the torus $\mathbb{T}^d$ obeying

$$\|f\|_{B^s_{p,q}} := \left\| \left\{ \left\| 2^{js} \Delta_j f \right\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q} < \infty. \tag{1.3}$$

2. The arguments

We first prove Theorem 1.1. Let $\{\phi_1, \cdots, \phi_n\}$ be a linearly independent set in V with

$$\|\phi_k\|_{B^s_{p,p}} = 1, \quad 1 \leq k \leq n.$$

We are going to demonstrate the fact that the number n is less than or equal to a constant C which is independent of n. Let $N_{p'}(r)$ be the finite dimensional p'-ball of radius r :

$$N_{p'}(r) = \left\{ (c_1, c_2, \cdots, c_n) : \left(\sum_{i=1}^n |c_i|^{p'} \right)^{1/p'} < r \right\} \quad \text{if} \quad 1 \leq p' < \infty$$

and

$$N_\infty(r) = \left\{ (c_1, c_2, \cdots, c_n) : \max_{1 \leq i \leq d} |c_i| < r \right\}$$

of $\mathbb{C}^n$ or $\mathbb{R}^n$ depending on the scalar field of the vector space $B^s_{p,p}(\mathbb{T}^d)$. We choose p' with $\frac{1}{p} + \frac{1}{p'} = 1$ and then select the greatest lower bound $r_0 > 0$ of the positive real numbers satisfying $N_1(r_0)$ contains $N_{p'}(1)$ in a finite dimensional Euclidean space $\mathbb{C}^n$ or $\mathbb{R}^n$. Let Q be a countable dense subset of $N_1(r_0)$. For $\mathbf{c} = (c_1, \cdots, c_n) \in N_1(r_0)$, we define

$$f_{\mathbf{c}} := \sum_{i=1}^n c_i \phi_i.$$

Then for any $j \geq -1$, we have

$$2^{js} \|\Delta_j f_{\mathbf{c}}\|_{L^p} \leq \sum_{i=1}^n |c_i| \cdot 2^{js} \|\Delta_j \phi_i\|_{L^p}.$$

The identity (1.2) implies that for any $j \geq -1$,

$$\begin{aligned}\|\Delta_j \phi_i\|^p_{B^s_{p,p}} &= 2^{s(j-1)p} \|\Delta_j \Delta_{j-1} \phi_i\|^p_{L^p} \\ &\quad + 2^{sjp} \|\Delta_j \Delta_j \phi_i\|^p_{L^p} + 2^{s(j+1)p} \|\Delta_j \Delta_{j+1} \phi_i\|^p_{L^p} \\ &\leq C_1^p \, \|\phi_i\|^p_{B^s_{p,p}}\end{aligned}$$

for some absolute positive constant C_1. Then since $\|\phi_i\|_{B^s_{p,p}} = 1$ $(1 \leq i \leq n)$, we have

$$\|\Delta_j f_{\mathbf{c}}\|_{B^s_{p,p}} \leq C_1 \sum_{i=1}^{n} |c_i| \cdot \|\phi_i\|_{B^s_{p,p}} \leq r_0 C_1 \tag{2.1}$$

and so we obtain that from (1.1)

$$\|\Delta_j f_{\mathbf{c}}\|_{L^\infty} \leq K r_0 C_1.$$

Since Q is countable, we can extract a subset T from $\mathbb{T}^d$ satisfying the measure of $\mathbb{T}^d - T$ is zero and

$$|\Delta_j f_{\mathbf{c}}(x)| \leq K r_0 C_1 \tag{2.2}$$

for every $\mathbf{c} \in Q$ and for every $x \in T$. The estimate (2.2), in fact, holds for all $\mathbf{c} \in N_{p'}(1) \subset N_1(r_0)$ and $x \in T$ due to the fact that for a fixed $x \in T$, the map

$$\mathbf{c} \mapsto \Delta_j f_{\mathbf{c}}(x)$$

is a continuous function on $N_1(r_0)$ (hence on $N_{p'}(1)$). It follows that

$$\begin{aligned}\sum_{i=1}^{n} 2^{sjp} |\Delta_j \phi_i(x)|^p &= 2^{sjp} \left[\sup\left\{|\Delta_j f_{\mathbf{c}}(x)| : \mathbf{c} \in N_{p'}(1)\right\}\right]^p \\ &\leq (2^{sj} K r_0 C_1)^p \end{aligned} \tag{2.3}$$

for every $x \in T$.

Case 1 : $1 \leq p < \infty$. The integration of the inequality (2.3) on $\mathbb{T}^d$ gives

$$\sum_{i=1}^{n} 2^{sjp} \|\Delta_j \phi_i(x)\|^p_{L^p} \leq (2^{sj} K r_0 C_1)^p |\mathbb{T}^d| \tag{2.4}$$

where $|\mathbb{T}^d|$ is the volume of the torus $|\mathbb{T}^d|$. This implies that

$$\begin{aligned} n = \sum_{i=1}^{n} \|\phi_i\|^p_{B^s_{p,p}} &= \sum_{i=1}^{n} \sum_{j=-1}^{\infty} 2^{sjp} \|\Delta_j \phi_i\|^p_{L^p} \\ &\leq \frac{(K r_0 C_1)^p}{2^{sp}(1 - 2^{sp})} |\mathbb{T}^d| := C. \end{aligned}$$

Case 2 : $p = \infty$. From the estimate (2.3), we have

$$n = \sum_{i=1}^{n} \|\phi_i\|_{B^s_{\infty,\infty}} \leq \sup_{-1 \leq j < \infty} (2^{sj} K r_0 C_1)^p |\mathbb{T}^d| := C.$$

We conclude that the dimension of V should be less than or equal to C. This completes the proof of Theorem 1.1. □

We now provide the proof of Corollary 1.2.

Proof of Corollary 1.2: By virtue of the closed graph theorem, the identity map from $B^s_{p,p}(\mathbb{T}^d)$ to $L^\infty(\mathbb{T}^d)$ is continuous. Therefore there is a constant $K > 0$ such that

$$\|f\|_{L^\infty} \leq K\|f\|_{B^s_{p,p}}$$

for every $f \in V$. $\square$

Acknowledgement

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2021R1I1A305247113).

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