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THE REICH TYPE CONTRACTION IN A WEIGHTED $b_\nu(\alpha)$-METRIC SPACE

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Abstract. In this paper, the concept of a weighted $b_\nu(\alpha)$-metric space is introduced as a generalization of the $b_\nu(s)$-metric space and ν-metric space. We prove some fixed point results of the Reich-type contraction in the weighted $b_\nu(\alpha)$-metric space. Furthermore, we generalize Reich's theorem by extending the result to a weighted $b_\nu(\alpha)$-metric space.

1. Introduction

In 1968, Kannan studied the following fixed point theorem, which is a generalization of Banach contraction principle and the mapping satisfying the contractive condition is known as Kannan-type contraction, which is interesting since the contraction mapping does not need to be continuous [3].

Definition 1.1. Let (X, d) be a metric space, $x_0 \in X$ and $T : X \to X$ be a given mapping. The sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$x_n = Tx_{n-1} = T^n x_0$$

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is the Picard iterative sequence, for $n \in \mathbb{N}$.

Theorem 1.2. ([3]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists $k < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leq k\left(d(x, Tx) + d(y, Ty)\right) \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z.

In 1971, Reich extended the Banach and Kannan fixed point theorems as follows [15].

Theorem 1.3. ([15]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists $a, b, c \geq 0$, $a + b + c < 1$ satisfying*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) \tag{1.2}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z.

Some authors explored the above line of thought by generalizing the type of contraction mappings while other authors explored the idea of generalizing the underlying space (see [11]). In 2000, Branciari in [1], introduced the following concept.

Definition 1.4. ([1]) Let X be a set and $d : X \times X \to [0, \infty)$ be a function that satisfies the following:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$ for all $x, u_1, u_2, ..., u_\nu, y \in X$ such that $x, u_1, u_2, ..., u_\nu, y$ are all different.

Then (X, d) is called a ν-generalized metric.

Suzuki et al. [17], provided a proof of the following fixed point theorem which is a generalization of the Banach contraction principle in ν-generalized metric space. Recent articles on fixed points results on contraction and Suzuki type mappings can be found in [4, 5, 6, 7, 8, 13].

Theorem 1.5. ([17]) *Let (X, d) be a complete ν-generalized metric space and let T be a contraction on X, that is, there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.3}$$

for $x, y \in X$. Then T has a unique fixed point in X.

2. Preliminaries

In [9], the authors introduced the concept of a $b_\nu(s)$-metric space as follows:

Definition 2.1. ([9]) Let X be a set and $d : X \times X \to [0,\infty)$ be a function that satisfies the following:

(i) $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(iii) there exists a real number $s \geq 1$ such that

$$d(x,y) \leq s[d(x,u_1) + d(u_1,u_2) + \cdots + d(u_\nu, y)]$$

for all $x, u_1, u_2, ..., u_\nu, y \in X$ such that $x, u_1, u_2, ..., u_\nu, y$ are all different and $\nu \in \mathbb{N}$.

Then (X,d) is called a $b_\nu(s)$-metric space.

In this paper, we introduce the concept of a weighted $b_\nu(\alpha)$-metric space.

Definition 2.2. Let X be a set and $d : X \times X \to [0,\infty)$ be a function that satisfies the following:

(i) $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(iii) there exists constants $\alpha_i \geq 1$ for all $i = 1, 2, \cdots, \nu$ such that

$$d(x,y) \leq \alpha_1 d(x,u_1) + \alpha_2 d(u_1,u_2) + \cdots + \alpha_\nu d(u_\nu, y)$$

for all $x, u_1, u_2, ..., u_\nu, y \in X$ such that $x, u_1, u_2, ..., u_\nu, y$ are all different and $\nu \in \mathbb{N}$.

Then (X,d) is called a weighted $b_\nu(\alpha)$-metric space, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\nu)$.

If $\alpha_i = 1$ for all $i = 1, \cdots, \nu$, then the weighted $b_\nu(\alpha)$-metric is a ν-generalized metric. If $\alpha_i = s$ for all $i = 1, \cdots, \nu$, then the weighted $b_\nu(\alpha)$-metric is a $b_\nu(s)$-generalized metric. If $\nu = 1$, then $b_1(\alpha)$, $\alpha = (\alpha_1, \alpha_2)$ is a generalized b-metric space introduced in [16].

Example 2.3. Let $X = (1,3)$. If we define

$$d(x,y) = \begin{cases} e^{|x-y|}, & if \quad x \neq y, \\ 0, & if \quad x = y, \end{cases} \tag{2.1}$$

then, the properties (i) and (ii) of Definition 2.2 can be easily verified. It remains to show property (iii) holds: Let $x, u_i, y \in X$ for $i = 1, 2, \cdots, \nu$. Then by Jensen's inequality [2],

$$\begin{aligned} d(x,y) &= e^{|x-y|} \\ &\le e^{|x-u_1|+|u_1-u_2|+\cdots+|u_\nu-y|} \\ &= e^{\frac{2}{\nu(\nu+1)}|x-u_1|+\frac{4}{\nu(\nu+1)}|u_1-u_2|+\cdots+\frac{2\nu}{\nu(\nu+1)}|u_\nu-y|} \\ &\quad \times e^{\left(1-\frac{2}{\nu(\nu+1)}\right)|x-u_1|+\cdots+\left(1-\frac{2\nu}{\nu(\nu+1)}\right)|u_\nu-y|} \\ &\le e^{2\nu-2}\left\{\frac{2}{\nu(\nu+1)}e^{|x-u_1|}+\cdots+\frac{2\nu}{\nu(\nu+1)}e^{|u_\nu-y|}\right\} \\ &= \alpha_1 d(x,u_1)+\alpha_2 d(u_1,u_2)+\cdots+\alpha_\nu d(u_\nu,y), \end{aligned}$$

where $\alpha_i = \frac{e^{2\nu-2}2i}{\nu(\nu+1)} \ge 1$ for all $i = 1, 2, \cdots, \nu$ and $\nu \ge 1$. It follows that (X, d) is a weighted $b_\nu(\alpha)$-metric space.

In Example 2.3, if we take $s = \max_{i=1,2,\cdots,\nu}\left\{\frac{e^{2\nu-2}2i}{\nu(\nu+1)}\right\}$ for $\nu \ge 1$, then d is a $b_\nu(s)$-metric.

Definition 2.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a weighted $b_\nu(\alpha)$-metric space.

(a) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

(b) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X if for $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0.$$

3. Main result

The following theorem is the analogue of the Reich contraction principle found in [14], in a weighted $b_\nu(\alpha)$-metric space, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\nu)$.

Theorem 3.1. *Let (X, d) be a complete $b_\nu(\alpha)$ metric space and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \le ad(x, y) + bd(x, Tx) + cd(y, Ty) \tag{3.1}$$

for all $x, y \in X$, where a, b, c are nonnegative constants with $a + b + c < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x = x_{n-1}$, $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le ad(x_{n-1}, x_n) + bd(x_{n-1}, Tx_{n-1}) + cd(x_n, Tx_n) \\ &= ad(x_{n-1}, x_n) + bd(x_{n-1}, x_n) + cd(x_n, x_{n+1}). \end{aligned}$$

It follows that

$$(1-c)d(x_n, x_{n+1}) \leq (a+b)d(x_{n-1}, x_n).$$

That is,

$$d(x_n, x_{n+1}) \leq \frac{(a+b)}{(1-c)} d(x_{n-1}, x_n), \tag{3.2}$$

where $\mu := \frac{(a+b)}{(1-c)} < 1$. Repeated use of inequality (3.2), we get

$$d(x_{n+1}, x_n) \leq \mu^n d(x_0, x_1) \tag{3.3}$$

for $n \geq 1$. Since $\mu < 1$, we obtain that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For $m > \nu \in \mathbb{N}$, we get

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha_{\nu+1} d(x_{n+\nu}, x_{n+\nu+1}) \\
&\leq \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha_{\nu+1} \left[\beta_{\nu+1} d(x_{n+\nu}, x_{n+\nu+1}) \right. \\
&\quad \left. + \beta_{\nu+2} d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + \beta_m d(x_{n+m-1}, x_{n+m})\right] \\
&\leq \gamma \left\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+\nu}, x_{n+\nu+1}) \right. \\
&\quad \left. + d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + d(x_{n+m-1}, x_{n+m})\right\} \\
&\leq \gamma \mu^n \left(1 + \mu + \cdots + \mu^{m-1}\right) d(x_0, x_1) \\
&\leq \gamma \frac{\mu^n}{1-\mu} d(x_0, x_1),
\end{aligned}$$

where $\gamma = \max_{1\leq i\leq \nu+1, \nu+1\leq j\leq m} \{\alpha_i, \alpha_{\nu+1}\beta_j\}$. Since $\mu < 1$, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X, d) is a complete $b_\nu(\alpha)$-metric space, there exists $x' \in X$ such that $d(x_n, x') \to 0$ as $n \to \infty$.

Now, we show that x' is a fixed point for T. Using inequality (3.1), we get

$$\begin{aligned}
d(x', Tx') &\leq \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha d(x_{n+\nu}, Tx') \\
&= \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha_\nu d(Tx_{n+\nu-1}, Tx') \\
&\leq \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha_\nu \left(a d(x_{n+\nu-1}, x') + b d(x_{n+\nu-1}, x_{n+\nu}) + c d(x', Tx')\right).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$d(x', Tx') \leq c d(x', Tx'). \tag{3.4}$$

Since $c < 1$, inequality is true, only if $d(x', Tx') = 0$, that is, $Tx' = x'$. For uniqueness, let x'' be another fixed point of T. Then

$$\begin{aligned} d(x', x'') &= d(Tx', Tx'') \\ &\leq ad(x', x'') + bd(x', Tx') + cd(x'', Tx'') \\ &\leq ad(x', x''). \end{aligned} \tag{3.5}$$

This is a contradiction, unless $d(x', x'') = 0$, that is, $x' = x''$. □

We have modified the class of functions introduced by Rakotch [12], in the following definition.

Definition 3.2. ([10, 12]) Let (X, d) be a weighted $b_\nu(\alpha)$-metric space. Define the family of functions $f : [0, \infty) \to (0, 1)$ which satisfies the following properties

(i) $f(x, y) = f(d(x, y))$,
(ii) f is a monotonically non-decreasing continuous function.

Denote this family of altering distance functions by $\mathbb{F}$.

Example 3.3. Define $\psi : [0, \infty) \to (0, 1)$ by

$$\psi(x) = \frac{e^x}{e^x + 1}.$$

Then ψ is a monotonically increasing function: for $x \leq y$, the exponential function is increasing thus, we get that $e^x \leq e^y$. It follows that $e^x + e^{x+y} \leq e^y + e^{x+y}$ which implies that $\frac{e^x}{1+e^x} \leq \frac{e^y}{1+e^y}$ thus ψ is an increasing function. For all $x \geq 0$, we have that $e^x < 1 + e^x$ since $e^x > 0$, which implies that $\frac{e^x}{1+e^x} < 1$, thus $0 < \psi(x) < 1$. It follows that $\psi \in \mathbb{F}$.

In this section, we generalize the Reich's theorem found in [15, 18] to a weighted $b_\nu(\alpha)$-metric space.

Theorem 3.4. *Let (X, d) be a complete weighted $b_\nu(\alpha)$-metric space. If $T : X \to X$ is a self-mapping and there exists $a, b, c \in \mathbb{F}$ such that*

$$d(Tx, Ty) \leq a(x, y)d(x, y) + b(x, y)d(x, Tx) + c(x, y)d(y, Ty) \tag{3.6}$$

for all $x, y \in X$ and $a(x, y) + b(x, y) + c(x, y) < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Then define a sequence $\{x_n\}_{n\in\mathbb{N}}$, as $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Then, we obtain from (3.6),

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le a(x_{n-1}, x_n)d(x_{n-1}, x_n) + b(x_{n-1}, x_n)d(x_{n-1}, Tx_{n-1}) \\ &\quad + c(x_{n-1}, x_n)d(x_n, Tx_n) \\ &= a(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\quad + b(x_{n-1}, x_n)d(x_{n-1}, x_n) + c(x_{n-1}, x_n)d(x_n, x_{n+1}). \end{aligned}$$

It follows that

$$\begin{aligned} (1 - c(x_{n-1}, x_n))d(x_n, x_{n+1}) &\le (a(x_{n-1}, x_n) + b(x_{n-1}, x_n)) \\ &\quad \times d(x_{n-1}, x_n)d(x_n, x_{n+1}) \\ &\le \frac{(a(x_{n-1}, x_n) + b(x_{n-1}, x_n))}{(1 - c(x_{n-1}, x_n))}d(x_{n-1}, x_n). \end{aligned}$$

If $f(x_{n-1}, x_n) = \frac{(a(x_{n-1},x_n)+b(x_{n-1},x_n))}{(1-c(x_{n-1},x_n))}$, then $0 < f(x_{n-1}, x_n) < 1$. It follows that

$$d(x_n, x_{n+1}) \le f(x_{n-1}, x_n)d(x_{n-1}, x_n). \tag{3.7}$$

Repeated use of (3.7), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\le f(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\vdots \\ &\le f(x_{n-1}, x_n)f(x_{n-2}, x_{n-1}) \cdots f(x_0, x_1)d(x_0, x_1). \end{aligned} \tag{3.8}$$

Now, if $d(x_k, x_{k+1}) \ge \varepsilon_0$ for some $\varepsilon_0 > 0$ and $k = 0, 1, 2, \cdots, n-1$, then by the monotonicity of f, it follows that $f(d(x_k, x_{k+1})) \le f(\varepsilon_0)$. Hence, we get

$$d(x_n, x_{n+1}) \le f^n(\varepsilon_0)d(x_0, x_1).$$

Since $0 < f^n(\varepsilon_0) < 1$, then it follows that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For $m > \nu \in \mathbb{N}$, we get

$$\begin{aligned} d(x_n, x_{n+m}) &\le \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + \alpha_{\nu+1} d(x_{n+\nu}, x_{n+\nu+1}) \\ &\le \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + \alpha_{\nu+1} [\beta_{\nu+1} d(x_{n+\nu}, x_{n+\nu+1}) \\ &\quad + \beta_{\nu+2} d(x_{n+\nu+1}, x_{n+\nu+2}) + \cdots + \beta_m d(x_{n+m-1}, x_{n+m})] \end{aligned}$$

$$
\begin{aligned}
&\leq \gamma \left\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\right. \\
&\quad + \cdots + d(x_{n+\nu}, x_{n+\nu+1}) + d(x_{n+\nu+1}, x_{n+\nu+2}) \\
&\quad \left. + \cdots + d(x_{n+m-1}, x_{n+m})\right\} \\
&\leq \gamma f^n(\varepsilon_0)\left(1 + f(\varepsilon_0) + \cdots + f^{m-1}(\varepsilon_0)\right) d(x_0, x_1) \\
&\leq \gamma \frac{f^n(\varepsilon_0)}{1 - f(\varepsilon_0)} d(x_0, x_1),
\end{aligned}
$$

where $\gamma = \max_{1\leq i\leq \nu+1, \nu+1\leq j\leq m}\{\alpha_i, \alpha_{\nu+1}\beta_j\}$. Since $0 < f(\varepsilon_0) < 1$, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X, d) is a complete $b_\nu(\alpha)$-metric space, there exists $x' \in X$ such that $d(x_n, x') \to 0$ as $n \to \infty$.

Now, we show that x' is a fixed point for T. Using inequality (3.6), we get

$$
\begin{aligned}
d(x', Tx') &\leq \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) + \cdots + \alpha d(x_{n+\nu}, Tx') \\
&= \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) + \cdots + \alpha_\nu d(Tx_{n+\nu-1}, Tx') \\
&\leq \alpha_1 d(x', x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + \alpha_\nu \left(a(x_{n+\nu-1}, x')d(x_{n+\nu-1}, x')\right. \\
&\quad \left. + b(x_{n+\nu-1}, x')d(x_{n+\nu-1}, x_{n+\nu}) + c(x_{n+\nu-1}, x')d(x', Tx')\right).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain

$$
d(x', Tx') \leq c(x', x')d(x', Tx'). \tag{3.9}
$$

Since $c < 1$, inequality is true, only if $d(x', Tx') = 0$, that is, $Tx' = x'$.

For uniqueness, let x'' be another fixed point of T. Then

$$
\begin{aligned}
d(x', x'') &= d(Tx', Tx'') \\
&\leq a(x', x'')d(x', x'') +; b(x', x'')d(x', Tx') + c(x', x'')d(x'', Tx'') \\
&\leq a(x', x'')d(x', x''). \qquad (3.10)
\end{aligned}
$$

Inequality (3.10) is a contradiction, unless $d(x', x'') = 0$, that is, $x' = x''$. □

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