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DEGREE OF APPROXIMATION OF A FUNCTION ASSOCIATED WITH HARDY-LITTLEWOOD SERIES IN WEIGHTED ZYGMUND $W(Z_r^{(\omega)})$-CLASS USING EULER-HAUSDORFF SUMMABILITY MEANS

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Abstract. Approximation of functions of Lipschitz and Zygmund classes have been considered by various researchers under different summability means. In the proposed study, we investigated an estimation of the order of convergence of a function associated with Hardy-Littlewood series in the weighted Zygmund class $W(Z_r^{(\omega)})$-class by applying Euler-Hausdorff summability means and subsequently established some (presumably new) results. Moreover, the results obtained here represent the generalization of several known results.

1. Introduction

Summability methods have been used in various fields of mathematics. For example, summability methods are applied in function theory in connection with the analytic continuation of holomorphic functions and the boundary behavior of a power series, in applied analysis for generation of iteration methods for finding solutions of a system of equations, and for acceleration of convergence in approximation theory. Also, it has been used in other fields

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of mathematics like probability theory (Markov chains) and number theory (Prime number theorem). The approximation analysis of signals (functions) has great importance in the field of science and engineering. It has also given a new aspect due to its exhaustive appliance in signal analysis, system design, modern telecommunications, radar and image processing system. The error estimation of functions in various function spaces such as Lipschitz, Hölder, Zygmund, Besov spaces using different summability techniques of Fourier series has been received a growing interest of several researchers in the last decades. Functions in L_r $(r \geq 1)$-spaces assumed to be most practicable in signal analysis. Particularly, L_1, L_2 and L_∞ spaces are used by engineers for designing digital filters. The generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is a generalization of $Z_{(\alpha)}, Z_{(\alpha),r}, Z^{(\omega)}$-class.

The generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is investigated by Leindler [7], Moricz [9], Moricz and Nemeth [10]. Lal and Shireen [6] established results on approximation of functions of generalized Zygmund class by Matrix-Euler summability mean of Fourier series. Pradhan *et al.* [14] studied on approximation of signals belonging to generalized Lipschitz class using $(\overline{N}, p_n, q_n)(E, s)$-summability mean of Fourier series. Singh *et al.* [16] studied approximation of functions in the generalized Zygmund class using Hausdorff means. Pradhan *et al.* [13] studied on approximation of signals in the generalized Zygmund class via $(E, 1)(\overline{N}, p_n)$ summability means of conjugate Fourier series. In 2019, Pradhan *et al.* [15] studied approximation of signals using generalized Zygmund class using $(E, 1)(\overline{N}, p_n)$ summability means of Fourier series. Das *et al.* [1] proved approximation of functions in the weighted Zygmund class via Euler-Hausdörff product summability means of Fourier series. Again, in 2020, Padhy *et al.* [12] estimated the degree of approximation of functions of generalized Zygmund class associated with Hardy-Littlewood series using Riesz mean. Very recently in 2023, Jena *et al.* [4] studied on the degree of approximation of Fourier series based on a certain class of product deferred summability means.

Motivated by the above mentioned works, to get best approximation and advance study in this direction, In this proposed paper, we give an estimation of degree of approximation of functions associated with Hardy-Littlewood series in weighted Zygmund class using Euler-Hausdorff summability means.

2. Preliminaries

Let $f(x)$ be a periodic function of period 2π, which is Lebesgue integrable in $[-\pi, \pi]$ and Fourier series associated with $f(x)$ is given by

$$\sum_{n=0}^{n} A_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.1}$$

Let $S_n^M(x)$ denotes the nth partial sum of the (2.1) is given by

$$S_n^M(x) = \sum_{k=0}^{n-1} A_n(x) + \frac{A_n(x)}{2}. \tag{2.2}$$

Then the Hardy-Littlewood series or HL-series associated with $f(x)$ is given by

$$\frac{c_0}{2} + \sum_{n=1}^{\infty} \frac{S_n^M(x) - f(x)}{n}, \tag{2.3}$$

where

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \phi(x,u)\frac{u}{2}\cot\frac{u}{2}du$$

and $\phi(x,u) = f(x+u) + f(x-u) - 2f(x)$. Let

$$\eta(u) = \int_u^{\pi} \phi(x,u)\frac{1}{2}\cot\frac{u}{2}du. \tag{2.4}$$

Clearly, $\eta(u)$ is an even function and Lebesgue integrable in $[-\pi,\pi]$. Also, the HL-series (2.3) is the Fourier series of $\eta(u)$ at $u=0$.

Let us write $\xi_n(f,x) = \frac{2}{\pi}\int_0^{\pi}\eta(u)\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}}du$, which represent the n^{th} partial sum of $\eta(u)$.

The L_r norm of a function η is defined by

$$\|\eta\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|\eta(x)|^r dx\right)^{\frac{1}{r}}, \ 1 \le r < \infty, \\ ess\sup_{0<x\le 2\pi}|\eta(x)|, \ r = \infty. \end{cases}$$

The degree of approximation of a function $\eta : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial t_n of order n under $\|\cdot\|_{L_\infty}$ norm is defined as

$$\|t_n - \eta(x)\|_{L_\infty} = \sup_{x\in\mathbb{R}}|t_n(x) - \eta(x)|$$

and let a function $\eta \in L_r$, its degree of approximation $E_n(\eta)$ is given by

$$E_n(\eta) = \min_{t_n}\|t_n - \eta\|_{L_r}.$$

Zygmund modulus of continuity [18] of η is defined by

$$\omega(\eta,h) = \sup_{o\le h, x\in\mathbb{R}}|\eta(x+t) + \eta(x-t) - 2\eta(x)|.$$

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[0, 2\pi]$ under the supremum norm. For $0 < \alpha \leq 1$, the function space

$$Z_{(\alpha)} = \{\eta \in \mathbb{C}_{2\pi} : |\eta(x+t) + \eta(x-t) - 2\eta(x)| = O(|t|^{\alpha})\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha)}$ is defined by

$$\|\eta\|_{(\alpha)} = \sup_{0\leq x\leq 2\pi} |\eta(x)| + \sup_{x,t\neq 0} \frac{|\eta(x+t) + \eta(x-t) - 2\eta(x)|}{|t|^{\alpha}}.$$

For $\eta \in L_r[0, 2\pi], r \geq 1$, the integral Zygmund modulus of continuity is defined by

$$\omega_r(\eta, h) = \sup_{0<t\leq h} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\eta(x+t) + \eta(x-t) - 2\eta(x)|^r dx \right\}^{\frac{1}{r}}.$$

Moreover, for $\eta \in \mathbb{C}_{2\pi}$ and $r = \infty$,

$$\omega_\infty(\eta, h) = \sup_{0<t\leq h} \max_x |\eta(x+t) + \eta(x-t) - 2\eta(x)|.$$

Also, it is known that $\omega_r(\eta, h) \to 0$ as $r \to 0$.

We now define,

$$Z_{(\alpha),r} = \left\{ \eta \in L_r[0, 2\pi] : \left(\int_0^{2\pi} |\eta(x+t) + \eta(x-t) - 2\eta(x)|^r dx \right)^{\frac{1}{r}} = O(|t|^{\alpha}) \right\}.$$

The space $Z_{(\alpha),r}, r \geq 1, 0 < \alpha \leq 1$ is a Banach space under the norm $\|\cdot\|_{(\alpha),r}$ and that,

$$\|\eta\|_{(\alpha),r} = \|\eta\|_r + \sup_{t\neq 0} \frac{\|\eta(\cdot+t) + \eta(\cdot-t) - 2\eta(\cdot)\|_r}{|t|^{\alpha}}.$$

The class of function $Z^{(\omega)}$ is defined as

$$Z^{(\omega)} = \{\eta \in \mathbb{C}_{2\pi} : |\eta(x+t) + \eta(x-t) - 2\eta(x)| = O(\omega(t))\},$$

where ω is a Zygmund modulus of continuity, that is, ω is positive, non-decreasing continuous function with the sub linearity property, that is,

(i) $\omega(0) = 0$,
(ii) $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.

Let $\omega : [0, 2\pi] \to \mathbb{R}$ be an arbitrary function with $\omega(t) > 0$ for $0 \leq t < 2\pi$ and let $\lim_{t\to 0^+} \omega(t) = \omega(0) = 0$, define

$$Z_r^{(\omega)} = \left\{ \eta \in L_r : 1 \leq r \leq \infty, \sup_{t\neq 0} \frac{\|\eta(\cdot+t) + \eta(\cdot-t) - 2\eta(\cdot)\|_r}{\omega(t)} < \infty \right\},$$

where

$$\|\eta\|_r^{(\omega)} = \|\eta\|_r + \sup_{t\neq 0} \frac{\|\eta(\cdot+t)+\eta(\cdot-t)-2\eta(\cdot)\|_r}{\omega(t)},\ r\geq 1.$$

Then, clearly $\|\cdot\|_r^{(\omega)}$ is a norm on $Z_r^{(\omega)}$. As we know L_r $(r\geq 1)$ is complete, the space $Z_r^{(\omega)}$ is also complete. Hence we can say $Z_r^{(\omega)}$ is a Banach space under the norm $\|\cdot\|_r^{(\omega)}$.

Now we define the weighted Zygmund class as

$$W(Z_r^{(\omega)}) = \left\{\eta\in W(Z_r^{(\omega)}) : 1\leq r\leq\infty, \sup_{t\neq 0} \frac{\|(\eta(\cdot+t)+\eta(\cdot-t)-2\eta(\cdot))\sin^{\beta}(\cdot)\|_r}{\omega(t)}\leq\infty\right\}, \tag{2.5}$$

where

$$\|\eta\|_r^{(\omega)} = \|\eta\|_r + \sup_{t\neq 0} \frac{\|(\eta(\cdot+t)+\eta(\cdot-t)-2\eta(\cdot))\sin^{\beta}(\cdot)\|_r}{\omega(t)},\ r\geq 1. \tag{2.6}$$

Clearly, $\|\cdot\|_r^{*(\omega)}$ is a norm of $Z_r^{(\omega)}$. The space $Z_r^{(\omega)}$ is complete because $L_r,\ r\geq 1$ is complete. Hence, we can say that $W(Z_r^{(\omega)})$ is complete.
As $Z_r^{(\omega)}$ is a Banach space under $\|\cdot\|_r^{(\omega)}$, so $W(Z_r^{(\omega)})$ is also a Banach space under $\|\cdot\|_r^{(\omega)}$ norm.

(i) If we put $\beta=0$ in $W(Z_r^{(\omega)})$ class, then it reduces to $Z_r^{(\omega)}$ class.
(ii) If we put $r\to\infty$, then the class $Z_r^{(\omega)}$ reduces to the $Z^{(\omega)}$ class.
(iii) If we put $\omega(t)=t^{\alpha}$ in $Z_r^{(\omega)}$ class, then it reduces to $Z_{(\alpha),r}$ class.
(iv) If we put $\omega(t)=t^{\alpha}$, the $Z^{(\omega)}$ class reduces to $Z_{(\alpha)}$ class.

Here $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\left(\frac{\omega(t)}{v(t)}\right)$ is positive, non-decreasing, then

$$\|\eta\|_r^{(v)} \leq \max\left(1, \frac{\omega(2\pi)}{v(2\pi)}\right)\|\eta\|_r^{(\omega)}\leq\infty.$$

Thus, we have

$$Z_r^{(\omega)}\subseteq Z_r^{(v)}\subseteq L_r\ (r\geq 1).$$

Hence,

$$W(Z_r^{(\omega)})\subseteq W(Z_r^{(v)})\subseteq W(L_r,\omega(t)).$$

Hausdorff matrices were first introduced by Hurwitz and Silverman [3] as the collection of lower triangular matrices that commute the Cesaro matrix of order one. A Hausdorff matrix $H \equiv (h_{n,k})$ is an infinite lower matrix defined by

$$h_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\mu_k, \text{ if } 0 \le k \le n, \\ 0, \text{ if } k > n, \end{cases}$$

where the operator Δ is defined by $\Delta\mu_n \equiv \mu_n - \mu_{n+1}$ and $\Delta^{k+1}\mu_n \equiv \Delta^k(\Delta\mu_n)$.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with partial sum $s_n = \sum_{k=0}^{n} u_k$. If $t_n^H = \sum_{k=0}^{n} h_{n,k}s_k \to s$ as $n \to \infty$, $\sum_{k=0}^{n} u_n$ is said to be summable to s by the Hausdorff matrix summability method (Δ_H means). The Hausdorff matrix H is regular, that is,, H preserves the limit of each convergent sequence if and only if

$$\int_0^1 |d(\alpha(z))| < \infty,$$

where the mass function $\alpha \in BV[0,1]$, $\alpha(0+) = \alpha(0) = 0$ and $\alpha(1) = 1$. In this case, the μ_n has the representation

$$\mu_n = \int_0^1 z^n d\alpha z.$$

Let $E_n^{(q)} = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k,\ q > 0$. If $E_n^{(q)} \to s$ as $n \to \infty$, $\sum_{k=0}^{\infty} u_n$ is said to be summable to s by Euler method, that is, the (E,q) method (see [1]). The (E,q) transform of t_n^H transform defines the $E^{(q)} \cdot \Delta_H$ transform of (s_n). It is denoted by T_n^{EH}. Thus,

$$T_n^{EH} = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}h_{k,v}s_v.$$

If $T_n^{EH} \to s$ as $n \to \infty$, $\sum_{k=0}^{\infty} u_n$ is said to be summable to s by Euler-Hausdorff summability means (see [5]), that is, the $E^{(q)} \cdot \Delta_H$ means. As the Euler method and Hausdorff methods are regular, $E^{(q)} \cdot \Delta_H$ method is regular.

We use the following notations throughout the papers,

$$\phi(x,t) = f(x+t) + f(x-t) - f(x),$$

$$\begin{aligned} K_n^{EH} &= \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}h_{k,v}\frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \\ &= \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}\int_0^1 z^k(1-z)^{k-v}d\alpha(z)\frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}. \end{aligned} \tag{2.7}$$

3. Main results

To prove the main results, we need the followings lemmas.

Lemma 3.1. $|K_n^{EH}(t)| = O(n+1)$ *for* $0 \leq t \leq \frac{1}{n+1}$.

Proof. For $\sin nt \leq n \sin t$, we have

$$\begin{aligned}
|K_n^{EH}(t)| &= \left| \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 z^k (1-z)^{k-v} d\alpha(z) \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right| \\
&\leq \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 z^k (1-z)^{k-v} d\alpha(z) \left| \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right| \\
&\leq \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 z^k (1-z)^{k-v} d\alpha(z) \frac{(2v+1)\sin\frac{1}{2}t}{\sin(\frac{t}{2})} \\
&= \frac{(2n+1)N}{2\pi(1+q)^n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 z^k (1-z)^{k-v} dz \\
&= O(n+1), \qquad (3.1)
\end{aligned}$$

where $N = \sup_{0<z\leq 1} |\alpha^{'}(z)|$. □

Lemma 3.2. $|K_n^{EH}(t)| = O\left(\frac{1}{(n+1)t^2}\right)$ *for* $\frac{1}{n+1} \leq t \leq \pi$.

Proof. For $|\sin nt| = 1$ and $\sin \frac{t}{2} \geq \frac{t}{\pi}$. First we calculate,

$$\begin{aligned}
&\int_0^1 \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} \sin(n+\frac{1}{2})t \; d\alpha(z) \\
&\quad = \int_0^1 Im \left[\sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} e^{i(n+\frac{1}{2})t} \; d\alpha(z) \right] \\
&\quad = \int_0^1 Im \left[e^{i\frac{t}{2}} (1-z)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{1-z} \right)^k e^{ikt} \; d\alpha(z) \right] \\
&\quad = \int_0^1 Im \left[e^{i\frac{t}{2}} (1-z+ze^{it})^n \; d\alpha(z) \right] \\
&\quad = N \; Im \left(\frac{(1-z+ze^{it})^{n+1}}{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}(n+1)} \right)_{z=0}^{z=1} \\
&\quad = N \; Im \left(\frac{e^{i(n+1)t} - 1}{2i \sin(\frac{t}{2})(n+1)} \right)
\end{aligned}$$

$$= \frac{N}{2n+1} Im\left(\frac{\cos(n+1)t + i\sin(n+1)t - 1}{2i\sin\frac{t}{2}}\right)$$
$$= \frac{N}{2n+1}\left(\frac{\sin(n+1)t}{i\sin\frac{t}{2}}\right)$$
$$= \frac{N}{2n+1} \times \frac{\pi}{t}$$
$$= O\left(\frac{1}{(n+1)t}\right). \tag{3.2}$$

Now applying Jordan's lemma, we have

$$|K_n^{EH}(t)| = \left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}\int_0^1 z^k(1-z)^{k-v}d\alpha(z)\frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right|$$
$$\leq \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left|\sum_{v=0}^{k}\int_0^1 z^k(1-z)^{k-v}\sin(v+\frac{1}{2})t\ d\alpha(z)\right|$$
$$= \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}O\left(\frac{1}{(k+1)t}\right)$$
$$= \left(\frac{1}{2t^2(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left(\frac{1}{(k+1)}\right)\right)$$
$$= O\left(\frac{1}{(n+1)t^2}\right). \tag{3.3}$$

□

Lemma 3.3. *Let $f \in Z_r^{(\omega)}$. Then for $0 < t \leq \pi$,*

(1) $\|\phi(\cdot,t)\|_r = O(\omega(t))$,

(2) $\|\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t)\|_r = \begin{cases} O(\omega(t)), \\ O(\omega(y)), \end{cases}$

(3) *If ω and v denotes the zygmund moduli such that $(\frac{\omega(t)}{v(t)})$ is positive and increasing, then we have*

$$\|\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t)\|_r = O\left(v(y)\frac{\omega(t)}{v(t)}\right),$$

where $\phi(x,t) = f(x+t)+f(x-t)-2f(x)$.

Lemma 3.4. $\|(\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t))\sin^{\beta}(\cdot)\|_r = O\left(t^{\beta}v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).$

Proof. For v is positive, nondecreasing, $t \leq y$, $|\sin^\beta t| \leq t^\beta$ and using Lemma 3.3, we obtained

$$\begin{aligned}\|(\phi(\cdot + y,t) + \phi(\cdot - y,t) - 2\phi(\cdot,t))\sin^\beta(\cdot)\|_r &= O(t^\beta\omega(t))\\ &= O\left(t^\beta v(t)\left(\frac{\omega(t)}{v(t)}\right)\right)\\ &\leq O\left(t^\beta v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).\end{aligned}$$

Since $\frac{\omega(t)}{v(t)}$ is positive, non-decreasing, if $t \geq y$, then $\frac{\omega(t)}{v(t)} \geq \frac{\omega(y)}{v(y)}$, so that

$$\begin{aligned}\|(\phi(\cdot + y,t) + \phi(\cdot - y,t) - 2\phi(\cdot,t))\sin^\beta(\cdot)\|_r &= O(t^\beta\omega(y))\\ &= O\left(t^\beta v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).\end{aligned}$$

□

The main objective of this paper is to prove the following theorems.

Theorem 3.5. *Let η be a 2π periodic function and Lebesgue integrable on $[-\pi,\pi]$ and belonging to weighted Zygmund class $W(Z_r^{(\omega)})$, $r \geq 1$. Then the degree of approximation of signal (function) η, using Euler-Hausdorff summability means of HL-series* (2.3) *is given by*

$$E_n(\eta) = \inf \|T_n^{EH} - \eta\|_r = O\left(\frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right), \tag{3.4}$$

where ω and v denotes the zygmund moduli such that $(\frac{\omega(t)}{v(t)})$ is positive and increasing.

Proof. Following the results of Titechmalch [17], the Euler-Hausdroff transform of $\{\xi_n(f,x)\}$ and is denoted by

$$\begin{aligned}\tau_n^{EH}(x) - \eta(x) &= \frac{2}{\pi}\int_0^\pi \phi(x,t)\frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\{t_n^H(x) - \eta(x)\}\\ &= \frac{2}{\pi}\int_0^\pi \phi(x,t)\frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\\ &\quad\times\left\{\frac{1}{2\pi}\int_0^\pi \phi(x;t)\sum_{v=0}^{k}\int_0^1\binom{k}{v}z^v(1-z)^{k-v}d\alpha(z)\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}dt\right\}\\ &= \int_0^\pi \phi(x;t)K_n^{EH}(t)dt\\ &= \mathcal{L}_n(x).\end{aligned} \tag{3.5}$$

Now

$$\mathcal{L}_n(x) = T_n^{EH}(x) - f(x) = \int_0^\pi \phi(x;t) K_n^{EH}(t)dt. \tag{3.6}$$

This implies

$$\begin{aligned} &\mathcal{L}_n(x+y) + \mathcal{L}_n(x-y) - 2\mathcal{L}_n(x) \\ &\qquad = \int_0^\pi \left[\phi(x+y;t) + \phi(x-y;t) - 2\phi(x;t)\right] K_n^{EH}(t)dt, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &(\mathcal{L}_n(\cdot+y) + \mathcal{L}_n(\cdot-y) - 2\mathcal{L}_n(\cdot)) \sin^\beta(\cdot) \\ &\quad = \int_0^\pi \left((\phi(\cdot+y;t) + \phi(\cdot-y;t) - 2\phi(\cdot;t)) \sin^\beta(\cdot) \right) K_n^{EH}(t)dt. \end{aligned} \tag{3.8}$$

Now we can write

$$\begin{aligned} &\|(\mathcal{L}_n(\cdot+y) + \mathcal{L}_n(\cdot-y) - 2\mathcal{L}_n(\cdot)) \sin^\beta(\cdot)\|_r \\ &\quad = \int_0^\pi \|(\phi(\cdot+y;t) + \phi(\cdot-y;t) - 2\phi(\cdot;t)) \sin^\beta(\cdot)\|_r K_n^{EH}(t)dt \\ &\quad = \int_0^{\frac{1}{n+1}} \|(\phi(\cdot+y;t) + \phi(\cdot-y;t) - 2\phi(\cdot;t)) \sin^\beta(\cdot)\|_r K_n^{EH}(t)dt \\ &\qquad + \int_{\frac{1}{n+1}}^\pi \|(\phi(\cdot+y;t) + \phi(\cdot-y;t) - 2\phi(\cdot;t)) \sin^\beta(\cdot)\|_r K_n^{EH}(t)dt \\ &\quad := I_1 + I_2. \end{aligned} \tag{3.9}$$

Further the function $f \in W(Z_r^{(\omega)})$ implies $\phi \in W(Z_r^{(\omega)})$ and applying Lemma 3.1, Lemma 3.4 and monotonicity of $\frac{\omega(t)}{v(t)}$ with respect to t, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} \|(\phi(\cdot+y;t) + \phi(\cdot-y;t) - 2\phi(\cdot;t)) \sin^\beta(\cdot)\|_r K_n^{EH}(t)dt \\ &= O\left(\int_0^{\frac{1}{n+1}} v(y) \frac{t^\beta \omega(t)}{v(t)} (n+1) dt \right) \\ &= O\left((n+1)\ v(y) \int_0^{\frac{1}{n+1}} \frac{t^\beta \omega(t)}{v(t)} dt \right) \\ &= O\left((n+1)\ v(y) \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \int_0^{\frac{1}{n+1}} t^\beta dt \right) \\ &= O\left(\frac{1}{(n+1)^\beta}\ v(y) \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \right). \end{aligned} \tag{3.10}$$

Next, using Lemma 3.2 and Lemma 3.4, we get

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{n+1}}^{\pi} \|(\phi(\cdot+y;t)+\phi(\cdot-y;t)-2\phi(\cdot;t))\sin^{\beta}(\cdot)\|_r K_n^{EH}(t)dt \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} v(y)\frac{t^{\beta}\omega(t)}{v(t)}\frac{1}{(n+1)t^2}dt\right) \\
&= O\left(\frac{1}{(n+1)}v(y)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \qquad (3.11)
\end{aligned}
$$

Thus using (3.9), (3.10) and (3.11) we can write

$$
\begin{aligned}
&\|(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y)-2\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r \\
&= O\left(\frac{1}{(n+1)^{\beta}}\ v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(\frac{1}{(n+1)}v(y)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \qquad (3.12)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\sup_{y\neq 0}\frac{\|\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y)-2\mathcal{L}_n(\cdot)\|_r}{v(y)} \\
&= O\left(\frac{1}{(n+1)^{\beta}}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \qquad (3.13)
\end{aligned}
$$

Clearly,

$$
\phi(x;t) = |f(x+t)+f(x-t)-2f(x)|.
$$

Now applying Minkowski's inequality, we have

$$
\|\phi(x;t)\|_r = \|f(x+t)+f(x-t)-2f(x)\|_r. \qquad (3.14)
$$

Now using Lemma 3.3, we have

$$
\begin{aligned}
&\|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r \\
&\leq \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\|(\phi(\cdot,t))\sin^{\beta}(\cdot)\|_r|K_n^{EH}(t)|dt \\
&= O\left((n+1)\int_0^{\frac{1}{n+1}} t^{\beta}\omega(t)dt\right)+O\left(\frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi}t^{\beta-2}\omega(t)dt\right) \\
&= O\left((n+1)\omega(\frac{1}{n+1})\int_0^{\frac{1}{n+1}}t^{\beta}dt\right)+O\left(\frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}dt\right) \\
&= O\left(\frac{1}{(n+1)^{\beta}}w(\frac{1}{n+1})\right)+O\left(\frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}dt\right). \qquad (3.15)
\end{aligned}
$$

Now from (3.14) and (3.15), we have

$$\begin{aligned}
\|(\mathcal{L}_n(\cdot))\sin^\beta(\cdot)\|_r^v &= \|(\mathcal{L}_n(\cdot))\sin^\beta(\cdot)\|_r \\
&\quad + \sup_{y\neq 0} \frac{\|(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y)-2\mathcal{L}_n(\cdot))\sin^\beta(\cdot)\|_r}{v(y)} \\
&= O\left(\frac{1}{(n+1)^\beta} w(\frac{1}{n+1})\right) + O\left(\frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t^{2-\beta}}dt\right) \\
&\quad + O\left(\frac{1}{(n+1)^\beta}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \\
&\quad + O\left(\frac{1}{(n+1)}\int_{\frac{1}{(n+1)^2}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt\right) \\
&= \sum_{i=1}^{4} O(J_i) \text{ (say).}
\end{aligned} \tag{3.16}$$

Now we write J_1 in terms of J_3 and further J_2, J_3 in terms of J_4. In view of monotonicity of $v(t)$ for $0 < t \leq \pi$, we have

$$\omega(t) = \frac{\omega(t)}{v(t)} \cdot v(t) \leq v(\pi)\frac{\omega(t)}{v(t)} \cdot v(t) = O\left(\frac{\omega(t)}{v(t)}\right) \text{ for } 0 < t \leq \pi.$$

Therefore, we can write for $t = (n+1)^{-1}$.

$$J_1 = O(J_3). \tag{3.17}$$

Again by using monotonicity of $v(t)$,

$$\begin{aligned}
J_2 &= \frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)} v(t)dt \\
&\leq \frac{1}{(n+1)^2} v(\pi)\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt \\
&\leq \frac{1}{(n+1)^2}\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt \\
&= O(J_4).
\end{aligned} \tag{3.18}$$

Now using $\left(\frac{\omega(t)}{v(t)}\right)$ is positive and non-decreasing, we have

$$\begin{aligned}
J_4 &= \frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)} dt \\
&\geq \frac{1}{(n+1)} \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \int_{\frac{1}{n+1}}^{\pi} t^{\beta-2} dt \\
&\geq \frac{1}{(n+1)} \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \frac{1}{(n+1)^{\beta-1}} \\
&\geq \frac{1}{(n+1)^{\beta}} \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}. \qquad (3.19)
\end{aligned}$$

Hence, we have

$$J_3 = O(J_4). \qquad (3.20)$$

Now combining (3.15) and (3.20), we get

$$\|(\mathcal{L}_n(\cdot)) \sin^{\beta}(\cdot)\|_r = O\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)} dt \right). \qquad (3.21)$$

Hence,

$$E_n(f) = \inf_n \|(\mathcal{L}_n(\cdot)) \sin^{\beta}(\cdot)\|_r^{(v)} = O\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)} dt \right). \qquad (3.22)$$

This completes the proof . □

Theorem 3.6. *Let η be a 2π periodic function and Lebesgue integrable on $[-\pi, \pi]$ and belonging to weighted Zygmund class $W(Z_r^{(\omega)})$, $r \geq 1$. Then the degree of approximation of signal (function) η, using Euler-Hausdorff summability means of HL-series* (2.3) *is given by*

$$E_n(\eta) = \inf \|T_n^{EH} - \eta\|_r = O\left(\frac{1}{(n+1)} \frac{t^{\beta}\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \right), \qquad (3.23)$$

where ω and v denotes the Zygmund moduli such that $(\frac{\omega(t)}{tv(t)})$ is positive and decreasing.

Proof. Following the proof of Theorem 3.5, we have

$$E_n(f) = O\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)} dt \right). \qquad (3.24)$$

From our assumption that $\left(\frac{\omega(t)}{tv(t)}\right)$ is positive and non-increasing with t, we have

$$E_n(f) = O\left(\frac{1}{(n+1)}(n+1)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\int_{\frac{1}{n+1}}^{\pi} t^{\beta-1}dt\right)$$
$$= O\left(\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}(\pi^{\beta} - \frac{1}{(n+1)^{\beta}})\right). \tag{3.25}$$

This completes the proof. □

4. Applications

Following corollaries can be obtained from Theorem 3.5.

Corollary 4.1. *If we replace Euler-Hausdorff mean by $(E,1)(C,1)$ mean* [11] *in Theorem* 3.5*, then the degree of approximation of a function $f \in W(Z_r^{\omega})$ by $(E,1)(C,1)$ mean of HL-series* (2.3) *is given by*

$$E_n(f) = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.1}$$

Corollary 4.2. *If we replace Euler-Hausdorff mean by $(E,q)(N,p_n,q_n)$ mean* [8] *in Theorem* 3.5*, then the degree of approximation of a function $f \in W(Z_r^{\omega})$ by $(E,q)(N,p_n,q_n)$ mean of HL-series* (2.3) *is given by*

$$E_n(f) = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.2}$$

Corollary 4.3. *If we replace Euler-Hausdorff mean by Hausdorff mean* [5] *in Theorem* 3.5*, then the degree of approximation of a function $f \in W(Z_r^{\omega})$ by Hausdorff mean of HL-series* (2.3) *is given by*

$$E_n(f) = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.3}$$

5. Conclusion

There are various types of results exist in the literature concerning the degree of approximations of periodic functions belonging to different Zygmund classes and weighted Zygmund classes. The established theorem in this paper is an attempt to study the approximation of signals (functions) belonging to weighted Zygmund class via Euler-Hausdorff summability means for Hardy-Littlewood series, which generalizes several known results. Further, the result can be extended for other functions belonging to weighted Zygmund class using Fourier series, conjugate Fourier series, derived Fourier series.

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