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EXISTENCE AND UNIQUENESS OF FIXED POINT OF SOME EXPANSIVE-TYPE MAPPINGS IN GENERALIZED MODULAR METRIC SPACES

This paper is dedicated to Golden Professor J. O. Olaleru on the occasion of the award of gold medal by the University of Lagos, Akoka, Lagos, Nigeria.

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Abstract. We define new classes of expansive-type mappings in the setting of modular ω^G-metric spaces and prove the existence of common unique fixed point for these classes of expansive-type mappings on ω^G-complete modular ω^G-metric spaces. The results established in this paper extend, improve, generalize and compliment many existing results in literature. We produce some examples to validate our results.

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1. Introduction

Expansiveness of mappings and their common fixed point results is an interesting and active research aspect of fixed point theory. The class of expansive mappings in complete metric spaces were introduced by Wang *et al.* [34]. They proved some interesting fixed point results for these class of mappings, thereby activating research in expansive mappings in metric spaces and related abstract spaces.

Kumar [12] proved some interesting theorems on expansive mappings in several settings, such as metric spaces, generalized metric spaces, probabilistic metric spaces and fuzzy metric spaces, which generalized the results of some authors like Ahmad et al. [1], Rhoades [30], Kang *et al.* [11], Wang *et al.* [34] and Vasuki [33]. Kumar [12] results contained some errors, which were corrected in [5]. However, Kumma [12] did not consider expansive mappings in the framework of modular ω^G-metric spaces, which is the main interest of the present paper. Gahler [10] proved some interesting results in complete 2-metric spaces, which is a generalization of the classical metric spaces. Baskaran *et al.* [4], established common fixed point theorems for expansive mappings by using compatibility and sequentially continuous mappings in 2-metric spaces. Dhage [9], extended the work in [10] and introduced the notion of D-metric spaces. These authors claimed that their results generalized the concept of classical metric spaces.

In 2010 Chistyakov [6] introduced the notion of modular metric spaces or parameterized metric spaces with the time parameter (λ, say) and his intension was to define the notion of a modular acting on an arbitrary set, and developed the theory of metric spaces generated by modulars, called modular metric spaces. Chistyakov [6], developed the theory of metric spaces generated by modulars, and extended the results given by Nakano [18], Musielak and Orlicz [28], Musielak [13] to modular metric spaces.

Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions. The introduction of the theory of metric spaces generated by modulars known as modular metric spaces received the attention of many mathematicians. Consequently, several interesting results were proved in this direction of research. Chistyakov [8] also established some fixed point theorems for contractive mappings in modular spaces and other fixed point results in modular metric spaces can be found in [7, 8, 24, 29] and the references therein.

Azizi *et al.* [3] studied some fixed point theorems for $S + T$, where T is ρ-expansive and $S(B)$ resides in a compact subset of X_ρ, where B is a closed, convex and nonempty subset of X_ρ and $T, S : B \to X_\rho$. Their results also improved the classical version of Krasnosel'skii fixed point theorems in modular spaces. However, as an application, they studied the existence of solution of some nonlinear integral equations in modular function spaces.

In 2001, Ahmad *et al.* [1] defined expansive mappings in the setting of D-metric spaces analogous to expansive mappings in complete metric spaces. They also extended some known results to two mappings in the setting of D-metric spaces.

In 2003, Mustafa and Sims [16] pointed out that the fundamental topological properties of D-metric spaces introduced by Dhage [9] were false. To remedy the drawbacks connected to D-metric spaces, Mustafa and Sims [17] introduced a generalization of metric spaces, called G-metric spaces and proved some interesting fixed point results in this framework. Mustafa *et al.* [14] defined the class of expansive mappings in the setting of G-metric spaces and proved some fixed point theorems for these class of mappings in G-metric spaces. Furthermore, Mustafa *et al.* [15] proved some fixed point results in the setting of complete G-metric spaces.

In 2013, Azadifar *et al.* [2] developed the concept of modular ω^G-metric spaces and obtained some fixed point theorems of contractive mappings defined on modular ω^G-metric spaces.

Very recently, Okeke and Francis [22] defined expansive mappings of types I and II in the setting of modular ω^G-metric spaces and proved that their fixed point exist. Also many fixed point results for the class of expansive mappings of type I and II defined on a complete modular ω^G-metric spaces were also proved by the authors. Furthermore, Okeke and Francis [19] proved the existence and uniqueness of fixed point of mappings satisfying Geraghty-type contractions in the setting of preordered modular ω^G-metric spaces and applied the results in solving nonlinear Volterra-Fredholm-type integral equations. For other interesting results on generalized modular metric spaces and extended modular b-metric spaces, interested readers should consult [22]-[27] and the references therein.

The purpose in this paper is to define three expansive-type mappings in the setting of modular ω^G-metric spaces and prove some common unique fixed point results for these class of expansive mappings on ω^G-complete modular ω^G-metric spaces. Furthermore, we construct some examples to support our claims.

2. Preliminaries

Throughout the article $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$ is the set of non-negative integers and $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ is the set of positive real numbers. We begin this section by recalling some definitions and results which will be useful in this paper.

Definition 2.1. ([34]) Let (X, d) be a complete metric space. If f is a mapping of X into itself, then f is called an expansive map if there exists a constant $q > 1$ such that

$$d(f(x), f(y)) \geq qd(x, y) \tag{2.1}$$

for each $x, y \in X$.

Definition 2.2. ([1]) Let X be a D-metric space and T be a self-mapping on X. Then T is called an expansive mapping if there exists a constant $a > 1$ such that for all $x, y, z \in X$, we have

$$D(Tx, Ty, Tz) \geq aD(x, y, z). \tag{2.2}$$

Definition 2.3. ([14]) Let (X, G) be a G-metric space and T be a self-mapping on X. Then T is called an expansive mapping if there exists a constant $a > 1$ such that for all $x, y, z \in X$, we have

$$G(Tx, Ty, Tz) \geq aG(x, y, z). \tag{2.3}$$

Definition 2.4. ([3]) Let X_ρ be a modular space and B a nonempty subset of X_ρ. The mapping $T : B \to X_\rho$ is called a ρ-expansive mapping, if there exist constants $c, k, l \in \mathbb{R}^+$ such that $c > l, k > 1$ and

$$\rho(l(Tx - Ty)) \geq k\rho(c(x - y)) \tag{2.4}$$

for all $x, y \in B$.

Definition 2.5. ([2]) Let X be a nonempty set, and let $\omega^G : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a function satisfying;

(1) $\omega^G_\lambda(x, y, z) = 0$ for all $x, y, z \in X$ and $\lambda > 0$ if $x = y = z$,
(2) $\omega^G_\lambda(x, x, y) > 0$ for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$,
(3) $\omega^G_\lambda(x, x, y) \leq \omega^G_\lambda(x, y, z)$ for all $x, y, z \in X$ and $\lambda > 0$ with $z \neq y$,
(4) $\omega^G_\lambda(x, y, z) = \omega^G_\lambda(x, z, y) = \omega^G_\lambda(y, z, x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),
(5) $\omega^G_{\lambda+\mu}(x, y, z) \leq \omega^G_\lambda(x, a, a) + \omega^G_\mu(a, y, z)$, for all $x, y, z, a \in X$ and $\lambda, \nu > 0$.

Then the function ω^G_λ is called a modular ω^G-metric on X. The pair (X, ω^G) is called a modular ω^G-metric space.

Without any confusion we will take X_{ω^G} as a modular ω^G-metric space.

Definition 2.6. ([2]) Let (X_ω, ω^G) be a modular ω^G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{ω^G} is modular ω^G-convergent to x, if it converges to x in the topology $\tau(\omega_\lambda^G)$. A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular ω^G-continuous if $\omega_\lambda^G(x_n, x, x) \to 0$, then $\omega_\lambda^G(Tx_n, Tx, Tx) \to 0$ for all $\lambda > 0$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is modular ω^G-convergent to x as $n \to \infty$, if $\lim_{n\to\infty} \omega_\lambda^G(x_n, x_m, x) = 0$. That is, for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x) < \epsilon$ for all $n, m \geq n_0$. Here we say that x is modular ω^G-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.7. ([2]) Let (X_ω, ω^G) be a modular ω^G-metric space. Then the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be modular ω^G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n_\epsilon$ and $\lambda > 0$. A modular G-metric space X_{ω^G} is said to be modular G-complete if every modular ω^G-Cauchy sequence in X_{ω^G} is modular ω^G-convergent in X_{ω^G}.

Proposition 2.8. ([2]) *Let (X_ω, ω^G) be a modular ω^G-metric space, for any $x, y, z, a \in X_{\omega^G}$, it follows that:*

(1) *If $\omega_\lambda^G(x, y, z) = 0$ for all $\lambda > 0$, then $x = y = z$.*
(2) $\omega_\lambda^G(x, y, z) \leq \omega_{\frac{\lambda}{2}}^G(x, x, y) + \omega_{\frac{\lambda}{2}}^G(x, x, z)$ *for all* $\lambda > 0$.
(3) $\omega_\lambda^G(x, y, y) \leq 2\omega_{\frac{\lambda}{2}}^G(y, x, x)$ *for all* $\lambda > 0$.
(4) $\omega_\lambda^G(x, y, z) \leq \omega_{\frac{\lambda}{2}}^G(x, a, z) + \omega_{\frac{\lambda}{2}}^G(a, y, z)$ *for all* $\lambda > 0$.
(5) $\omega_\lambda^G(x, y, z) \leq \frac{2}{3}(\omega_{\frac{\lambda}{2}}^G(x, y, a) + \omega_{\frac{\lambda}{2}}^G(x, a, z) + \omega_{\frac{\lambda}{2}}^G(a, y, z))$ *for all* $\lambda > 0$.
(6) $\omega_\lambda^G(x, y, z) \leq \omega_{\frac{\lambda}{2}}^G(x, a, a) + \omega_{\frac{\lambda}{4}}^G(y, a, a) + \omega_{\frac{\lambda}{4}}^G(z, a, a)$ *for all* $\lambda > 0$.

Proposition 2.9. ([2]) *Let (X_ω, ω^G) be a modular ω^G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω. Then the following are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ *is ω^G-convergent to x,*
(2) $\omega_\lambda^G(x_n, x) \to 0$ *as $n \to \infty$, that is, $\{x_n\}_{n\in\mathbb{N}}$ converges to x relative to modular metric $\omega_\lambda^G(.)$,*
(3) $\omega_\lambda^G(x_n, x_n, x) \to 0$ *as $n \to \infty$ for all $\lambda > 0$,*
(4) $\omega_\lambda^G(x_n, x, x) \to 0$ *as $n \to \infty$ for all $\lambda > 0$,*
(5) $\omega_\lambda^G(x_m, x_n, x) \to 0$ *as $m, n \to \infty$ for all $\lambda > 0$.*

Next, we give the following two definitions, following [1], [32] which will play some vital roles in Section 3 of this paper.

Definition 2.10. Let (X_ω, ω^G) be a modular ω^G-metric space and $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three mappings. Then the mappings T, S, R are called expansive type I mappings if there exists a constant $a > 1$ such that for all

$x \neq y \neq z \neq x \in X_{\omega^G}$ and for any $\lambda > 0$, we have

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a\omega_\lambda^G(x, y, z). \tag{2.5}$$

Definition 2.11. Let (X_ω, ω^G) be a modular ω^G-metric space and $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three mappings. Then the mappings T, S, R are called expansive type II mappings if there exists a constant $a > 1$ such that for all $x, y \in X_{\omega^G}$ and for any $\lambda > 0$, we have

$$\omega_\lambda^G(Tx, Sy, Ry) \geq a\omega_\lambda^G(x, y, y). \tag{2.6}$$

Remark 2.12. Examples of the class of expansive mappings defined in Definitions 2.10 and 2.11 above will be given after Theorem 3.1 and Theorem 3.10, respectively.

A point $x \in M$ is said to be a fixed point of T if $x = Tx$. And the set of fixed points of T will be denoted by $Fix(T)$, that is, $Fix(T) = \{x \in M : x = Tx\}$.

3. Main results

We begin this section with the following results.

Theorem 3.1. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a\omega_\lambda^G(x, y, z), \quad \forall\, \lambda > 0. \tag{3.1}$$

Then T, S, R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_\omega$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$, $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$, continuing this process, we generate a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$.

Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$,

$$\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0,$$

so that from inequality (3.1), we have

$$\begin{aligned} \omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\ &\geq a\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \quad \forall\, \lambda > 0. \end{aligned} \tag{3.2}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \mu\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.3}$$

where $\mu = \frac{1}{a}$ and for all $\lambda > 0$. On continuing the process above, we have

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \le \mu^n \omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) \tag{3.4}$$

for $\lambda > 0$ and $n \in \mathbb{N}$.

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, that is, condition (5) of Definition 2.5 we have

$$\begin{aligned}
\omega^G_\lambda(x_{3n}, x_{3m}, x_{3m}) &\le \omega^G_{\frac{\lambda}{m-n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) \\
&\quad + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
&\le \omega^G_{\frac{\lambda}{n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \omega^G_{\frac{\lambda}{n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) \\
&\quad + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
&\le (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1}) \omega^G_\lambda(x_0, x_1, x_2) \\
&\le \frac{\mu^n}{1-\mu} \omega^G_\lambda(x_0, x_1, x_2)
\end{aligned} \tag{3.5}$$

for all $m > n \ge N \in \mathbb{N}$, then

$$\omega^G_\lambda(x_{3n}, x_{3m}, x_{3m}) \le \frac{\mu^n}{1-\mu} \omega^G_\lambda(x_0, x_1, x_2) \tag{3.6}$$

for all $m, l, n \ge N$ for some $N \in \mathbb{N}$, so that by condition (2) of Proposition 2.8, we have

$$\omega^G_\lambda(x_{3n}, x_{3m}, x_{3l}) \le \omega^G_{\frac{\lambda}{2}}(x_{3n}, x_{3m}, x_{3m}) + \omega^G_{\frac{\lambda}{2}}(x_{3l}, x_{3m}, x_{3m}), \tag{3.7}$$

so that

$$\begin{aligned}
\omega^G_\lambda(x_{3n}, x_{3m}, x_{3l}) &\le \omega^G_{\frac{\lambda}{2}}(x_{3n}, x_{3m}, x_{3m}) + \omega^G_{\frac{\lambda}{2}}(x_{3l}, x_{3m}, x_{3m}) \\
&\le \omega^G_\lambda(x_{3n}, x_{3m}, x_{3m}) + \omega^G_\lambda(x_{3l}, x_{3m}, x_{3m}) \\
&\le \frac{\mu^n}{1-\mu} \omega^G_\lambda(x_0, x_1, x_2) + \frac{\mu^n}{1-\mu} \omega^G_\lambda(x_0, x_1, x_2) \\
&= \left(\frac{2\mu^n}{1-\mu} \right) \omega^G_\lambda(x_0, x_1, x_2).
\end{aligned} \tag{3.8}$$

Thus, we have

$$\lim_{n,m,l \to \infty} \omega^G_\lambda(x_{3n}, x_{3m}, x_{3l}) = 0, \ \forall \ \lambda > 0. \tag{3.9}$$

Therefore, we can see clearly that $\{x_n\}_{n \in \mathbb{N}}$ is modular ω^G-Cauchy sequence in X_{ω^G}.

The modular ω^G-completeness of (X_ω, ω^G) implies that for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, u) = 0$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} x_n \to u \in X_{\omega^G}$ as $n \to \infty$, or by applying condition (5) of Proposition 2.9. As T, S, R are onto mappings, there exists $w, z^*, v \in X_{\omega^G}$ such that $u = Tw, u = Sz^*$ and $u = Rv$. We claim that $u = w = z^* = v$.

First, from inequality (3.1) with $x = x_{3n+1}$ and $y = z^*$ and $z = v$, we have that for all $n \geq 1, \lambda > 0$;

$$\begin{aligned} \omega_\lambda^G(x_{3n}, u, u) &= \omega_\lambda^G(Tx_{3n+1}, Sz^*, Rv) \\ &\geq a\omega_\lambda^G(x_{3n+1}, z^*, v), \ \forall\, \lambda > 0. \end{aligned} \tag{3.10}$$

As $n \to \infty$, we have $\omega_\lambda^G(u, z^*, v) = 0$, that is, $u = z^* = v$.

Secondly, using inequality (3.1) with $x = w$, $y = x_{3n+2}$ and $z = v$, we have that for all $n \geq 1, \lambda > 0$;

$$\begin{aligned} \omega_\lambda^G(u, x_{3n+1}, u) &= \omega_\lambda^G(Tw, Sx_{3n+2}, Rv) \\ &\geq a\omega_\lambda^G(w, x_{3n+2}, v), \ \forall\, \lambda > 0. \end{aligned} \tag{3.11}$$

As $n \to \infty$, we have $\omega_\lambda^G(w, u, v) = 0$, that is, $w = u = v$.

Lastly, from inequality (3.1) with $x = w$, $y = z^*$ and $z = x_{3n+3}$, we have that for all $n \geq 1, \lambda > 0$;

$$\begin{aligned} \omega_\lambda^G(u, u, x_{3n+2}) &= \omega_\lambda^G(Tw, Sz^*, Rx_{3n+3}) \\ &\geq a\omega_\lambda^G(w, z^*, x_{3n+3}), \ \forall\, \lambda > 0. \end{aligned} \tag{3.12}$$

As $n \to \infty$, we have $\omega_\lambda^G(w, z^*, u) = 0$, that is, $w = z^* = u$.

We can see clearly that in the three cases above, $u = w = z^* = v$, so that u is a common fixed point of T, S, R, that is, $u = Tu = Su = Ru$.

To prove uniqueness, suppose that there exists an another common fixed point of T, S, R, that is, there is a $u^* \in X_{\omega^G}$ such that $u^* = Tu^* = Su^* = Ru^*$. Suppose it is not the case, that is $u \neq u^*$, and for all $\lambda > 0$, again inequality (3.1) becomes;

$$\begin{aligned} \omega_\lambda^G(u, u^*, u^*) &= \omega_\lambda^G(Tu, Su^*, Ru^*) \\ &\geq a\omega_\lambda^G(u, u^*, u^*) \\ &> \omega_\lambda^G(u, u^*, u^*), \end{aligned} \tag{3.13}$$

which is a contradiction since $a > 1$, hence $u = u^*$. Therefore T, S, R has a common unique fixed point in X_{ω^G}. □

Remark 3.2. Theorem 3.1 is an expansive form of Theorem 1 in [31]. Meanwhile, Theorem 3.1 is a generalization of Theorem 3.1 in Okeke and Francis [22].

We know that the following remark from the paper in Remark 3.3 (see Okeke and Francis [22]).

Remark 3.3. If we let $T = S = R$, we get a result we have given in [22]. Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. If there exists a constant $a > 1$. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_1) < \infty$, for which the following condition holds;

$$\omega^G_\lambda(Tx, Ty, Tz) \geq a\omega^G_\lambda(x, y, z),\ \forall\, \lambda > 0. \tag{3.14}$$

Then T has a unique fixed point in X_{ω^G}.

Next, we prove the following corollary.

Corollary 3.4. *Let (X_ω, ω^G) be a G-complete modular G-metric space and there exists a constant $a > 1$. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_1) < \infty$, for which the following condition holds;*

$$\omega^G_\lambda(Tx, Ty, Tz) \geq a\omega^G_\lambda(x, y, z),\ \forall\, \lambda > 0. \tag{3.15}$$

Then T has a unique fixed point in X_{ω^G}.

Proof. It follows from Theorem 3.1 by taken $T = S = R$. Hence, T has a unique fixed point in X_{ω^G}. □

Remark 3.5. Observe that in Theorem 3.1 above, if $T = S = R$, we get an extension of Theorem 2.1 in [14] which is our Corollary 3.4 in modular ω^G-metric space.

Example 3.6. Let $X_{\omega^G} = \mathbb{R}^+ \cup \{\infty\}$. Define mappings $T, S, R : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by $Tx = x^n + 4x, Sx = x^n + 4x - 1$ and $Rx = x^n + 4x - 2$ for all $x \in \mathbb{R}^+ \cup \{\infty\}$ and $n \in \mathbb{N}$. Then T, S, R are expansive maps with nontrivial common fixed point of T, S, R. Indeed, Define modular G-metric by $\omega^G_\lambda : (0, \infty) \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$. For all distinct $x, y, z \in \mathbb{R}^+ \cup \{\infty\}$ and $\lambda > 0, n \in \mathbb{N}$, then

$$\begin{aligned}
\omega_\lambda^G(Tx, Sy, Rz) &= \frac{1}{\lambda}\bigg(\|Tx - Sy\| + \|Sy - Rz\| + \|Tx - Rz\| \bigg) \\
&= \frac{1}{\lambda}\bigg(\big\|x^n + 4x - (y^n + 4y - 1)\big\| \\
&\quad + \big\|y^n + 4y + 1 - (z^n + 4z - 2)\big\| \\
&\quad + \big\|x^n + 4x - (z^n + 4z - 2)\big\| \bigg) \\
&= \frac{1}{\lambda}\bigg(\big\|x^n - y^n + 4(x-y) + 1\big\| \\
&\quad + \big\|y^n - z^n + 4(y-z) + 3\big\| + \big\|x^n - z^n + 4(x-z) + 2\big\| \bigg) \\
&= \frac{1}{\lambda}\bigg(\Big\|(x-y)(x^{n-1} + yx^{n-2} + \cdots + y^{n-1}) + 4(x-y) + 1\Big\| \\
&\quad + \Big\|(y-z)(y^{n-1} + zy^{n-2} + \cdots + z^{n-1}) + 4(y-z) + 3\Big\| \\
&\quad + \Big\|(x-z)(x^{n-1} + zx^{n-2} + \cdots + z^{n-1}) + 4(x-z) + 2\Big\| \bigg) \\
&\geq \frac{1}{\lambda}\bigg\{ 4\|x-y\| + 4\|y-z\| + 4\|x-z\| \bigg\} \\
&= 4\omega_\lambda^G(x, y, z). \qquad (3.16)
\end{aligned}$$

Therefore,

$$\omega_\lambda^G(Tx, Sy, Rz) \geq 4\omega_\lambda^G(x, y, z), \qquad (3.17)$$

which justifies that T, S, R are expansive mappings with a common expansive constant 4. Hence inequality (3.1) is satisfied with $a = 4 > 1$.

Corollary 3.7. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2)$ is finite, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a\bigg(\omega_{\frac{\lambda}{2}}^G(x, x, y) + \omega_{\frac{\lambda}{2}}^G(x, x, z)\bigg), \ \forall\, \lambda > 0. \qquad (3.18)$$

Then T, S, R has a common unique fixed point in X_{ω^G}.

Proof. By condition (2) of Proposition 2.8, we have that

$$\omega_{\frac{\lambda}{2}}^G(x, x, y) + \omega_{\frac{\lambda}{2}}^G(x, x, z) \geq \omega_\lambda^G(x, y, z)$$

for all $\lambda > 0$. Therefore, from inequality (3.18), we have

$$\begin{aligned}\omega_\lambda^G(Tx, Sy, Rz) &\geq a\bigg(\omega_{\frac{\lambda}{2}}^G(x, x, y) + \omega_{\frac{\lambda}{2}}^G(x, x, z)\bigg)\\ &\geq a\omega_\lambda^G(x, y, z). \end{aligned} \tag{3.19}$$

So that for all $\lambda > 0$ and $a > 1$, we have

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a\omega_\lambda^G(x, y, z). \tag{3.20}$$

By proof of Theorem 3.1, T, S, R have a unique common fixed point in X_{ω^G}. □

The next corollary is a variant form of Theorem 3.1 which reads as follows;

Corollary 3.8. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds; for some positive integer, $m \geq 1$;*

$$\omega_\lambda^G(T^m x, S^m y, R^m z) \geq a\omega_\lambda^G(x, y, z),\ \forall\ \lambda > 0. \tag{3.21}$$

Then T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Theorem 3.1, T^m, S^m, R^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.21).

Now, $T^m(Tu^*) = T^{m+1}u^* = T(T^m u^*) = Tu^*$, so Tu^* is a fixed point of $T^m u^*$. Similarly, Su^* is a fixed point of $S^m u^*$ and Ru^* is a fixed point of $R^m u^*$.

For the uniqueness, suppose if possible that there exists another common fixed point of T^m, S^m, R^m say $v^* \in X_\omega$, that is, $T^m v^* = S^m v^* = R^m v^* = v^*$. We show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, v^*, v^*) > 0$, from inequality (3.21), we have

$$\begin{aligned}\omega_\lambda^G(u^*, v^*, v^*) &= \omega_\lambda^G(T^m u^*, S^m v^*, R^m v^*)\\ &\geq a\omega_\lambda^G(u^*, v^*, v^*),\ \forall\ \lambda > 0. \end{aligned} \tag{3.22}$$

So that

$$\begin{aligned}\omega_\lambda^G(u^*, v^*, v^*) &\geq a\omega_\lambda^G(u^*, v^*, v^*)\\ &> \omega_\lambda^G(u^*, v^*, v^*), \end{aligned} \tag{3.23}$$

which is a contradiction since $a > 1$, hence T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Corollary 3.9. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_2) < \infty$, for which the following condition holds; for some positive integer, $m \geq 1$;*

$$\omega^G_\lambda(T^m x, S^m y, R^m z) \geq a\left(\omega^G_{\frac{\lambda}{2}}(x, x, y) + \omega^G_{\frac{\lambda}{2}}(x, x, z)\right), \ \forall\, \lambda > 0. \tag{3.24}$$

Then T, S, R has a unique common fixed point in X_{ω^G}, for some positive integer, $m \geq 1$.

Proof. By proof of Corollary 3.8, T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Theorem 3.10. *Let (X_ω, ω^G) be a G-complete modular G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x, y \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_1) < \infty$, for which the following condition holds;*

$$\omega^G_\lambda(Tx, Sy, Ry) \geq a\omega^G_\lambda(x, y, y), \ \forall\, \lambda > 0. \tag{3.25}$$

Then T, S, R has a unique common fixed point in X_{ω^G}

Proof. Let $x_0, x_1 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_1 = Rx_2$ for $x_2 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2} = Rx_{3n+2}$.

Now, since $x_{3n} \neq x_{3n+1}$ implies that for any $\lambda > 0$, $\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+1}) > 0$, so that from inequality (3.25), we have

$$\begin{aligned} \omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+1}) &= \omega^G_\lambda(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+2}) \\ &\geq a\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+2}), \ \forall\, \lambda > 0. \end{aligned} \tag{3.26}$$

Therefore,

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+2}) \leq \beta\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+1}), \tag{3.27}$$

where $\beta = \frac{1}{a}$ and for all $\lambda > 0$. On continuing the process above, we have

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+2}) \leq \beta^n\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+1}) \tag{3.28}$$

for $\lambda > 0$ and $n \in \mathbb{N}$, where $\beta = \frac{1}{a} < 1$.

Following proof of Theorem 3.1 carefully, we see clearly that u is a unique common fixed point of T, S, R in X_{ω^G}. □

Example 3.11. Let $X_{\omega^G} = \mathbb{R}^+ \cup \{\infty\}$. Define mappings $T, S, R : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by $Tx = x^p + 1, Sx = x^p$ and $Rx = x^p - 1$ for all $x \in \mathbb{R}^+ \cup \{\infty\}$ and $p \in \mathbb{N}$. Then T, S, R are expansive maps with nontrivial common fixed point of T, S, R.

Remark 3.12. If we take $p = 1$, then the Example 3.11 is clear. In fact, define modular ω^G-metric by $\omega^G_\lambda : (0, \infty) \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$.

Now, for all $x, y \in \mathbb{R}^+ \cup \{\infty\}$ and $\lambda > 0$,

$$\begin{aligned}
\omega^G_\lambda(x^p + 1, y^p, y^p - 1) &= \omega^G_\lambda(Tx, Sy, Ry) \\
&= \frac{1}{\lambda}\Big(\|Tx - Sy\| + \|Sy - Ry\| + \|Tx - Sy\| \Big) \\
&= \frac{1}{\lambda}\Big(\|x^p + 1 - y^p\| + \big\|y^p - (y^p - 1)\big\| \\
&\quad + \big\|x^p + 1 - (y^p - 1)\big\| \Big) \\
&= \frac{1}{\lambda}\Big(\|x^p - y^p + 1\| + \|1\| + \|x^p - y^p + 2\| \Big) \\
&\geq \frac{1}{\lambda}\Big(\|x^p - y^p\| + \|x^p - y^p\| + 1 \Big) \\
&= \frac{1}{\lambda}\Big(2\|x^p - y^p\| + 1 \Big) \\
&\geq \frac{2}{\lambda}\|x^p - y^p\| \\
&= \frac{2}{\lambda}\Big\|(x - y)(x^{p-1} + yx^{p-2} + \cdots + y^{p-1})\Big\| \\
&\geq \frac{2}{\lambda}\|x - y\| \\
&= 2\omega^G_\lambda(x, y, y). \qquad (3.29)
\end{aligned}$$

Therefore,

$$\omega^G_\lambda(Tx, Sy, Ry) \geq 2\omega^G_\lambda(x, y, y), \ \forall\, \lambda > 0, \qquad (3.30)$$

which shows that T, S, R are expansive mappings with common expansive constant 2. Hence, inequality (3.25) is satisfied with $a = 2 > 1$.

Corollary 3.13. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x, y, z \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that*

$\omega_\lambda^G(x_0, x_1, x_1) < \infty$, *for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a\Bigg(\omega_\lambda^G(x, z, z) + \omega_\lambda^G(z, z, y)\Bigg),\ \forall\, \lambda > 0. \tag{3.31}$$

Then T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Observe that by putting $y = z$ in inequality (3.31), we have

$$\omega_\lambda^G(Tx, Sy, Ry) \geq a\omega_\lambda^G(x, y, y),\ \forall\, \lambda > 0. \tag{3.32}$$

By proof of Theorem 3.10, T, S, R has a unique common fixed point in X_{ω^G}. $\square$

Corollary 3.14. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x, y \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_1) < \infty$, for which the following condition holds, for some positive integer, $m \geq 1$;*

$$\omega_\lambda^G(T^m x, S^m y, R^m y) \geq a\omega_\lambda^G(x, y, y),\ \forall\, \lambda > 0. \tag{3.33}$$

Then T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Theorem 3.10, T^m, S^m, R^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.33). Now $T^m(Tu^*) = T^{m+1}u^* = T(T^m u^*) = Tu^*$, so Tu^* is a fixed point of $T^m u^*$. Similarly, Su^* is a fixed point of $S^m u^*$ and Ru^* is a fixed point of $R^m u^*$. For the uniqueness, suppose, if possible that there exists another common fixed point of T^m, S^m, R^m say $v^* \in X_{\omega^G}$ that is $T^m v^* = S^m v^* = R^m v^* = v^*$. We show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, v^*, v^*) > 0$, from inequality (3.33), we get

$$\begin{aligned} \omega_\lambda^G(u^*, v^*, v^*) &= \omega_\lambda^G(T^m u^*, S^m v^*, R^m v^*) \\ &\geq a\omega_\lambda^G(u^*, v^*, v^*),\ \forall\, \lambda > 0. \end{aligned} \tag{3.34}$$

So that

$$\begin{aligned} \omega_\lambda^G(u^*, v^*, v^*) &\geq a\omega_\lambda^G(u^*, v^*, v^*) \\ &> \omega_\lambda^G(u^*, v^*, v^*), \end{aligned} \tag{3.35}$$

which is a contradiction since $a > 1$, hence T, S, R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$. $\square$

Corollary 3.15. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a \max \left\{ \begin{array}{c} \omega^G_{\frac{\lambda}{2}}(x, z, z) + \omega^G_{\frac{\lambda}{2}}(z, z, y), \\ \omega^G_{\frac{\lambda}{2}}(z, y, y) + \omega^G_{\frac{\lambda}{2}}(y, y, x), \\ \omega^G_{\frac{\lambda}{2}}(z, x, x) + \omega^G_{\frac{\lambda}{2}}(x, x, y) \end{array} \right\}. \tag{3.36}$$

Then T, S, R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0, x_1, x_2 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. Continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$, so that from inequality (3.36), we have

$$\begin{aligned} &\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &= \omega^G_\lambda(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\ &\geq a \max \left\{ \begin{array}{c} \omega^G_{\frac{\lambda}{2}}(x_{3n+1}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+3}, x_{3n+2}), \\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+2}, x_{3n+2}) + \omega^G_{\frac{\lambda}{2}}(x_{3n+2}, x_{3n+2}, x_{3n+1}), \\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{2}}(x_{3n+1}, x_{3n+1}, x_{3n+2}) \end{array} \right\}. \end{aligned} \tag{3.37}$$

By condition (2) of Proposition 2.8, we have

$$a \max \left\{ \begin{array}{l} \omega^G_{\frac{\lambda}{2}}(x_{3n+1}, x_{3n+3}, x_{3n+3}) \\ \quad + \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+3}, x_{3n+2}), \\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+2}, x_{3n+2}) \\ \quad + \omega^G_{\frac{\lambda}{2}}(x_{3n+2}, x_{3n+2}, x_{3n+1}), \\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3}, x_{3n+1}, x_{3n+1}) \\ \quad + \omega^G_{\frac{\lambda}{2}}(x_{3n+1}, x_{3n+1}, x_{3n+2}) \end{array} \right\} \geq a\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{3.38}$$

Therefore, we have from inequality (3.38) for all $\lambda > 0$,

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \gamma\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.39}$$

where $\gamma = \frac{1}{a} < 1$. Following the proof of Theorem 3.1, T, S, R has a common unique fixed point in X_{ω^G}. $\square$

Corollary 3.16. *Let (X_ω, ω^G) be a ω^G-complete non-symmetric modular ω^G-metric space and there exists a constant $a > 2$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x, y, z \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_1) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a \max \left\{ \begin{array}{l} \omega_{\frac{\lambda}{2}}^G(y, x, x) + \dfrac{1}{2}\omega_{\frac{\lambda}{2}}^G(y, z, z), \\ \omega_{\frac{\lambda}{2}}^G(z, x, x) + \dfrac{1}{2}\omega_{\frac{\lambda}{2}}^G(y, y, z), \\ \omega_{\frac{\lambda}{2}}^G(z, z, y) + \dfrac{1}{2}\omega_{\frac{\lambda}{2}}^G(z, y, z) \end{array} \right\}. \tag{3.40}$$

Then T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Observe that if $z = y$, inequality (3.40) becomes

$$\omega_\lambda^G(Tx, Sy, Ry) \geq a\omega_{\frac{\lambda}{2}}^G(y, x, x). \tag{3.41}$$

Now, we consider the right hand side of inequality (3.40) by applying condition (3) of Proposition 2.8, we get $\omega_\lambda^G(x, y, y) \leq 2\omega_{\frac{\lambda}{2}}^G(y, x, x)$ for all $\lambda > 0$, or, putting $z = y$ in condition (2) of Proposition 2.8, we have $\omega_\lambda^G(x, y, y) \leq \omega_{\frac{\lambda}{2}}^G(y, x, x) + \omega_{\frac{\lambda}{2}}^G(y, x, x)$ for all $\lambda > 0$. So that $\frac{1}{2}\omega_\lambda^G(x, y, y) \leq \omega_{\frac{\lambda}{2}}^G(y, x, x)$ for all $\lambda > 0$. From inequality (3.41), we have that

$$\omega_\lambda^G(Tx, Sy, Ry) \geq a\omega_{\frac{\lambda}{2}}^G(y, x, x) \geq \frac{a}{2}\omega_\lambda^G(x, y, y). \tag{3.42}$$

By proof of Theorem 3.10 we are done. Hence, T, S, R has a unique common fixed point in X_{ω^G}. $\square$

Corollary 3.17. *Let (X_ω, ω^G) be a ω^G-complete non-symmetric modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x, y, z \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a \max \left\{ \begin{array}{l} 2\omega_{\frac{\lambda}{2}}^G(y, x, x) + \omega_{\frac{\lambda}{2}}^G(y, z, z), \\ 2\omega_{\frac{\lambda}{2}}^G(z, x, x) + \omega_{\frac{\lambda}{2}}^G(y, y, z), \\ 2\omega_{\frac{\lambda}{2}}^G(z, z, y) + \omega_{\frac{\lambda}{2}}^G(z, y, z) \end{array} \right\}. \tag{3.43}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Following the proof of corollary 3.16, we get

$$\omega_\lambda^G(Tx, Sy, Ry) \geq 2a\omega_{\frac{\lambda}{2}}^G(y, x, x) \geq a\omega_\lambda^G(x, y, y). \tag{3.44}$$

By Theorem 3.10, the proof is completed. □

Corollary 3.18. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\left(\omega_{\frac{\lambda}{2}}^G(x, Tx, Tx) + \omega_{\frac{\lambda}{2}}^G(Tx, y, z)\right), \ \forall\, \lambda > 0. \tag{3.45}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have $\omega_{\frac{\lambda}{2}}^G(x, Tx, Tx) + \omega_{\frac{\lambda}{2}}^G(Tx, y, z) \geq \omega_\lambda^G(x, y, z)$. Therefore, for all $\lambda > 0$, inequality (3.45) becomes

$$\begin{aligned} \omega_\lambda^G(Tx, Sy, Rz) &\geq k\left(\omega_{\frac{\lambda}{2}}^G(x, Tx, Tx) + \omega_{\frac{\lambda}{2}}^G(Tx, y, z)\right) \\ &\geq k\omega_\lambda^G(x, y, z), \ \forall\, \lambda > 0. \end{aligned} \tag{3.46}$$

Hence,

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\omega_\lambda^G(x, y, z), \ \forall\, \lambda > 0, \tag{3.47}$$

where $k > 1$. By proof of Corollary 3.7, the proof is completed. □

Corollary 3.19. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\left(\omega_{\frac{\lambda}{2}}^G(x, Sx, Sx) + \omega_{\frac{\lambda}{2}}^G(Sx, y, z)\right), \ \forall\, \lambda > 0. \tag{3.48}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have $\omega_{\frac{\lambda}{2}}^G(x, Sx, Sx) + \omega_{\frac{\lambda}{2}}^G(Sx, y, z) \geq \omega_\lambda^G(x, y, z)$. Therefore, for all $\lambda > 0$, inequality (3.48) becomes

$$\begin{aligned} \omega_\lambda^G(Tx, Sy, Rz) &\geq k\left(\omega_{\frac{\lambda}{2}}^G(x, Sx, Sx) + \omega_{\frac{\lambda}{2}}^G(Sx, y, z)\right) \\ &\geq k\omega_\lambda^G(x, y, z), \ \forall\, \lambda > 0. \end{aligned} \tag{3.49}$$

Hence,

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\omega_\lambda^G(x, y, z),\ \forall\, \lambda > 0, \tag{3.50}$$

where $k > 1$. By proof of Corollary 3.7, the proof is completed. □

Corollary 3.20. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space and there exists a constant $a > 1$. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all$x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\Big(\omega_{\frac{\lambda}{2}}^G(x, Rx, Rx) + \omega_{\frac{\lambda}{2}}^G(Rx, y, z)\Big),\ \forall\, \lambda > 0. \tag{3.51}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have

$$\omega_\lambda^G(Tx, Sy, Rz) \geq k\omega_\lambda^G(x, y, z),\ \forall\, \lambda > 0, \tag{3.52}$$

where $k > 1$. By proof of Corollary 3.7, the proof is completed. □

Corollary 3.21. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\begin{aligned}\omega_\lambda^G(Tx, Sy, Rz) \geq\ & \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(Tx, x, y)\\ &+ \gamma\omega_\lambda^G(Sy, y, z) + \delta\omega_\lambda^G(x, Rz, z),\end{aligned} \tag{3.53}$$

where $\alpha + \beta + \gamma + \delta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Let $x_0, x_1, x_2 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$, so that from inequality (3.53), we have

$$
\begin{aligned}
\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\
&\geq \alpha\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta\omega_\lambda^G(Tx_{3n+1}, x_{3n+1}, x_{3n+2}) \\
&\quad + \gamma\omega_\lambda^G(Sx_{3n+2}, x_{3n+2}, x_{3n+3}) \\
&\quad + \delta\omega_\lambda^G(x_{3n+1}, Rx_{3n+3}, x_{3n+3}) \\
&= \alpha\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) \\
&\quad + \gamma\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \delta\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&= (\alpha+\gamma+\delta)\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}).
\end{aligned}
$$

Therefore,

$$
\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq h\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.54}
$$

where $h = \frac{1-\beta}{(\alpha+\gamma+\delta)} < 1$, $\beta < 1$ and $\lambda > 0$. By continuing this process, we get

$$
\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq h^n\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}),\ \forall\ \lambda > 0 \tag{3.55}
$$

and $n \geq 1$. Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, that is, condition (5) of Definition 2.5 we have

$$
\begin{aligned}
\omega_\lambda^G(x_{3n}, x_{3m}, x_{3m}) &\leq \omega^G_{\frac{\lambda}{m-n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) \\
&\quad + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
&\leq \omega^G_{\frac{\lambda}{n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \omega^G_{\frac{\lambda}{n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) \\
&\quad + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
&\leq (h^n + h^{n+1} + \cdots + h^{m-1})\omega_\lambda^G(x_0, x_1, x_2) \\
&\leq \frac{h^n}{1-h}\omega_\lambda^G(x_0, x_1, x_2) \qquad (3.56)
\end{aligned}
$$

for all $m > n \geq N \in \mathbb{N}$, then

$$
\omega_\lambda^G(x_{3n}, x_{3m}, x_{3m}) \leq \frac{h^n}{1-h}\omega_\lambda^G(x_0, x_1, x_2) \tag{3.57}
$$

for all $m, l, n \geq N$ for some $N \in \mathbb{N}$, so that by condition (2) of Proposition 2.8, we have

$$\omega_\lambda^G(x_{3n}, x_{3m}, x_{3l}) \leq \omega_{\frac{\lambda}{2}}^G(x_{3n}, x_{3m}, x_{3m}) + \omega_{\frac{\lambda}{2}}^G(x_{3l}, x_{3m}, x_{3m}), \tag{3.58}$$

so that

$$\begin{aligned}\omega_\lambda^G(x_{3n}, x_{3m}, x_{3l}) &\leq \omega_{\frac{\lambda}{2}}^G(x_{3n}, x_{3m}, x_{3m}) + \omega_{\frac{\lambda}{2}}^G(x_{3l}, x_{3m}, x_{3m}) \\ &\leq \omega_\lambda^G(x_{3n}, x_{3m}, x_{3m}) + \omega_\lambda^G(x_{3l}, x_{3m}, x_{3m}) \\ &\leq \frac{h^n}{1-h}\omega_\lambda^G(x_0, x_1, x_2) + \frac{h^n}{1-h}\omega_\lambda^G(x_0, x_1, x_2) \\ &= \left(\frac{2h^n}{1-h}\right)\omega_\lambda^G(x_0, x_1, x_2). \end{aligned} \tag{3.59}$$

Thus, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(x_n, x_m, x_l) = 0, \ \forall\, \lambda > 0. \tag{3.60}$$

Therefore, we can see clearly that $\{x_n\}_{n\in\mathbb{N}}$ is modular ω^G-Cauchy sequence. The modular ω^G-completeness of (X_ω, ω^G) implies that for any $\lambda > 0$,

$$\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, u) = 0,$$

that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} x_n \to u \in X_{\omega^G}$ as $n \to \infty$, or by applying condition (5) of Proposition 2.9.

As T, S, R are onto mappings, there exists $w, p, v \in X_{\omega^G}$ such that $u = Tw, u = Sp$ and $u = Rv$. We claim that $u = w = p = v$.

First, from inequality (3.53) with $x = x_{3n+1}$ and $y = p$, $z = v$, we have that for all $n \geq 1, \lambda > 0$;

$$\begin{aligned}\omega_\lambda^G(x_{3n}, u, u) &= \omega_\lambda^G(Tx_{3n+1}, Sp, Rv) \\ &\geq \alpha\omega_\lambda^G(x_{3n+1}, p, v) + \beta\omega_\lambda^G(Tx_{3n+1}, x_{3n+1}, p) \\ &\quad + \gamma\omega_\lambda^G(Sp, p, v) + \delta\omega_\lambda^G(x_{3n+1}, Rv, v) \\ &= \alpha\omega_\lambda^G(x_{3n+1}, p, v) + \beta\omega_\lambda^G(x_{3n}, x_{3n+1}, p) \\ &\quad + \gamma\omega_\lambda^G(Sp, p, v) + \delta\omega_\lambda^G(x_{3n+1}, Rv, v) \\ &= \alpha\omega_\lambda^G(x_{3n+1}, p, v) + \beta\omega_\lambda^G(x_{3n}, x_{3n+1}, p) \\ &\quad + \gamma\omega_\lambda^G(u, p, v) + \delta\omega_\lambda^G(x_{3n+1}, u, v), \ \forall\, \lambda > 0. \end{aligned} \tag{3.61}$$

As $n \to \infty$, inequality (3.61) becomes

$$\alpha\omega_\lambda^G(u, p, v) + \beta\omega_\lambda^G(u, u, p) + \gamma\omega_\lambda^G(u, p, v) + \delta\omega_\lambda^G(u, u, v) \leq 0,$$

so that

$$(\alpha+\gamma)\omega_\lambda^G(u,p,v)+\beta\omega_\lambda^G(u,u,p)+\delta\omega_\lambda^G(u,u,v)=0.$$

Therefore, since $\alpha+\gamma\neq 0$, $\omega_\lambda^G(u,p,v)=0$, that is, $u=p=v$, similarly, since $\beta,\delta\neq 0$, $\omega_\lambda^G(u,u,p)=0$ and $\omega_\lambda^G(u,u,v)=0$, that is, $u=p=v$.

Secondly, using inequality (3.53) with $x=w$ and $y=x_{3n+2}$ and $z=v$, we have that for all $n\geq 1,\lambda>0$;

$$\begin{aligned}\omega_\lambda^G(u,x_{3n+1},u)&=\omega_\lambda^G(Tw,Sx_{3n+2},Rv)\\&\geq\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(Tw,w,x_{3n+2})\\&\quad+\gamma\omega_\lambda^G(Sx_{3n+2},x_{3n+2},v)+\delta\omega_\lambda^G(w,Rv,v)\\&=\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(Tw,w,x_{3n+2})\\&\quad+\gamma\omega_\lambda^G(x_{3n+1},x_{3n+2},v)+\delta\omega_\lambda^G(w,Rv,v)\\&=\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(u,w,x_{3n+2})\\&\quad+\gamma\omega_\lambda^G(x_{3n+1},x_{3n+2},v)+\delta\omega_\lambda^G(w,u,v),\ \forall\,\lambda>0.\end{aligned}\tag{3.62}$$

As $n\to\infty$, we have

$$(\alpha+\delta)\omega_\lambda^G(w,u,v)+\beta\omega_\lambda^G(u,w,u)+\gamma\omega_\lambda^G(u,u,v)\leq 0.$$

Since $\alpha+\delta\neq 0$, $\beta\neq 0$ and $\gamma\neq 0$, $w=u=v$.

Lastly, from inequality (3.53) with $x=w$ and $y=p$ and $z=x_{3n+3}$, we have that for all $n\geq 1,\lambda>0$;

$$\begin{aligned}\omega_\lambda^G(u,u,x_{3n+2})&=\omega_\lambda^G(Tw,Sp,Rx_{3n+3})\\&\geq\alpha\omega_\lambda^G(w,p,x_{3n+3})+\beta\omega_\lambda^G(Tw,w,p)\\&\quad+\gamma\omega_\lambda^G(Sp,p,x_{3n+3})+\delta\omega_\lambda^G(w,Rx_{3n+3},x_{3n+3})\\&=\alpha\omega_\lambda^G(w,p,x_{3n+2})+\beta\omega_\lambda^G(Tw,w,p)\\&\quad+\gamma\omega_\lambda^G(Sp,p,x_{3n+3})+\delta\omega_\lambda^G(w,x_{3n+2},x_{3n+3})\\&=\alpha\omega_\lambda^G(w,p,x_{3n+3})+\beta\omega_\lambda^G(u,w,p)\\&\quad+\gamma\omega_\lambda^G(u,p,x_{3n+3})+\delta\omega_\lambda^G(w,x_{3n+2},x_{3n+3}),\ \forall\,\lambda>0.\end{aligned}\tag{3.63}$$

As $n\to\infty$, inequality (3.63) becomes

$$(\alpha+\beta)\omega_\lambda^G(u,w,p)+\gamma\omega_\lambda^G(u,p,u)+\delta\omega_\lambda^G(w,u,u)\leq 0,$$

hence, $\omega_\lambda^G(u,w,p)=0$, i.e., $u=w=p$. We can see clearly that in the three cases above, $u=w=p=v$, so that u is a common fixed point of T,S,R, that is, $u=Tu=Su=Ru$.

To prove uniqueness, suppose that there exists an another common fixed point of T,S,R, that is, there is a $u^*\in X_{\omega^G}$ such that $u^*=Tu^*=Su^*=Ru^*$.

Suppose that $u \neq u^*$, and for all $\lambda > 0$, again inequality (3.53) becomes;

$$\begin{aligned}\omega_\lambda^G(u, u^*, u^*) &= \omega_\lambda^G(Tu, Su^*, Ru^*)\\ &\geq \alpha\omega_\lambda^G(u, u^*, u^*) + \beta\omega_\lambda^G(Tu, u, u^*)\\ &\quad + \gamma\omega_\lambda^G(Su^*, u^*, u^*) + \delta\omega_\lambda^G(u, Ru^*, u^*)\\ &= \alpha\omega_\lambda^G(u, u^*, u^*) + \beta\omega_\lambda^G(u, u, u^*)\\ &\quad + \gamma\omega_\lambda^G(u^*, u^*, u^*) + \delta\omega_\lambda^G(u, u^*, u^*)\\ &= (\alpha+\delta)\omega_\lambda^G(u, u^*, u^*) + \beta\omega_\lambda^G(u, u, u^*)\\ &\geq (\alpha+\delta)\omega_\lambda^G(u, u^*, u^*)\\ &> \omega_\lambda^G(u, u^*, u^*), \end{aligned} \tag{3.64}$$

which is a contradiction, hence $u = u^*$. □

Remark 3.22. Corollary 3.21 is an extension of Theorem 3.11 in Okeke and Francis [22].

Corollary 3.23. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds for some positive integer, $m \geq 1$;*

$$\begin{aligned}\omega_\lambda^G(T^m x, S^m y, R^m z) \geq\ & \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(T^m x, x, y)\\ &+ \gamma\omega_\lambda^G(S^m y, y, z) + \delta\omega_\lambda^G(x, R^m z, z),\end{aligned} \tag{3.65}$$

where $\alpha + \beta + \gamma + \delta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By corollary 3.21, T^m, S^m and R^m has common fixed point say $u \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.65), we have that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. For uniqueness, suppose that there exist another common fixed point $u^* \in X_{\omega^G}$ of T^m, S^m and R^m for some positive integer, $m \geq 1$ such that $T^m u^* = S^m u^* = R^m u^* = u^*$. Suppose that $u \neq v$, which implies that for any $\lambda > 0$, from inequality (3.65), for some positive integer, $m \geq 1$, we get

$$\begin{aligned}\omega_\lambda^G(u,u^*,u^*) &= \omega_\lambda^G(T^m u, S^m u^*, R^m u^*)\\ &\geq \alpha\omega_\lambda^G(u,u^*,u^*) + \beta\omega_\lambda^G(T^m u,u,u^*)\\ &\quad + \gamma\omega_\lambda^G(S^m u^*,u^*,u^*) + \delta\omega_\lambda^G(u,R^m u^*,u^*)\\ &= \alpha\omega_\lambda^G(u,u^*,u^*) + \beta\omega_\lambda^G(u,u,u^*)\\ &\quad + \gamma\omega_\lambda^G(u^*,u^*,u^*) + \delta\omega_\lambda^G(u,u^*,u^*)\\ &= (\alpha+\delta)\omega_\lambda^G(u,u^*,u^*) + \beta\omega_\lambda^G(u,u,u^*)\\ &\geq (\alpha+\delta)\omega_\lambda^G(u,u^*,u^*)\\ &> \omega_\lambda^G(u,u^*,u^*), \end{aligned} \tag{3.66}$$

which is a contradiction, hence $u = u^*$. □

Remark 3.24. Corollary 3.23 is an extension of Theorem 3.12 in Okeke and Francis [22].

Corollary 3.25. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0,x_1,x_2) < \infty$, for which the following condition holds;*

$$\begin{aligned}\omega_\lambda^G(Tx,Sy,Rz) &\geq \alpha\omega_\lambda^G(x,y,z)\\ &\quad + \beta\Big(\omega_\lambda^G(Tx,x,y) + \omega_\lambda^G(Sy,y,z) + \omega_\lambda^G(x,Rz,z)\Big), \end{aligned} \tag{3.67}$$

where $\alpha + 3\beta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Putting $\beta = \gamma = \delta$, then by proof Corollary 3.21, T, S, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.26. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0,x_1,x_2) < \infty$, for which the following condition holds for some positive integer, $m \geq 1$;*

$$\begin{aligned}\omega_\lambda^G(T^m x,S^m y,R^m z) &\geq \alpha\omega_\lambda^G(x,y,z)\\ &\quad + \beta\Big(\omega_\lambda^G(T^m x,x,y) + \omega_\lambda^G(S^m y,y,z) + \omega_\lambda^G(x,R^m z,z)\Big), \end{aligned} \tag{3.68}$$

where $\alpha + 3\beta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By proof Corollary 3.25, T, S, R has a unique common point in X_{ω^G} for some positive integer, $m \geq 1$. □

Corollary 3.27. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $(x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega^G_\lambda(Tx, Sy, Rz) \geq \alpha\omega^G_\lambda(x, y, z) + \beta\omega^G_\lambda(x, Rz, Tx) + \gamma\omega^G_\lambda(y, Sy, z) + \delta\omega^G_\lambda(z, Sy, Rz), \tag{3.69}$$

where $\alpha + \beta + \gamma + \delta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Let $(x_0, x_1, x_2 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$ By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$, so that from inequality (3.69), and after some simplifications, we get

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq k\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.70}$$

where $k = \frac{1-\beta}{(\alpha+\gamma+\delta)} < 1$, $\beta < 1$ and $\lambda > 0$. Following proof of Corollary 3.21, we conclude that T, S, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.28. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega^G_\lambda(x_0, x_1, x_2) < \infty$, for which the following condition holds for some positive integer, $m \geq 1$;*

$$\omega^G_\lambda(T^m x, S^m y, R^m z) \geq \alpha\omega^G_\lambda(x, y, z) + \beta\omega^G_\lambda(x, R^m z, Tx) + \gamma\omega^G_\lambda(y, S^m y, z) + \delta\omega^G_\lambda(z, S^m y, R^m z), \tag{3.71}$$

where $\alpha + \beta + \gamma + \delta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By proof Corollary 3.27, we can conclude that T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Corollary 3.29. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. If there exists a constant $a > 1$ and let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto* *mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a \max \begin{Bmatrix} \omega_\lambda^G(x, y, z), \omega_\lambda^G(Tx, y, y), \\ \omega_\lambda^G(Sy, y, z), \omega_\lambda^G(x, Rz, z) \end{Bmatrix}. \tag{3.72}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now, since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$, so that from inequality (3.72), we have, with $x = x_{3n+1}$ and $y = x_{3n+2}$ and $z = x_{3n+3}$ for all $n \geq 1$, $\lambda > 0$,

$$\begin{aligned} \omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\ &\geq a \max \begin{Bmatrix} \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(Tx_{3n+1}, x_{3n+2}, x_{3n+2}), \\ \omega_\lambda^G(Sx_{3n+2}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(x_{3n+1}, Rx_{3n+3}, x_{3n+3}) \end{Bmatrix}. \end{aligned} \tag{3.73}$$

So that

$$\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) \geq a \max \begin{Bmatrix} \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(x_{3n}, x_{3n+2}, x_{3n+2}), \\ \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \end{Bmatrix}. \tag{3.74}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq b\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) \tag{3.75}$$

for all $\lambda > 0$ and $b = \frac{1}{a} < 1$. By proof of Corollary 3.21, T, S, R has a unique common fixed point in X_{ω^G}. $\square$

Corollary 3.30. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. If there exists a constant $a > 1$ and let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds for some positive integer $m \geq 1$;*

$$\omega_\lambda^G(T^m x, S^m y, R^m z) \geq a \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(T^m x, y, y), \\ \omega_\lambda^G(S^m y, y, z), \omega_\lambda^G(x, R^m z, z) \end{array} \right\}. \tag{3.76}$$

Then, T, S, R has a unique common fixed point in X_{ω^G} for some positive integer $m \geq 1$.

Proof. By Corollary 3.29, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now, we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u, v, v) > 0$, then for uniqueness, inequality (3.76) we get a contradiction, since $a > 1$, hence $u = v$. □

Corollary 3.31. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\begin{aligned} \omega_\lambda^G(Tx, Sy, Rz) \geq {} & \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(Sx, Tx, Tx) \\ & + \gamma\omega_\lambda^G(Ry, Sy, Sy) + \delta\omega_\lambda^G(Tz, Rz, Rz), \end{aligned} \tag{3.77}$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Let $x_0, x_1, x_2 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$. From inequality (3.77), with $x = x_{3n+1}$ and $y = x_{3n+2}$ and $z = x_{3n+3}$, we have that for all $n \geq 1$, $\lambda > 0$,

$$\begin{aligned} \omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\ &\geq \alpha\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &\quad + \beta\omega_\lambda^G(Sx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}) \\ &\quad + \gamma\omega_\lambda^G(Rx_{3n+2}, Sx_{3n+2}, Sx_{3n+2}) \\ &\quad + \delta\omega_\lambda^G(Tx_{3n+3}, Rx_{3n+3}, Rx_{3n+3}) \end{aligned}$$

$$
\begin{aligned}
&= \alpha\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta\omega_\lambda^G(x_{3n}, x_{3n}, x_{3n}) \\
&\quad + \gamma\omega_\lambda^G(x_{3n+1}, x_{3n+1}, x_{3n+1}) \\
&\quad + \delta\omega_\lambda^G(x_{3n+2}, x_{3n+2}, x_{3n+2}) \\
&= \alpha\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \qquad (3.78)
\end{aligned}
$$

Therefore,

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq r\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}), \qquad (3.79)$$

where $r = \frac{1}{\alpha}$ and for all $\lambda > 0$, $r < 1$. By continuing this process, we get

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq r^n\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}), \ \forall\, \lambda > 0 \qquad (3.80)$$

and $n \geq 1$. By Corollary 3.21, we are done. □

Remark 3.32. Corollary 3.31 is an extension of Corollary 3.5 in [32]. Corollary 3.31 is an extension of Corollary 3.16 in Okeke and Francis [22].

Corollary 3.33. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$
\begin{aligned}
\omega_\lambda^G(Tx, Sy, z) \geq{}& \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(Sx, Tx, Tx) \\
&+ \gamma\omega_\lambda^G(y, Sy, Sy) + \delta\omega_\lambda^G(Tz, z, z), \qquad (3.81)
\end{aligned}
$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, T, S has a unique common fixed point in X_{ω^G}.

Proof. Take $R = I$ in Corollary 3.31, we can conclude that T, S has a unique common fixed point in X_{ω^G}. □

Corollary 3.34. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $S, R : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$
\begin{aligned}
\omega_\lambda^G(x, Sy, Rz) \geq{}& \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(Sx, x, x) \\
&+ \gamma\omega_\lambda^G(Ry, Sy, Sy) + \delta\omega_\lambda^G(z, Rz, Rz), \qquad (3.82)
\end{aligned}
$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, S, R has a unique common fixed point in X_{ω^G}.

Proof. Take $T = I$ in Corollary 3.31, we can conclude that S, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.35. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, R : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\begin{aligned} \omega_\lambda^G(Tx, y, Rz) \geq \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(x, Tx, Tx) \\ + \gamma\omega_\lambda^G(Ry, y, y) + \delta\omega_\lambda^G(Tz, Rz, Rz), \end{aligned} \tag{3.83}$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Take $S = I$ in Corollary 3.31, we can conclude that T, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.36. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $R : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_1) < \infty$, for which the following condition holds;*

$$\begin{aligned} \omega_\lambda^G(x, y, Rz) \geq \alpha\omega_\lambda^G(x, y, z) \\ + \gamma\omega_\lambda^G(Ry, y, y) + \delta\omega_\lambda^G(z, Rz, Rz), \end{aligned} \tag{3.84}$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, R has a unique fixed point in X_ω.

Proof. Take $S = T = I$ in Corollary 3.31, we can conclude that R has a unique fixed point in X_{ω^G}. □

Corollary 3.37. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_1) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, y, z) \geq \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(x, Tx, Tx) + \delta\omega_\lambda^G(Tz, z, z), \tag{3.85}$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, T has unique fixed point in X_{ω^G}.

Proof. Take $R = S = I$ in Corollary 3.31, we can conclude that T has a unique fixed point in X_{ω^G}. □

Corollary 3.38. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds for some positive integer, $m \geq 1$;*

$$\begin{aligned} \omega_\lambda^G(T^m x, S^m y, R^m z) \geq \alpha\omega_\lambda^G(x, y, z) + \beta\omega_\lambda^G(S^m x, T^m x, T^m x) \\ + \gamma\omega_\lambda^G(R^m y, S^m y, S^m y) + \delta\omega_\lambda^G(T^m z, R^m z, R^m z), \end{aligned} \tag{3.86}$$

where $\alpha > 1$ and for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Corollary 3.31, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now, we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u, v, v) > 0$, then for uniqueness, inequality (3.86) we get a contradiction, hence $u = v$. □

Corollary 3.39. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\begin{aligned}\omega_\lambda^G(Tx, Sy, Rz) &\geq \alpha\omega_\lambda^G(x, y, z)\\ &\quad + \beta\Big(\omega_\lambda^G(Sx, Tx, Tx) + \omega_\lambda^G(Ry, Sy, Sy) + \omega_\lambda^G(Tz, Rz, Rz)\Big),\end{aligned} \tag{3.87}$$

where $\alpha > 1$ for all $\lambda > 0$. Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Putting $\beta = \gamma = \delta$, then by Corollary 3.31, T, S, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.40. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. If there exists a constant $a > 1$ and let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds;*

$$\omega_\lambda^G(Tx, Sy, Rz) \geq a \max\left\{\begin{array}{l}\omega_\lambda^G(x, y, z), \omega_\lambda^G(Sx, Tx, Tx),\\ \omega_\lambda^G(Ry, Sy, Sy), \omega_\lambda^G(Tz, Rz, Rz)\end{array}\right\}. \tag{3.88}$$

Then, T, S, R has a unique common fixed point in X_{ω^G}.

Proof. Let $x_0, x_1, x_2 \in X_{\omega^G}$. Since T, S, R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now, since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$. From inequality (3.88) with $x = x_{3n+1}$ and $y = x_{3n+2}$ and $z = x_{3n+3}$, we have that for all $n \geq 1$, $\lambda > 0$,

$$\begin{aligned}\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \\ &\geq a \max \left\{ \begin{array}{l} \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(Sx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\ \omega_\lambda^G(Rx_{3n+2}, Sx_{3n+2}, Sx_{3n+2}), \\ \omega_\lambda^G(Tx_{3n+3}, Rx_{3n+3}, Rx_{3n+3}) \end{array} \right\}, \end{aligned} \tag{3.89}$$

hence,

$$\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}) \geq a \max \left\{ \begin{array}{l} \omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ \omega_\lambda^G(x_{3n}, x_{3n}, x_{3n}), \\ \omega_\lambda^G(x_{3n+1}, x_{3n+1}, x_{3n+1}), \\ \omega_\lambda^G(x_{3n+2}, x_{3n+2}, x_{3n+2}) \end{array} \right\}. \tag{3.90}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \kappa \omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.91}$$

for all $\lambda > 0$ and $\kappa = \frac{1}{a} < 1$. Proof of Corollary 3.31 completes Corollary 3.40. Hence T, S, R has a unique common fixed point in X_{ω^G}. □

Corollary 3.41. *Let (X_ω, ω^G) be a ω^G-complete modular ω^G-metric space. If there exists a constant $a > 1$ and let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G} for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there are $x_0, x_1, x_2 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, x_1, x_2) < \infty$, for which the following condition holds for some positive integer $m \geq 1$;*

$$\omega_\lambda^G(T^m x, S^m y, R^m z) \geq a \max \left\{ \begin{array}{l} \omega_\lambda^G(x, y, z), \\ \omega_\lambda^G(S^m x, T^m x, T^m x), \\ \omega_\lambda^G(R^m y, S^m y, S^m y), \\ \omega_\lambda^G(T^m z, R^m z, R^m z) \end{array} \right\}. \tag{3.92}$$

Then, T, S, R has unique common fixed point in X_{ω^G} for some positive integer $m \geq 1$.

Proof. By Corollary 3.40, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now, we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u, v, v) > 0$, then for uniqueness, inequality (3.92) we have

$$\omega_\lambda^G(u,v,v)=\omega_\lambda^G(T^m u,S^m v,R^m v)\geq a\max\left\{\begin{matrix}\omega_\lambda^G(u,v,v),\\ \omega_\lambda^G(S^m u,T^m u,T^m u),\\ \omega_\lambda^G(R^m v,S^m v,S^m v),\\ \omega_\lambda^G(T^m v,R^m v,R^m v)\end{matrix}\right\}, \quad (3.93)$$

which implies that

$$\omega_\lambda^G(u,v,v)\geq a\max\left\{\begin{matrix}\omega_\lambda^G(u,v,v),\\ \omega_\lambda^G(u,u,u),\\ \omega_\lambda^G(v,v,v),\\ \omega_\lambda^G(v,v,v)\end{matrix}\right\}=a\omega_\lambda^G(u,v,v)>\omega_\lambda^G(u,v,v). \quad (3.94)$$

This is a contradiction, since $a>1$, hence $u=v$. □

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