Nonlinear Functional Analysis and Applications
Vol. 28, No. 4 (2023), pp. 929-942
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.04.06
http://nfaa.kyungnam.ac.kr/journal-nfaa

TOPOLOGICAL ERGODIC SHADOWING AND TOPOLOGICAL PSEUDO-ORBITAL SPECIFICATION OF IFS ON UNIFORM SPACES

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Abstract. In this paper, we discuss topological ergodic shadowing property and topological pseudo-orbital specification property of iterated function systems(IFS) on uniform spaces. We show that an IFS on a sequentially compact uniform space with topological ergodic shadowing property has topological shadowing property. We define the notion of topological pseudo-orbital specification property and investigate its relation to topological ergodic shadowing property. We find that a topologically mixing IFS on a compact and sequentially compact uniform space with topological shadowing property has topological pseudo-orbital specification property and thus has topological ergodic shadowing property.

1. Introduction

Consider (X, f) to be a dynamical system on a compact metric space (X, ρ). For any $\delta > 0$, an infinite sequence $\{x_i\}_{i=0}^{\infty}$ is said to be a δ-pseudo orbit if

[0]Received February 8, 2023. Revised March 8, 2023. Accepted March 11, 2023.
[0]2020 Mathematics Subject Classification: 37B65, 54E15, 26A18.
[0]Keywords: Iterated function systems, shadowing, ergodic shadowing, pseudo-orbital specification, uniform spaces.
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$\rho(f(x_i), x_{i+1}) < \delta$ for all $i \geq 0$. For an $\epsilon > 0$, a point $x \in X$ ϵ-shadows a δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ if $\rho(f^i(x), x_i) < \epsilon$ for all $i \geq 0$. Therefore, the existence of a shadowing point for a pseudo-orbit implies that it is followed by the orbit of the shadowing point. A dynamical system (X, f) is said to have shadowing property if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit is ϵ-shadowed by a point in X.

The theory of shadowing introduced independently by Anosov [4] and Bowen [9] in the 1970s is a classical notion. It plays an important role in the development of the general qualitative theory of dynamical systems. In recent years, shadowing has been developed intensively and has become a notion of great interest. Many researchers have introduced different new notions of shadowing in dynamical systems, including average shadowing [17], h-shadowing [8], ergodic shadowing [13] and $\underline{d}$-shadowing [10].

Generally, we consider a dynamical system as a pair (X, f) where X is a compact metric space and f is a continuous self-map on X. However, in general, many notions of a dynamical system, including the shadowing property, depends on the corresponding metric. Dynamical systems on non-compact metric spaces may have these properties for one metric but not for another metric that induces the same topology. Therefore, there is an urge to study these dynamical notions in non-compact non-metrizable spaces. Two natural candidates for such extensions are uniform spaces and Hausdorff (but not necessarily compact or metric) spaces. In uniform spaces, one can mimic the existing proofs in metric spaces. Therefore, many authors have recently extended many dynamical notions to uniform spaces and Hausdorff spaces.

Alcaraz and Sanchis [3], first extended the study of dynamical systems to the totally bounded uniform space. They showed that the notions of minimality, transitivity, and chaos (in Devaney's sense) could be extended to uniform spaces. While Good and Macías [15] first defined various dynamical notions in Hausdorff spaces. Devi and Mangang [12] extended the notion of the pseudo-orbital specification to Hausdorff spaces. Recently, Akoijam and Mangang [2] have introduced the notions of ergodic shadowing, $\underline{\text{d}}$-shadowing and eventual shadowing to Hausdorff spaces.

Iterated function system (IFS), $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ which was first introduced by Hutchinson [16] and popularized by Barnsley [7] is a family of continuous self-maps on X, where X is a compact metric space and Λ is a non-empty finite set. If Λ is a singleton set, then $\mathcal{F}$ is simply a dynamical system. Hence, many dynamical notions such as transitivity, minimality, expansivity, and shadowing could be extended to iterated function systems. The most generalized case for an iterated function system would be when X is any set. Researchers have recently started discussing the notions of iterated function systems in uniform spaces.

Firstly, Samuel and Tetenov [20] have discussed the notion of attractors of iterated function systems on uniform spaces. In [22], Vasisht et al. have extended the notions of shadowing and ergodic shadowing to iterated function systems on uniform spaces. Recently, Devi and Mangang [11] have also studied the shadowing and chain properties of iterated function systems on uniform spaces. Motivated by the above works, in this paper, we aim to extend some results of ergodic shadowing and introduce the notion of the pseudo orbital specification to iterated function systems on uniform spaces.

This paper is arranged as follows. In section 2, we give some preliminaries of iterated function systems and uniform spaces that are used in the paper. In section 3, we discuss the topological ergodic shadowing property (*TESP*) of iterated function systems and prove some related results. We give an example of an iterated function system on a complete uniform space having *TESP*. In section 4, we introduce the notion of topological pseudo-orbital specification property (*TPOSP*) to iterated functions systems and study its relation to *TESP* on sequentially compact uniform spaces.

2. Preliminaries

Hutchinson [16] first introduced the concept of iterated function systems in the year 1982. An iterated function system (*IFS*), $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ is a family of self continuous mappings f_λ on X, where Λ is a non-empty finite set and (X, ρ), a compact metric space. Let $\mathbb{Z}_+$, $\mathbb{N}$ denote the sets of non-negative integers and positive integers, respectively. We denote by $\Lambda^{\mathbb{Z}_+}$, the set of all sequences $\{\lambda_i\}_{i\in\mathbb{Z}_+}$ of symbols in Λ. Let $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ be a typical element of $\Lambda^{\mathbb{Z}_+}$. Thus, we use the short notation $\mathcal{F}_{\sigma_i} = f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_1} \circ f_{\lambda_0}$. A sequence $\{x_i\}_{i\in\mathbb{Z}_+}$ in X is said to be an orbit of $\mathcal{F}$ if there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $x_i = \mathcal{F}_{\sigma_i}(x_0)$ for any $i \in \mathbb{Z}_+$, where $\mathcal{F}_{\sigma_i}(x_0) = f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_1} \circ f_{\lambda_0}(x_0)$ and $\mathcal{F}_{\sigma_0}(x_0) = x_0$. Thus, for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, orbit of a point $x \in X$ is defined as $O_\sigma(x) = \{\mathcal{F}_{\sigma_i}(x) : i \in \mathbb{Z}_+\}$.

Bahabadi [6] first extended the notion of shadowing to iterated function systems. For any $\delta > 0$, a finite sequence $\{x_i\}_{i=0}^n$ is said to be a δ-chain if there exists a finite sequence $\{\lambda_i\}_{i=0}^{n-1}$ where $\lambda_i \in \Lambda$ such that $\rho(f_{\lambda_i}(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$. A δ-pseudo orbit is an infinite δ-chain. Let $\epsilon > 0$, a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ with respect to $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+}$ is said to be ϵ-shadowed by a point $x \in X$ if $\rho(\mathcal{F}_{\sigma_i}(x), x_i) < \epsilon$ for all $i \in \mathbb{Z}_+$. An *IFS* $\mathcal{F}$ is said to have shadowing property if for any $\epsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit is ϵ-shadowed by some point in X. Many authors have also extended various types of shadowing property to iterated function systems. Fatehi Nia have studied the notion of average shadowing property in [14].

An *IFS* $\mathcal{F}$ is said to be

(i) *topologically transitive (TT)* if for any pair of nonempty open sets $U, V \subset X$, there exists $n > 0$ such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$ for some $\sigma \in \Lambda^{\mathbb{Z}_+}$.
(ii) *topologically mixing (TM)* if for any pair of non-empty open sets $U, V \subset X$, there exists $N > 0$ such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$ for every $n \geq N$ and for some $\sigma \in \Lambda^{\mathbb{Z}_+}$.
(iii) *chain transitive (CT)* if for any pair of points $x, y \in X$ and for any $\delta > 0$, there exists a δ-chain from x to y.
(iv) *chain mixing (CM)* if for any pair of points $x, y \in X$ and for any $\delta > 0$, there is $N > 0$ such that for all $n \geq N$, there exists a δ-chain from x to y of length exactly n.

The notion of uniform spaces was first introduced by Weil in [24]. By a uniform space, we mean a pair $(X, \mathcal{U})$, where X is a non-empty set and $\mathcal{U}$ is a non-empty family of subsets of $X \times X$ satisfying the following conditions:

(1) Every member E of $\mathcal{U}$ contains the diagonal Δ, where $\Delta = \{(x,x) \in X \times X : x \in X\}$.
(2) $E^{-1} \in \mathcal{U}$ if $E \in \mathcal{U}$, where $E^{-1} = \{(y,x) : (x,y) \in E\}$.
(3) For every $E \in \mathcal{U}$, there is some $D \in \mathcal{U}$ for which $D \circ D \subset E$ where $D \circ D = \{(x,y) : \exists\, z \in X$ such that $(x,z) \in D$ and $(z,y) \in D\}$.
(4) If $E, D \in \mathcal{U}$, then $E \cap D \in \mathcal{U}$.
(5) If $E \in \mathcal{U}$ and $E \subset D \subset X \times X$, then $D \in \mathcal{U}$.

$\mathcal{U}$ is said to be a uniformity of X. The members of $\mathcal{U}$ are called entourages. $E \subset X \times X$ is considered to be symmetric if $E = E^{-1}$. For every entourage $E \in \mathcal{U}$, we can find a symmetric entourage $D \in \mathcal{U}$ for which $D \circ D \subset E$. The uniform topology $\mathcal{T}_u$ is given by $\mathcal{T}_u = \{U \subset X :$ for every $x \in U$ there is $E \in \mathcal{U}$ with $E[x] \subset U\}$, where $E[x] = \{y \in X : (x,y) \in E\}$ is known as the cross section of E at x. We consider a topological space X to be uniformizable if there is a uniformity on X such that the uniform topology is the given topology. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. A function $f : X \to Y$ is considered to be uniformly continuous if for every $E \in \mathcal{V}$, there is some $D \in \mathcal{U}$ such that $(f(x), f(y)) \in E$ whenever $(x,y) \in D$. Throughout the paper, we consider the *IFS* $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ to be a collection of uniformly continuous self-maps on a uniform space X and Λ, a non-empty finite set.

Vasisht et al. [22] introduced the notions of topological shadowing, topological ergodic shadowing and topological chain properties of *IFS* on uniform spaces. For any $D \in \mathcal{U}$, a finite sequence $\{x_i\}_{i=0}^{n}$ in X is said to be a D-chain of length n if there is a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ satisfying $(f_{\lambda_i}(x_i), x_{i+1}) \in D$, where $\lambda_i \in \Lambda$, $0 \leq i \leq n-1$. A D-pseudo orbit is an infinite D-chain. Let $E \in \mathcal{U}$ be an entourage. A D-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ in $\mathcal{F}$ with respect to the infinite sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$ is said to be E-shadowed by some point z in X, if $(\mathcal{F}_{\sigma_i}(z), x_i) \in E$, $\forall\, i \in \mathbb{Z}_+$.

An iterated function system (*IFS*) $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ on a uniform space $(X,\mathcal{U})$

- (i) has the topological shadowing property (*TSP*) if for any entourage $E \in \mathcal{U}$, there exists an entourage $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by some point in X.
- (ii) is said to be topological chain transitive (*TCT*) if for any entourage $D \in \mathcal{U}$ and any pair of points $x, y \in X$, there exists a D-chain from x to y.
- (iii) is said to be topological chain mixing (*TCM*) if for any entourage $D \in \mathcal{U}$ and any pair of points $x, y \in X$, there exists positive integer N such that for all $n \geq N$ there is a D-chain from x to y of length exactly n.

For any $A \subseteq \mathbb{Z}_+$, the cardinal number of A is denoted by $|A|$. The *upper density of* A is defined by

$$\overline{d} = \limsup_{n\to\infty} \tfrac{1}{n}|A \cap \{0, 1, \cdots, n-1\}|$$

and the *lower density of* A is defined by

$$\underline{d} = \liminf_{n\to\infty} \tfrac{1}{n}|A \cap \{0, 1, \cdots, n-1\}|.$$

If $\overline{d} = \underline{d} = \mathfrak{d}$(say), then the density of A is defined as $d(A) = \mathfrak{d}$.

Let $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ be an *IFS* on a uniform space $(X,\mathcal{U})$. For any $D \in \mathcal{U}$, an infinite sequence $\{x_i\}_{i\in\mathbb{Z}_+}$ is called a D-ergodic pseudo orbit in $\mathcal{F}$ if there exists $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ such that $d(\{i \in \mathbb{Z}_+ : (f_{\lambda_i}(x_i), x_{i+1}) \notin D\}) = 0$. Let $E \in \mathcal{U}$ be an entourage. A D-ergodic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ in $\mathcal{F}$ with respect to the infinite sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$ is said to be E-ergodic shadowed by some point z in X, if $d(\{i \in \mathbb{Z}_+ : (\mathcal{F}_{\sigma_i}(z), x_i) \notin E\}) = 0$.

An *IFS* $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ on a uniform space $(X,\mathcal{U})$ has the topological ergodic shadowing property (*TESP*) if for any entourage $E \in \mathcal{U}$, there exists an entourage $D \in \mathcal{U}$ such that every D-ergodic pseudo orbit is E-ergodic shadowed by some point in X.

3. Topological ergodic shadowing property

Fakhari and Ghane [13] first introduced the concept of ergodic shadowing to dynamical systems. The main purpose of ergodic pseudo orbits is to use the notions of lower density and upper density for elements of Furstenburg families so that we can measure the set of error times in pseudo orbits [18, 19, 25]. In [21], Shabani extended the notion of ergodic shadowing to semigroup actions. Wang and Liu [23] extended this notion to iterated function systems. Recently, Ahmadi [1] extended this concept to dynamical systems in uniform spaces. Vasisht et al. [22] extended it to iterated function systems on uniform spaces. In this section, we prove that an *IFS* on a sequentially compact uniform space with *TESP* has *TSP* if one of the constituent maps is surjective.

First, we give an example of an *IFS* with *TESP*.

Example 3.1. Let $n \geq 1$ be a finite number. Consider the complete uniform space $X = \{a_0, a_1, \cdots, a_n\}$ with discrete uniformity $\mathcal{U}$. For any $0 \leq \lambda \leq n$, let $f_\lambda : X \to X$ be a uniformly continuous map defined by $f_\lambda(x) = a_\lambda$. Then, we know that the *IFS* $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$, where $\Lambda = \{0, 1, \cdots, n\}$ has *TESP*. In fact, let $E \in \mathcal{U}$ be an entourage. Since X is a discrete uniform space, $\Delta \in \mathcal{U}$ is an entourage. Let $\{x_i\}_{i\in\mathbb{Z}_+}$ be any Δ-ergodic pseudo orbit. Then, there exists some $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ such that

$$d(\{i \in \mathbb{N} : (f_{\lambda_{i-1}}(x_{i-1}), x_i) \notin \Delta\}) = 0.$$

Thus, for any $j \in \mathbb{N}\backslash\{i \in \mathbb{N} : (f_{\lambda_{i-1}}(x_{i-1}), x_i) \notin \Delta\}$, we have $(f_{\lambda_{j-1}}(x_{j-1}), x_j) \in \Delta$. This implies that $f_{\lambda_{j-1}}(x_{j-1}) = x_j$ for any $j \in \mathbb{N}\backslash\{i \in \mathbb{N} : (f_{\lambda_{i-1}}(x_{i-1}), x_i) \notin \Delta\}$. Since $f_\lambda,\ \lambda \in \Lambda$ are constant functions, for any $x \in X$ and $j \in \mathbb{N}\backslash\{i \in \mathbb{N} : (f_{\lambda_{i-1}}(x_{i-1}), x_i) \notin \Delta\}$, we have $f_{\lambda_{j-1}}(x) = x_j$. Let $z \in X$ be a point. Then, for any $j \in \mathbb{N}\backslash\{i \in \mathbb{N} : (f_{\lambda_{i-1}}(x_{i-1}), x_i) \notin \Delta\}$,

$$\begin{aligned}(\mathcal{F}_{\sigma_j}(z), x_j) &= (f_{\lambda_{j-1}} \circ f_{\lambda_{j-2}} \circ \cdots \circ f_{\lambda_0}(z), x_j)\\ &= (f_{\lambda_{j-1}}(f_{\lambda_{j-2}} \circ \cdots \circ f_{\lambda_0}(z)), x_j)\\ &= (x_j, x_j) \in \Delta \subset E.\end{aligned}$$

It follows that $d(\{j \in \mathbb{Z}_+ : (\mathcal{F}_{\sigma_j}(z), x_j) \notin E\}) = 0$. Hence, $\mathcal{F}$ has *TESP*.

The following lemma is an extension of Lemma 3.1 in [21] to iterated function systems on uniform spaces.

Lemma 3.2. *Let $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ be an IFS on a compact uniform space $(X,\mathcal{U})$ with TESP. If one of the $f_\lambda,\ \lambda \in \Lambda$ is surjective, then $\mathcal{F}$ is TCT.*

Proof. Let $E \in \mathcal{U}$ and $x, y \in X$ be any two points. Let $D \in \mathcal{U}$ be a symmetric entourage such that $D \circ D \subset E$. Since $\mathcal{F}$ has *TESP*, we can find an entourage $C \in \mathcal{U}$ such that every C-ergodic pseudo orbit is D-ergodic shadowed by some point in X. Let $a_0 = 0,\ b_0 = 1$. For any $p \geq 1$, let $a_p = b_{p-1} + p$ and $b_p = a_p + p + 1$. Suppose $f_\lambda,\ \lambda \in \Lambda$ is surjective. Then, for any $p \geq 0$, there exists a point $z_p \in X$ with $f_\lambda^p(z_p) = y$. Thus, for each $p \geq 0$ and for a fixed sequence $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$, we define

$$x_i = \begin{cases} \mathcal{F}_{\sigma_{i-a_p}}(x), & \text{if} \quad a_p \le i < b_p, \\ f_\lambda^{i-b_p}(z_p), & \text{if} \quad b_p \le i < a_{p+1}. \end{cases}$$

Then, $x_{a_p} = x,\ x_{a_p+1} = \mathcal{F}_{\sigma_1}(x),\ x_{a_p+2} = \mathcal{F}_{\sigma_2}(x), \cdots, x_{b_p-1} = \mathcal{F}_{\sigma_{b_p-a_p-1}}(x)$ and $x_{b_p} = z_p, x_{b_p+1} = f_\lambda(z_p), \cdots, x_{a_{p+1}-1} = f_\lambda^p(z_p) = y$. Therefore,

$$\{x_i\}_{i\in\mathbb{Z}_+} = \{x, y, x, \mathcal{F}_{\sigma_1}(x), z_1, y, x, \mathcal{F}_{\sigma_1}(x), \mathcal{F}_{\sigma_2}(x), z_2, f_\lambda(z_2), y, \cdots\}.$$

Again, on $[a_p, b_p)$, $\{\mathcal{F}_{\sigma_i}(x)\}_{i=0}^p$ is a piece of orbit starting from x of length p and on $[b_p, a_{p+1})$, $\{f_\lambda^i(z_p)\}_{i=0}^p$ is a piece of orbit of length p that ends at y. Thus, with respect to some $\sigma^{'} = \{\lambda_i^{'}\}_{i\in\mathbb{Z}_+}$ where

$$\lambda_i^{'} = \begin{cases} \lambda^{'}, & \text{if} \ \ i = 0,\ 1, \\ \lambda_{i-a_p}, & \text{if} \ \ a_p \le i < b_p - 1, \\ \lambda^{'}, & \text{if} \ \ i = b_p - 1, \\ \lambda, & \text{if} \ \ b_p \le i < a_{p+1} - 1, \\ \lambda^{'}, & \text{if} \ \ i = a_{p+1} - 1 \end{cases}$$

for any $p \ge 1$ and any $\lambda^{'} \in \Lambda$, we find that $\{x_i\}_{i\in\mathbb{Z}_+}$ is a C-ergodic pseudo orbit. Clearly, for $\lambda_i^{'} \in \sigma^{'}$, $d(\{i \in \mathbb{Z}_+ : (f_{\lambda_i^{'}}(x_i), x_{i+1}) \notin C\}) = 0$ as $\{i \in \mathbb{Z}_+ : (f_{\lambda_i^{'}}(x_i), x_{i+1}) \notin C\} \subset \{i \in \mathbb{Z}_+ : \lambda_i^{'} = \lambda^{'}\}$ and

$$d(\{i \in \mathbb{Z}_+ : \lambda_i^{'} = \lambda^{'}\}) \le \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0.$$

If z D-ergodic shadows this C-ergodic pseudo orbit, then

$$d(\{i \in \mathbb{Z}_+ : (\mathcal{F}_{\sigma_i^{'}}(z), x_i) \notin D\}) = 0.$$

Therefore, $\{i \in \mathbb{Z}_+ : (\mathcal{F}_{\sigma_i^{'}}(z), x_i) \in D\}$ must intersect infinitely many intervals $[a_p, b_p)$ and infinitely many intervals $[b_p, a_{p+1})$. Thus, we can find $s \in [a_p, b_p)$ such that $(x_s, \mathcal{F}_{\sigma_s^{'}}(z)) \in D$ which implies that $(\mathcal{F}_{\sigma_{s-a_p}}(x), \mathcal{F}_{\sigma_s^{'}}(z)) \in D$.

Also, we can find $t \in [b_{p^{'}}, a_{p^{'}+1})$ for some $p^{'} \ge p$ such that $(\mathcal{F}_{\sigma_t^{'}}(z), x_t) \in D$. It follows that $(\mathcal{F}_{\sigma_t^{'}}(z), f_\lambda^{t-b_{p^{'}}}(z_{p^{'}})) \in D$. Since $D = \Delta \circ D \subset D \circ D \subset E$, it follows that $(\mathcal{F}_{\sigma_{s-a_p}}(x), \mathcal{F}_{\sigma_s^{'}}(z)) \in E$ and $(\mathcal{F}_{\sigma_t^{'}}(z), f_\lambda^{t-b_{p^{'}}}(z_{p^{'}})) \in E$. Hence, we have an E-chain $\{x, \mathcal{F}_{\sigma_1}(x), \mathcal{F}_{\sigma_2}(x), \cdots, \mathcal{F}_{\sigma_{s-a_p-1}}(x), \mathcal{F}_{\sigma_s^{'}}(z), \mathcal{F}_{\sigma_{s+1}^{'}}(z), \cdots,$ $\mathcal{F}_{\sigma_{t-1}^{'}}(z), f_\lambda^{t-b_{p^{'}}}(z_{p^{'}}), f_\lambda^{t+1-b_{p^{'}}}(z_{p^{'}}), \cdots . f_\lambda^{p^{'}-1}(z_{p^{'}}), f_\lambda^{p^{'}}(z_{p^{'}}) = y\}$ from x to y. □

Shabani [21] showed that a finitely generated semigroup with ergodic shadowing property enjoys the ordinary shadowing property if one of the generators is surjective. In [23] Proposition 5.5, the authors showed that an *IFS* with ergodic shadowing property has shadowing property if one of the constituent maps is surjective. In the following theorem, we extend this result to iterated function systems on sequentially compact uniform spaces using Lemma 3.2.

Theorem 3.3. *Let $\mathcal{F} = \{X; f_\lambda : \lambda \in \Lambda\}$ be an IFS on a sequentially compact uniform space $(X,\mathcal{U})$ with TESP. Suppose one of the f_λ, $\lambda \in \Lambda$ is surjective, then $\mathcal{F}$ has TSP.*

Proof. Let E be an entourage and D, a symmetric entourage such that $D \circ D \subset E$. Since $\mathcal{F}$ has *TESP*, there exists $C \in \mathcal{U}$ such that every C-ergodic pseudo orbit is D-ergodic shadowed by some point in X. Let $\{x_i\}_{i\in\mathbb{Z}_+}$ be a C-pseudo orbit. Then, there exists $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ such that $(f_{\lambda_i}(x_i), x_{i+1}) \in C$ for all $\lambda_i \in \sigma$. Therefore, for each $n \geq 0$, $\{x_0, x_1, \cdots, x_n\}$ is a C-chain with respect to the finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$. By Lemma 3.2, we can find a C-chain from x_n to x_0, say $\{y_0 = x_n, y_1, \cdots, y_p = x_0\}$. That is, there exists a finite sequence $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{p-1}^{'}\}$ for which $(f_{\lambda_i^{'}}(y_i), y_{i+1}) \in C$, $i = 0, 1, \cdots, p-1$. If we consider $\{z_j\}_{j\in\mathbb{Z}_+} = \{x_0, x_1, \cdots, x_n, y_1, y_2, \cdots, y_{p-1}, x_0, x_1, \cdots\}$, then it is a C-pseudo orbit with respect to

$$\sigma^{''} = \{\lambda_i^{''}\}_{i\in\mathbb{Z}_+} = \{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{p-1}^{'}, \lambda_0, \lambda_1, \cdots\}.$$

Since every C-pseudo orbit is a C-ergodic pseudo orbit, for each $n \geq 0$, there exists a point $z_n \in X$ which D-ergodic shadows this C-ergodic pseudo orbit. Thus, $\{j \in \mathbb{Z}_+ : (\mathcal{F}_{\sigma_j^{''}}(z_n), z_j) \notin D\}$ cannot meet every interval containing $\{x_0, x_1, \cdots, x_n\}$, otherwise it would have positive density. Hence, for each $n \geq 0$, at least one $\{x_0, x_1, \cdots, x_n\}$ is entirely D-shadowed by a piece of orbit of z_n. Therefore, for each $n \geq 0$, we can find some $w_n \in \{\mathcal{F}_{\sigma_m^{''}}(z_n) : m \in \mathbb{Z}_+\}$ such that $(\mathcal{F}_{\sigma_i}(w_n), x_i) \in D$ for $i = 0, 1, \cdots, n$. Since X is sequentially compact, we can find a subsequence $\{w_{n_k}\}_{k\in\mathbb{Z}_+}$ of $\{w_n\}_{n\in\mathbb{Z}_+}$ and a point $w \in X$ such that $w_{n_k} \to w$ as $k \to \infty$. By uniform continuity of f_λ, $\lambda \in \Lambda$, for each $i \geq 0$, there exists $N_i > 0$ such that for every $k \geq N_i$, we have $(\mathcal{F}_{\sigma_i}(w_{n_k}), \mathcal{F}_{\sigma_i}(w)) \in D$. Also, $(\mathcal{F}_{\sigma_i}(w_{n_k}), x_i) \in D$. This implies that $(\mathcal{F}_{\sigma_i}(w), x_i) \in E$ for all $i \geq 0$ as $k \to \infty$. Hence, $\mathcal{F}$ has *TSP*. □

4. Topological pseudo-orbital specification property

Specification property introduced by Bowen in [9] to study the ergodic property of Axiom A diffeomorphisms is another variant of shadowing. A continuous map f on a compact metric space X is said to have specification property if one can approximate distinct finite pieces of orbits by an actual orbit with a certain uniformity. In [5], it has shown that a mapping with the specification property is chaotic in the sense of Devaney. Pseudo-orbital specification property is some kind of specification property introduced by Fakhari and Ghane in [13]. Here, they have investigated its relation with ergodic shadowing property. In [21], Shabani extended this notion to semigroup actions and studied its relation to ergodic shadowing property. In this section, we define topological pseudo-orbital specification property for iterated function systems on uniform spaces and study its relation to topological ergodic shadowing property.

Definition 4.1. An *IFS* $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ has topological pseudo-orbital specification property ($TPOSP$) if for any entourage $E \in \mathcal{U}$, there exist an entourage $D \in \mathcal{U}$ and $K > 0$ such that for any non-negative integers $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_n \le b_n$ with $a_{j+1} - b_j \ge K$ and D-chains ξ_j with respect to the finite sequences $\{\lambda^j_{a_j}, \cdots, \lambda^j_{b_j-1}\}$ where $\xi_j = \{x_{(j,i)}\}$, $i \in I_j = [a_j, b_j]$ and $1 \le j \le n$, we can find a point $z \in X$ and $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+}$ with $\lambda_i = \lambda^j_i$ for $i \in [a_j, b_j - 1]$ and $1 \le j \le n$ such that $(\mathcal{F}_{\sigma_i}(z), x_{(j,i)}) \in E$, $i \in I$ *and* $1 \le j \le n$.

Theorem 4.2. *Let $\mathcal{F}$ be a TM IFS on a compact uniform space $(X,\mathcal{U})$ with TSP. Then $\mathcal{F}$ has TPOSP.*

Proof. Let $E \in \mathcal{U}$. Since $\mathcal{F}$ has TSP, there exists $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by some point in X. Since $\{f_\lambda : \lambda \in \Lambda\}$ is a finite family of uniformly continuous maps, it is uniformly equicontinuous. Therefore, we can find an entourage $C \subset D \in \mathcal{U}$ such that for any $x, y \in X$ with $(x, y) \in C$, we have $(f_\lambda(x), f_\lambda(y)) \in D$ for all $\lambda \in \Lambda$. Let $B \in \mathcal{U}$ be a symmetric entourage such that $B \circ B \subset C$. As X is compact, we can find a finite set of points $\{x_1, x_2, \cdots, x_N\}$ such that $\mathfrak{C} = \{int_X B[x_1], int_X B[x_2], \cdots, int_X B[x_N]\}$ is an open cover of X. Since $\mathcal{F}$ is topologically mixing, for any pair of non-empty open sets $int_X B[x_p]$ and $int_X B[x_q]$ in $\mathfrak{C}$, there exist $N_{pq} > 0$ and $\sigma^{pq} \in \Lambda^{\mathbb{Z}_+}$ such that $\mathcal{F}_{\sigma^{pq}_n}(int_X B[x_p]) \cap int_X B[x_q] \neq \phi$ for all $n \ge N_{pq}$. Let

$$K = \max\{N_{pq} : 1 \le p, q \le N\}.$$

Let $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_n \le b_n$ be any non-negative integers with $a_{j+1} - b_j \ge K$ and let $\xi_j = \{x_{(j,i)}\}$, $i \in I_j = [a_j, b_j]$ be D-chains with respect to the finite sequence $\{\lambda^j_{a_j}, \cdots, \lambda^j_{b_j-1}\}$ for $1 \le j \le n$. Let $k_j = a_{j+1} - b_j$. Then by the choice of K, there are $int_X B[x_p]$, $int_X B[x_q] \in \mathfrak{C}$ with $x_{(j,b_j)} \in int_X B[x_p]$ and $x_{(j+1,a_{j+1})} \in int_X B[x_q]$ and there is some $\sigma^{pq} = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ such that

$$\mathcal{F}_{\sigma^{pq}_{k_j}}(int_X B[x_p]) \cap int_X B[x_q] \neq \phi.$$

That is, there exists $y_j \in int_X B[x_p]$ such that $\mathcal{F}_{\sigma^{pq}_{k_j}}(y_j) \in int_X B[x_q]$.

Consider

$$\lambda_i' = \begin{cases} \lambda_i^j, & \text{if} \quad a_j \leq i \leq b_j - 1, \\ \lambda_{i-b_j}, & \text{if} \quad b_j \leq i \leq a_{j+1} - 1, \end{cases}$$

and $\eta_j = \{f_{\lambda_0}(y_j) = \mathcal{F}_{\sigma_1^{pq}}(y_j), \cdots, \mathcal{F}_{\sigma_{k_j-1}^{pq}}(y_j)\}$ for $1 \leq j \leq n$. Then, $\{\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_{n-1}, \eta_{n-1}, \xi_n, \eta_n\}$ is a D-chain with respect to the finite sequence $\{\lambda_i'\}$ which can be extended to a D-pseudo orbit. Thus, it can be E-shadowed by some point $z \in X$. Hence, $\mathcal{F}$ has *TPOSP*. □

In the following, we give an example of a *TM IFS* with *TSP* which has *TPOSP*.

Example 4.3. Let $X = \{a, b, c, d\}$. Consider the subset $D = \Delta_X \cup \{(a, b), (b, a), (c, d), (d, c)\}$ of $X \times X$. Then, $\mathcal{U} = \{E \subset X \times X : D \subseteq E\}$ is a uniformity of X which induces the topology $\mathcal{T}_{\mathcal{U}} = \{\phi, \{a, b\}, \{c, d\}, X\}$. Let $f_1, f_2 : X \to X$ be uniformly continuous mappings defined by

$$f_1(a) = b,\ f_1(b) = a,\ f_1(c) = d,\ f_1(d) = c;$$
$$f_2(a) = c,\ f_2(b) = d,\ f_2(c) = a,\ f_2(d) = b.$$

Then, the *IFS*, $\mathcal{F} = \{X; f_1, f_2\}$ is *TM* with *TSP* which has *TPOSP*.

Proof. First, we show that $\mathcal{F}$ is *TM*. Let U, V be any pair of non-empty open sets in X. If $U \neq V$, then we can find a sequence

$$\sigma = \{\lambda_i\}_{i \in \mathbb{Z}_+} = \begin{cases} 2, & \text{if } i = 2, \\ 1, & \text{otherwise} \end{cases}$$

such that $\mathcal{F}_{\sigma_i}(U) \cap V \neq \phi$ for all $i \geq 3$. If $U = V$, then we can find a sequence

$$\sigma = \{\lambda_i\}_{i \in \mathbb{Z}_+} = \begin{cases} 2, & \text{if } i = 0, 1, \\ 1, & \text{otherwise} \end{cases}$$

such that $\mathcal{F}_{\sigma_i}(U) \cap V \neq \phi$ for all $i \geq 3$. Thus, $\mathcal{F}$ is *TM*.

Now, we show that $\mathcal{F}$ has *TSP*. Let $E \in \mathcal{U}$ be any entourage. Suppose $\{x_i\}_{i \in \mathbb{Z}_+}$ is any D-pseudo orbit in X. Then, there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$ where $\Lambda = \{1, 2\}$ such that $(f_{\lambda_i}(x_i), x_{i+1}) \in D$ for all $i \geq 0$. By induction, we claim that $(\mathcal{F}_{\sigma_i}(x_0), x_i) \in D$ for all $i \geq 0$. For $i = 0$, it is obvious. Suppose the claim holds true for $i = n$. That is, $(\mathcal{F}_{\sigma_n}(x_0), x_n) \in D$. We know that, $x_{n+1} \in \{a, b, c, d\}$. First, let $x_{n+1} \in \{a, b\}$. Since $(f_{\lambda_n}(x_n), x_{n+1}) \in D$, we have $f_{\lambda_n}(x_n) \in \{a, b\}$. This implies that $x_n \in \{a, b\}$ if $\lambda_n = 1$ and $x_n \in \{c, d\}$ if $\lambda_n = 2$. Also, $(\mathcal{F}_{\sigma_n}(x_0), x_n) \in D$ gives that $\mathcal{F}_{\sigma_n}(x_0) \in \{a, b\}$ if $\lambda_n = 1$ and $\mathcal{F}_{\sigma_n}(x_0) \in \{c, d\}$ if $\lambda_n = 2$. Thus, $\mathcal{F}_{\sigma_{n+1}}(x_0) = f_{\lambda_n}(\mathcal{F}_{\sigma_n}(x_0)) \in \{a, b\}$ for any λ_n. This follows that $(\mathcal{F}_{\sigma_{n+1}}(x_0), x_{n+1}) \in \{(a, a), (b, b), (a, b), (b, a)\} \subset D$ if $x_{n+1} \in \{a, b\}$.

Similarly, it can be shown that $(\mathcal{F}_{\sigma_{n+1}}(x_0), x_{n+1}) \in \{(c, c), (d, d), (c, d), (d, c)\} \subset D$ if $x_{n+1} \in \{c, d\}$. Thus, by induction, we have $(\mathcal{F}_{\sigma_i}(x_0), x_i) \in D$ for all

$i \geq 0$. Since $D \subseteq E$, we have $(\mathcal{F}_{\sigma_i}(x_0), x_i) \in E$ for all $i \geq 0$. Therefore, any D-pseudo orbit in X is E-shadowed by x_0. Hence, $\mathcal{F}$ has *TSP*.

Lastly, we show that $\mathcal{F}$ has *TPOSP*. Let $E \in \mathcal{U}$ be any entourage. Then, $D \subseteq E$. Also, for any $x, y \in X$ with $(x, y) \in D$, we can easily see that $(f_\lambda(x), f_\lambda(y)) \in D$ for all $\lambda \in \Lambda$. $\mathfrak{C} = \{\{a, b\}, \{c, d\}\}$ is a finite open cover of X. Let $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ be any non-negative integers with $a_{j+1} - b_j \geq 3$ and let $\xi_j = \{x_{(j,i)}\}$, $i \in I_j = [a_j, b_j]$ be D-chains with respect to the finite sequences $\{\lambda^j_{a_j}, \cdots, \lambda^j_{b_j-1}\}$ for $1 \leq j \leq n$. Put $k_j = a_{j+1} - b_j$, then $k_j \geq 3$. Suppose $x_{(j,b_j)} \in \{a, b\} = U$ and $x_{(j+1,a_{j+1})} \in \{c, d\} = V$ $(U \neq V)$. As, $k_j \geq 3$, we have

$$\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} = \begin{cases} 2, & \text{if } i = 2, \\ 1, & \text{otherwise} \end{cases}$$

such that $\mathcal{F}_{\sigma_{k_j}}(\{a, b\}) \cap \{c, d\} \neq \phi$. That is, there exists $y_j \in \{a, b\}$ such that $\mathcal{F}_{\sigma_{k_j}}(y_j) \in \{c, d\}$.

Consider

$$\lambda'_i = \begin{cases} \lambda^j_i, & \text{if } a_j \leq i \leq b_j - 1, \\ \lambda_{i-b_j}, & \text{if } b_j \leq i \leq a_{j+1} - 1, \end{cases}$$

and $\eta_j = \{f_{\lambda_0}(y_j) = \mathcal{F}_{\sigma_1}(y_j), \cdots, \mathcal{F}_{\sigma_{k_j-1}}(y_j)\}$ for $1 \leq j \leq n$. Now, $y_j \in \{a, b\}$ and $x_{(j,b_j)} \in \{a, b\}$ implies that $(x_{(j,b_j)}, y_j) \in \{(a.a), (b, b), (a, b), (b, a)\} \subset D$. Again, $\mathcal{F}_{\sigma_{k_j}}(y_j), x_{(j+1,a_{j+1})} \in \{c, d\}$ implies that $(\mathcal{F}_{\sigma_{k_j}}(y_j), x_{(j+1,a_{j+1})}) \in \{(c, c), (d, d), (c, d), (d, c)\} \subset D$. Therefore, $\{\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \eta_{n-1}, \xi_n, \eta_n\}$ is a D-chain with respect to the finite sequence $\{\lambda'_i\}$ which can be extended to a D-pseudo orbit. Thus, it can be E-shadowed by $x_{(1,a_1)}$.

By proceeding similarly, we can obtain similar results when $x_{(j,b_j)} \in U$ and $x_{(j+1,a_{j+1})} \in V$ for any pair U, $V \in \mathfrak{C}$. Hence, $\mathcal{F}$ has *TPOSP*. □

Theorem 4.4. *Let $\mathcal{F}$ be an IFS on a sequentially compact uniform space $(X, \mathcal{U})$ with TPOSP. Then $\mathcal{F}$ has TESP.*

Proof. Let $E \in \mathcal{U}$ be an entourage and let $D \in \mathcal{U}$ and $K > 0$ be as in the definition of *TPOSP* with respect to E. Let $\{x_i\}_{i\in\mathbb{Z}_+}$ be a D-ergodic pseudo orbit. Then, there exists an infinite sequence $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ for which $d(\{i \in \mathbb{Z}_+ : (f_{\lambda_i}(x_i), x_{i+1}) \notin D\}) = 0$. Let $N = \{i \in \mathbb{Z}_+ : (f_{\lambda_i}(x_i), x_{i+1}) \in D\}$ and take a sequence $a_1 < b_1 < a_2 < b_2 < \cdots$ of non-negative integers satisfying the following properties:

(1) for any n, $[a_n, b_n] \subset N$.
(2) for any n, $a_{n+1} - b_n \geq K$.
(3) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{(a_{k+1}-b_k)}{b_n} \to 0$.

Take $\xi_j = \{x_{(j,i)}\} := \{x_i\}$ where $i \in I_j = [a_j, b_j]$ and $j \in \mathbb{N}$. Then, each ξ_j is a D-chain in X defined on the sub-interval I_j, $j \in \mathbb{N}$ with respect to the finite sequence $\{\lambda^j_{a_j}, \cdots, \lambda^j_{b_j-1}\} \subset \sigma$. For any n, by *TPOSP*, there exists a point $z_n \in X$ which E-shadows the n pieces of D-ergodic pseudo orbits corresponding to n intervals $[a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n]$. Since X is sequentially compact, we can find a convergent subsequence $\{z_{n_k}\}_{k\in\mathbb{Z}_+}$ of $\{z_n\}_{n\in\mathbb{Z}_+}$ which E-shadows the n_k pieces of D-ergodic pseudo orbits corresponding to n_k intervals $[a_1, b_1], [a_2, b_2], \cdots, [a_{n_k}, b_{n_k}]$. Without loss of generality, if $z_{n_k} \to z$, then $\{x_i\}_{i\in\mathbb{Z}_+}$ is E-ergodic shadowed by z. Hence, $\mathcal{F}$ has *TESP*. □

Corollary 4.5. *Let $\mathcal{F}$ be a TM IFS on a sequentially compact uniform space $(X,\mathcal{U})$ with TSP. Then $\mathcal{F}$ has TPOSP and thus has TESP.*

In [21] (see Lemma 5.2), it has been proved that a semigroup with pseudo-orbital specification property is topologically mixing and has the shadowing property. In the following theorem, we show that an *IFS* on a sequentially compact uniform space with *TPOSP* is *TM* and has *TSP*.

Theorem 4.6. *Let $\mathcal{F}$ be an IFS on a sequentially compact uniform space $(X,\mathcal{U})$ with TPOSP. Then $\mathcal{F}$ is TM and has TSP.*

Proof. First, we claim that $\mathcal{F}$ is *TM*. Let $U, V \subset X$ be non-empty open sets and $E \in \mathcal{U}$ be a symmetric entourage for which $E[x] \subset U$ and $E[y] \subset V$ where $x, y \in X$. Let $D \in \mathcal{U}$ and $K > 0$ be as in the definition of *TPOSP* with respect to E. Put $a_1 = 0$ and take a number $b_1 > 0 = a_1$, then put $a_2 = K + b_1$. Take a number $b_2 > a_2 = K + b_1$. Let $x_{(1,i)} = \mathcal{F}_{\sigma^1_{i-a_1}}(x)$ and $x_{(2,i)} = \mathcal{F}_{\sigma^2_{i-a_2}}(y)$ where $\sigma^j = \{\lambda^j_{a_j}, \cdots, \lambda^j_{b_j-1}\}$ for $j = 1, 2$ and $i \in I_j = [a_j, b_j]$. Clearly, $\xi_j = x_{(j,i)}$, $i \in I_j = [a_j, b_j]$, $j = 1, 2$ are D-chains. Therefore, by *TPOSP*, there exists a point $z \in X$ and $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ with $\lambda_i = \lambda^j_i$ for $i \in [a_j, b_j - 1]$ and $j = 1, 2$ such that $\big(\mathcal{F}_{\sigma_i}(z), x_{(j,i)}\big) \in E$. Particularly, $\big(\mathcal{F}_{\sigma_{a_1}}(z), x_{(1,a_1)}\big) = (z, x) \in E$ and $\big(\mathcal{F}_{\sigma_{a_2}}(z), x_{(2,a_2)}\big) = \big(\mathcal{F}_{\sigma_{a_2}}(z), y\big) \in E$. This implies that $z \in E[x] \subset U$ and $\mathcal{F}_{\sigma_{a_2}}(z) \in E[y] \subset V$ which gives that $\mathcal{F}_{\sigma_{b_1+K}}(U) \cap V \neq \phi$. With similar argument, we can show that for any $n > b_1 + K$, there exists a sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$. Hence $\mathcal{F}$ is *TM*.

Next, we show that $\mathcal{F}$ has *TSP*. Let $E \in \mathcal{U}$, and let $D \in \mathcal{U}$ be a symmetric entourage such that $D \circ D \subset E$. Consider $C \in \mathcal{U}$ as in the definition of *TPOSP* with respect to D. Suppose $\{x_i\}_{i\in\mathbb{Z}_+}$ is a C-pseudo orbit. Then, there exists an infinite sequence $\sigma = \{\lambda_i\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ for which $(f_{\lambda_i}(x_i), x_{i+1}) \in C$. Thus, for any $n \in \mathbb{N}$, $\{x_0, x_1, x_2, \cdots, x_n\}$ is a C-chain with respect to the finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$. Take $a_1 = 0, b_1 = n$ and $x_{(1,i)} = x_i$, $i \in [0, n]$. Then, $\xi_1 = \{x_{(1,i)}\}$ is a C-chain. By *TPOSP*, there is a point $z_n \in X$ and

$\sigma' = \{\lambda_i'\}_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$ with $\lambda_i' = \lambda_i$ for $i \in [0, n-1]$ such that $(\mathcal{F}_{\sigma_i'}(z_n), x_i) \in D$, $i \in [0, n]$. Since X is compact, we can find a convergent subsequence $\{z_{n_k}\}_{k\in\mathbb{Z}_+}$ of $\{z_n\}_{n\in\mathbb{Z}_+}$ such that $(\mathcal{F}_{\sigma_i'}(z_{n_k}), x_i) \in D$, $i \in [0, n_k]$. If $z_{n_k} \to z$, then $(\mathcal{F}_{\sigma_i'}(z), x_i) \in D \circ D \subset E$ for all $i \geq 0$. Hence, $\mathcal{F}$ has *TSP*. □

5. Conclusion

In this paper, we find that an *IFS* $\mathcal{F}$ on a compact uniform space $(X,\mathcal{U})$ with *TESP* is topological chain transitive (*TCT*) if one of the constituent maps f_λ is surjective. We also obtain that an *IFS* $\mathcal{F}$ on a sequentially compact uniform space $(X,\mathcal{U})$ with *TESP* has topological shadowing property *TSP* if one of the constituent maps f_λ is surjective. We also find that a topologically mixing (*TM*) *IFS* on a compact uniform space with *TSP* has *TPOSP*. In particular, on a compact and sequentially compact uniform space, we show that a topologically mixing *IFS* with the topological shadowing property has the topological pseudo-orbital specification property and thus has the topological ergodic shadowing property.

Acknowledgments: We want to thank the anonymous referees for their valuable comments, suggestions, and remarks that contributed to the improvement of the initial version of the manuscript. We also thank University Grants Commission (UGC), New Delhi, India, for funding this article.

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