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IMPROVED VERSION ON SOME INEQUALITIES OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree n and $P(z) \neq 0$ in $|z| < 1$. Then for every real α and $R > 1$, Aziz [1] proved that

$$\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}},$$

where

$$M_\alpha = \max_{1 \leq k \leq n} |P(e^{i(\alpha+2k\pi)n})|.$$

In this paper, we establish some improvements and generalizations of the above inequality concerning the polynomials and their ordinary derivatives.

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1. Introduction

If $P(z)$ is a polynomial of degree n, then according to a well-known classical result due to Bernstein [2]

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Also a simple deduction from the maximum modulus principle yields that,

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

Both inequalities (1.1) and (1.2) are sharp and equality hold if $p(z)$ has all its zeros at the origin.

As a refinement of (1.1) Frappier et al. [3] proved that

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{1\leq k\leq n} |P(e^{ik\pi n})|. \tag{1.3}$$

It is evident that inequality (1.3) is a refinement of (1.1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z) = z^n + ia, a > 0$.

Aziz [1] improved the bound of inequality (1.3) by proving that, for every real α

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{1.4}$$

where

$$M_\alpha = \max_{1\leq k\leq n} |P(e^{i(\alpha+2k\pi)n})|. \tag{1.5}$$

If we restrict ourselves to the class of polynomials $P(z)$ having no zero in $|z| < 1$, then Erdös conjectured and later Lax [4] proved that

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.6}$$

For the same class of polynomials, inequality (1.6) was improved by Aziz [1] by proving that, for every real α,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.7}$$

where M_α is as defined in (1.5).

Further, if $P(z)$ is a polynomial having no zero in $|z| < 1$, Aziz [1] also proved that, for every real α and $R > 1$,

$$\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}. \tag{1.8}$$

Rather et al. [6] obtained a generalization of inequality (1.8) by proving that if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for every real α and $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.9}$$

where M_α is as defined in (1.5) (see also [7, 8]).

They further improved the bound of (1.9) by involving $m = \min_{|z|=k} |P(z)|$ and obtained

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}, \tag{1.10}$$

where M_α is as defined in (1.5). Further, by involving some coefficients of the polynomial, Rather et al. [6] improved the bound in (1.9) and proved that, if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for every real α and $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.11}$$

where M_α is as defined in (1.5). Rather et al. [6] improved inequality (1.10) under the same hypothesis and obtained,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}, \tag{1.12}$$

where $m = \min_{|z|=k} |P(z)|$ and M_α is as defined in (1.5).

2. Lemmas

We need the following lemmas to prove our theorems. This lemma is a special case of a result due to Frappier et al. [3].

Lemma 2.1. *If* $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n*, then for* $R\geq 1$*,*

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)|-\left(R^n-R^{n-2}\right)|P(0)| \text{ if } n>2 \tag{2.1}$$

and

$$\max_{|z|=R}|P(z)|\leq R^2\max_{|z|=1}|P(z)|-\left(R^2-1\right)|P(0)| \text{ if } n=2. \tag{2.2}$$

Lemma 2.2. *If* $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n *having no zero in* $|z|<k$*,* $k\geq 1$*, then*

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu\leq 1. \tag{2.3}$$

This lemma is due to Qazi [5].

The following four lemmas are due to Rather et al. [6].

Lemma 2.3. *If* $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j, 1\leq\mu\leq n$*, is a polynomial of degree* $n\geq 2$ *having no zero in* $|z|<k, k\geq 1$*, then for every real* α*,*

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{\sqrt{2(1+k^{2\mu})}}(M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{2.4}$$

where M_α *is as defined in* (1.5).

Lemma 2.4. *If* $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j, 1\leq\mu\leq n$*, is a polynomial of degree* $n\geq 2$ *having no zero in* $|z|<k, k\geq 1$ *and* $m=\min_{|z|=k}|P(z)|$*, then for every real* α*,*

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{\sqrt{2(1+k^{2\mu})}}(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}, \tag{2.5}$$

where M_α *is as defined in* (1.5).

Lemma 2.5. *If* $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j, 1\leq\mu\leq n$*, is a polynomial of degree* $n\geq 2$ *having no zero in* $|z|<k, k\geq 1$*, then for every real* α*,*

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{\sqrt{2\left\{1+k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}}(M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{2.6}$$

where M_α *is as defined in* (1.5).

Lemma 2.6. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree* $n \ge 2$ *having no zero in* $|z| < k, k \ge 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for every real* α,

$$\max_{|z|=1} |P^{'}(z)| \le \frac{n}{\sqrt{2\left\{1+k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}}(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}, \tag{2.7}$$

where M_α *is as defined in* (1.5).

3. Main results

In this paper, we not only obtain improvement and generalization of inequality (1.8) but also improves inequalities (1.9), (1.10), (1.11) and (1.12) respectively. In fact, we proved

Theorem 3.1. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree* $n \ge 3$ *having no zero in* $|z| < k, k \ge 1$, *then for every real* α *and* $R > 1$,

$$\begin{aligned} |P(Rz) - P(z)| \quad \le \quad & \frac{R^n - 1}{\sqrt{2(1+k^{2\mu})}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ & - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P^{'}(0)| \quad \text{for} \ \ n > 3 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} |P(Rz) - P(z)| \quad \le \quad & \frac{R^3 - 1}{\sqrt{2(1+k^{2\mu})}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ & - \left[\frac{R^3-1}{3} - (R-1)\right]|P^{'}(0)| \quad \text{for} \ \ n = 3, \end{aligned} \tag{3.2}$$

where M_α *is as defined in* (1.5).

Proof. Applying Lemma 2.1 to the polynomial $P^{'}(z)$ which is of degree greater than or equal to 3 and using inequality (2.4) of Lemma 2.3, we obtain for $t \ge 1$

and $0 \leq \theta < 2\pi$,

$$\begin{aligned}\left|P'(te^{i\theta})\right| &\leq t^{n-1} \max_{|z|=1} |P'(z)| - \left(t^{n-1} - t^{n-3}\right) |P'(0)| \\ &\leq t^{n-1} \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \left(t^{n-1} - t^{n-3}\right) |P'(0)|. \quad (3.3)\end{aligned}$$

Hence for each $\theta, 0 \leq \theta < 2\pi$ and $R > 1$, we have

$$\begin{aligned}\left|P(Re^{i\theta}) - P(e^{i\theta})\right| &= \left|\int_1^R e^{i\theta} P'(te^{i\theta}) dt\right| \\ &\leq \int_1^R \left|P'(te^{i\theta})\right| dt. \quad (3.4)\end{aligned}$$

Equivalently, (3.3) becomes

$$\begin{aligned}\left|P(Re^{i\theta}) - P(e^{i\theta})\right| &\leq \int_1^R t^{n-1} dt \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ &\quad - \int_1^R \left(t^{n-1} - t^{n-3}\right) dt |P'(0)| \\ &\leq \frac{R^n - 1}{n} \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ &\quad - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |P'(0)|. \quad (3.5)\end{aligned}$$

This implies for $|z| = 1$ and $R > 1$,

$$\begin{aligned}|P(Rz) - P(z)| &\leq \frac{R^n - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ &\quad - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |P'(0)| \text{ for } n > 3,\end{aligned}$$

and similarly, for $n = 3$,

$$\begin{aligned}|P(Rz) - P(z)| &\leq \frac{R^3 - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ &\quad - \left[\frac{R^3 - 1}{3} - (R-1)\right] |P'(0)|,\end{aligned}$$

which completes the proof. □

Remark 3.2. Inequalities (3.1) and (3.2) provide an improvement of inequality (1.9) when $P'(0) \neq 0$.

Theorem 3.3. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree* $n \ge 3$ *having no zero in* $|z| < k, k \ge 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for every real* α *and* $R > 1$,

$$\begin{aligned} |P(Rz) - P(z)| \quad &\le \quad \frac{R^n - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} \\ &\quad - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P^{'}(0)| \quad \text{for} \quad n > 3 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} |P(Rz) - P(z)| \quad &\le \quad \frac{R^3 - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} \\ &\quad - \left[\frac{R^3 - 1}{3} - (R-1) \right] |P^{'}(0)| \quad \text{for} \quad n = 3, \end{aligned} \tag{3.7}$$

where M_α *is as defined in* (1.5).

Proof. Proof of this theorem follows on the same line as that of the proof of Theorem 3.1 but instead of using Lemma 2.3, we used Lemma 2.4. We omit the proof. □

Remark 3.4. Inequalities (3.6) and (3.7) give an improvement of inequality (1.10) when $P^{'}(0) \neq 0$.

We next prove the following two results, which not only gives improvements of inequalities (1.11) and (1.12) but also provide an improvement and generalization of Theorems 3.1 and 3.3 under the same hypotheses.

Theorem 3.5. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree* $n \ge 3$ *having no zero in* $|z| < k, k \ge 1$, *then for every real* α *and* $R > 1$,

$$\begin{aligned} |P(Rz) - P(z)| \le \quad &\frac{R^n - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}} \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} \\ &- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P^{'}(0)| \quad \text{for} \quad n \ge 3, \end{aligned} \tag{3.8}$$

and

$$|P(Rz) - P(z)| \leq \frac{R^3 - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \left[\frac{R^3 - 1}{3} - (R - 1)\right]|P^{'}(0)| \quad for \quad n = 3, \tag{3.9}$$

where M_α *is as defined in* (1.5).

Proof. Proof of this theorem follows on the same line as that of the proof of Theorem 3.1 but instead of using Lemma 2.3, we used Lemma 2.5. We omit the proof. □

Remark 3.6. Inequalities (3.8) and (3.9) yield an improvement of inequality (1.11) when $P^{'}(0) \neq 0$.

Remark 3.7. To show that Theorem 3.5 yields a better bound than that of Theorem 3.1, it is sufficient to show that

$$\frac{1}{\sqrt{2(1 + k^{2\mu})}} \geq \frac{1}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}} \tag{3.10}$$

that is,

$$1 + k^{2\mu} \leq 1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2$$

or

$$k^\mu \leq k^{(\mu+1)}\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}$$

which is obvious by (2.3) of Lemma 2.2.

Theorem 3.8. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$, *is a polynomial of degree* $n \geq 3$ *having no zero in* $|z| < k, k \geq 1$ *and* $m = \min_{|z|=k}|P(z)|$, *then for every* *real* α *and* $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}} \times (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P^{'}(0)| \quad \textit{for} \;\; n > 3, \tag{3.11}$$

and

$$|P(Rz) - P(z)| \leq \frac{R^3 - 1}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}} \times (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \left[\frac{R^3 - 1}{3} - (R-1)\right]|P^{'}(0)| \quad \textit{for} \;\; n = 3, \tag{3.12}$$

where M_α *is as defined in* (1.5).

Proof. Proof of this theorem follows on the same line as that of the proof of Theorem 3.1 but instead of using Lemma 2.3, we used Lemma 2.6. We omit the proof. □

Remark 3.9. Inequalities (3.11) and (3.12) yield an improvement of inequality (1.12) when $P^{'}(0) \neq 0$.

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