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GENERAL SYSTEM OF MULTI-SEXTIC MAPPINGS AND STABILITY RESULTS

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Abstract. In this study, we characterize the structure of the multivariable mappings which are sextic in each component. Indeed, we unify the general system of multi-sextic functional equations defining a multi-sextic mapping to a single equation. We also establish the Hyers-Ulam and Găvruţa stability of multi-sextic mappings by a fixed point theorem in non-Archimedean normed spaces. Moreover, we generalize some known stability results in the setting of quasi-β-normed spaces. Using a characterization result, we indicate an example for the case that a multi-sextic mapping is non-stable.

1. Introduction

We say a functional equation Γ is *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution of Γ. Moreover, Γ is *hyperstable* if any function ϕ fulfilling Γ approximately (under some conditions), then it is an exact solution of Γ.

In two last decade, the stability problem for functional equations which was initiated by Ulam [24] for group homomorphisms (answered by Hyers [14], Aoki [2], Th. M. Rassias [20] and Găvruţa [23]) has been studied for multiple variable mappings such as multi-additive, multi-quadratic, multi-cubic and multi-quartic mappings which can be found for instance in [8], [10], [11], [18] and [27]. We state their definitions as follows:

Let $(V,+)$ be a commutative group, W be a linear space over rationals, and n be an integer with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called

(i) *multi-additive* if it satisfies the Cauchy's functional equation $A(x+y) = A(x) + A(y)$ in each variable [10];
(ii) *multi-quadratic* if it fulfills the quadratic functional equation $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ in all components [4, 11, 27];
(iii) *multi-cubic* if it satisfies the cubic equation $C(2x+y) + C(2x-y) = 2C(x+y) + 2C(x-y) + 12C(x)$ in each variable [8];

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(iv) *multi-quartic* if it satisfies one of the following quartic equation in all variables [1, 6, 17].

$$\mathfrak{Q}(x+2y)+\mathfrak{Q}(x-2y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)-6\mathfrak{Q}(x)+24\mathfrak{Q}(y);$$
$$\mathfrak{Q}(2x+y)+\mathfrak{Q}(2x-y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)+24\mathfrak{Q}(x)-6\mathfrak{Q}(y).$$

Note that the equations above have been introduced in [19] and [16], respectively.

In [25], Xu et al. obtained the general solution of the sextic functional equation

$$\begin{aligned}&f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)\\&-6f(x-2y)+f(x-3y)=720f(y)\end{aligned}$$

for the first time. They also investigated the Ulam stability problem for it in quasi-β-normed spaces via a fixed point method. Recall that the Hyers-Ulam stability of the sextic functional equation

$$\begin{aligned}S(2x+y)+S(2x-y)+S(x+2y)+S(x-2y)\\=20[S(x+y)+S(x-y)]+90[S(x)+S(y)]\end{aligned} \tag{1.1}$$

has been studied by Ravi et al. [21].

It is worth mentioning that an alternative fixed point theorem presented in [9] have been considered as a tool for the stability of multivariable mappings such as multi-Jensen, multi-additive, multi-quadratic, multi-cubic and multi-quartic mappings in non-Archimedean spaces which are available for instance in [1], [3], [7], [12] and [26].

In this article, we introduce the multi-sextic mappings (taken from (1.1)). We also include a characterization of such mappings. In fact, we prove that every multi-sextic mapping can be shown a single functional equation and vice versa (under some extra conditions). Moreover, we investigate the Hyers-Ulam and Găvruţa stability for the multi-sextic mappings by applying two fixed point methods in non-Archimedean normed and quasi-β-normed spaces [9]. As a result, we show that under some mild conditions a multi-sextic functional equation can be hyperstable. Lastly, an appropriate counterexample is supplied to invalidate the results in the case of singularity for the multi-sextic mappings.

2. Characterization of multi-sextic mappings

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ are the set of all positive integers and rationals, respectively, and moreover $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{R}_+ := [0,\infty)$. For any $l\in\mathbb{N}_0$, $n\in\mathbb{N}$, $t=(t_1,\dots,t_n)\in\{-1,1\}^n$ and $x=(x_1,\dots,x_n)\in V^n$ we write $lx := (lx_1,\dots,lx_n)$ and $tx := (t_1x_1,\dots,t_nx_n)$, where lx stands, as usual, for the scaler product of l on x in the commutative group V.

Definition 2.1. Let V and W be vector spaces over $\mathbb{Q}$, $n\in\mathbb{N}$. A multivariable mapping $f: V^n \longrightarrow W$ is called *n-sextic* or *multi-sextic* if f satisfies (1.1) in each of its n arguments, that is

$$\begin{aligned}
&f(v_1,\dots,v_{i-1},2v_i+v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},2v_i-v_i',v_{i+1},\dots,v_n)\\
&+f(v_1,\dots,v_{i-1},v_i+2v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},v_i-2v_i',v_{i+1},\dots,v_n)\\
&=20[f(v_1,\dots,v_{i-1},v_i+v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},v_i-v_i',v_{i+1},\dots,v_n)]\\
&+90[f(v_1,\dots,v_{i-1},v_i,v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},v_i',v_{i+1},\dots,v_n)].
\end{aligned}$$

Let $n\in\mathbb{N}$ with $n\geq 2$ and $x_i^n=(x_{i1},\dots,x_{in})\in V^n$, where $i\in\{1,2\}$. We shall denote x_i^n by x_i when no confusion can arise. For $x_1,x_2\in V^n$ set

$$\mathbb{A}^n=\{\mathfrak{A}_n=(A_1,\dots,A_n)\,|\,A_j\in\{x_{1j},x_{2j},x_{1j}\pm x_{2j},x_{1j}\pm 2x_{2j}\}\}$$

for all $j\in\{1,\dots,n\}$. Moreover, for $p_l\in\mathbb{N}_0$ with $0\leq p_l\leq n$, where $l\in\{1,2,3\}$, consider the subset $\mathbb{A}^n_{(p_1,p_2,p_3)}$ of $\mathbb{A}^n$ as follows:

$$\begin{aligned}
\mathbb{A}^n_{(p_1,p_2,p_3)}:=\{\mathfrak{A}_n\in\mathbb{A}^n\,|\,&\mathrm{Card}\{A_j:\ A_j=x_{1j}\}=p_1,\mathrm{Card}\{A_j:\ A_j=x_{2j}\}=p_2,\\
&\mathrm{Card}\{A_j:\ A_j=x_{1j}\pm x_{2j}\}=p_3\}.
\end{aligned}$$

From now on, for the multi-sextic mappings, we use the following notations:

$$f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right):=\sum_{\mathfrak{A}_n\in\mathbb{A}^n_{(p_1,p_2,p_3)}}f\left(\mathfrak{A}_n\right),\tag{2.1}$$

and

$$f\left(\mathbb{A}^n_{(p_1,p_2,p_3)},z\right):=\sum_{\mathfrak{A}_n\in\mathbb{A}^n_{(p_1,p_2,p_3)}}f\left(\mathfrak{A}_n,z\right)\qquad(z\in V).$$

In the next theorem, we describe a multi-sextic mapping as an equation.

Theorem 2.2. *If $f:V^n\longrightarrow W$ is a multi-sextic mapping, then it fulfills the equation*

$$\begin{aligned}
&\sum_{q\in\{-1,1\}^n}f(2x_1+qx_2)\\
&=\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)
\end{aligned}\tag{2.2}$$

for all $x_1,x_2\in V^n$, where $f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)$ is defined in (2.1).

Proof. We prove f satisfies equation (2.2) by induction on n. For $n=1$, it is trivial that f fulfills equation (1.1). Suppose that (2.2) holds for some positive integer $n>1$. Then

$$\begin{aligned}
&\sum_{q\in\{-1,1\}^{n+1}}f\left(2x_1^{n+1}+qx_2^{n+1}\right)\\
&=-\sum_{q\in\{-1,1\}^n}\sum_{t\in\{-1,1\}}f\left(x_1^n+qx_2^n,x_{1,n+1}+2tx_{2,n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+20 \sum_{q\in\{-1,1\}^n} \sum_{t\in\{-1,1\}} f\left(x_1^n+qx_2^n, x_{1,n+1}+tx_{2,n+1}\right)\\
&+90\left[\sum_{q\in\{-1,1\}^n} f\left(x_1^n+qx_2^n, x_{1,n+1}\right)+\sum_{q\in\{-1,1\}^n} f\left(x_1^n+qx_2^n, x_{2,n+1}\right)\right]\\
=&-\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}\sum_{t\in\{-1,1\}}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}\\
&\qquad\times f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1}+2tx_{2,n+1}\right)\\
&+20\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}\sum_{t\in\{-1,1\}}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}\\
&\qquad\times f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1}+tx_{2,n+1}\right)\\
&+90\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{1,n+1}\right)\\
&+90\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}, x_{2,n+1}\right)\\
=&\sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1}\sum_{p_3=0}^{n+1-p_1-p_2}(-1)^{n+1-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}f\left(\mathbb{A}^{n+1}_{(p_1,p_2,p_3)}\right).
\end{aligned}
$$

Comparing the first and the last terms, we can obtain the desired result. □

We remember that the binomial coefficient for all $n, r \in \mathbb{N}_0$ with $n \geq r$ is defined and denoted by $\binom{n}{r} := \frac{n!}{r!(n-r)!}$.

Definition 2.3. Given a mapping $f : V^n \longrightarrow W$.

(i) We say f satisfies (has) the 6-*power condition* in the jth variable if

$$f(z_1,\dots,z_{j-1},2z_j,z_{j+1},\dots,z_n)=2^6 f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n)$$

for all $z_1,\dots,z_n \in V^n$. The 6-power condition is also called the *sextic condition*.

(ii) If $f(z_1,\dots,z_n) = 0$ when the fixed z_j is zero, then we say that f has *zero functional equation* in the jth variable. Moreover, if $f(z_1,\dots,z_n) = 0$ for any $(z_1,\dots,z_n) \in V^n$ with at least one z_j is zero, we say f has *zero functional equation*.

It is clear that every multi-sextic mapping fulfills the sextic condition in each variable and thus it has zero functional equation. In other words, if a mapping $f : V^n \longrightarrow W$ satisfies the sextic condition in the jth variable, then it has zero functional equation in the same component. Under the sextic condition in all components, every mapping satisfying equation (2.2) can be multi-sextic as follows.

Theorem 2.4. *If $f : V^n \longrightarrow W$ fulfills equation* (2.2) *and has the sextic condition in each variable, then f is a multi-sextic mapping.*

Proof. Fix $j \in \{1, \dots, n\}$. Set

$$\begin{aligned} f^*(2x_{1j}, x_{2j}) &:= f\left(x_{11}, \dots, x_{1,j-1}, 2x_{1j} + x_{2j}, x_{1,j+1}, \dots, x_{1n}\right) \\ &\quad + f\left(x_{11}, \dots, x_{1,j-1}, 2x_{1j} - x_{2j}, x_{1,j+1}, \dots, x_{1n}\right), \end{aligned}$$

$$\begin{aligned} f^*(x_{1j}, 2x_{2j}) &:= f\left(x_{11}, \dots, x_{1,j-1}, x_{1j} + 2x_{2j}, x_{1,j+1}, \dots, x_{1n}\right) \\ &\quad + f\left(x_{11}, \dots, x_{1,j-1}, x_{1j} - 2x_{2j}, x_{1,j+1}, \dots, x_{1n}\right), \end{aligned}$$

$$\begin{aligned} f^*(x_{1j}, x_{2j}) &:= f\left(x_{11}, \dots, x_{1,j-1}, x_{1j} + x_{2j}, x_{1,j+1}, \dots, x_{1n}\right) \\ &\quad + f\left(x_{11}, \dots, x_{1,j-1}, x_{1j} - x_{2j}, x_{1,j+1}, \dots, x_{1n}\right), \end{aligned}$$

$$f^*(x_{1j}) := f(x_1) = f\left(x_{11}, \dots, x_{1n}\right),$$

and

$$f^*(x_{2j}) := f\left(x_{11}, \dots, x_{1,j-1}, x_{2j}, x_{1,j+1}, \dots, x_{1n}\right).$$

Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$ in (2.2) and using the property of having the sextic condition in each component, we get

(2.3)

$$\begin{aligned}
&2^{n-1} \times 2^{6(n-1)} f^*(2x_{1j}, x_{2j}) \\
&= 2^{n-1}[f\left(2x_{11}, \dots, 2x_{1,j-1}, 2x_{1j} + x_{2j}, 2x_{1,j+1}, \dots, 2x_{1n}\right) \\
&\quad + f\left(2x_{11}, \dots, 2x_{1,j-1}, 2x_{1j} - x_{2j}, 2x_{1,j+1}, \dots, 2x_{1n}\right)] \\
&= \sum_{p_1=0}^{n-1} \sum_{p_2=0}^{n-1-p_1} \left[\binom{n-1}{p_1} \binom{n-1-p_1}{p_2} 2^{n-1-p_1-p_2} 2^{p_2} (-1)^{n-p_1-p_2} 20^{p_2} 90^{p_1} \right] f^*(x_{1j}, 2x_{2j}) \\
&\quad + \sum_{p_1=0}^{n-1} \sum_{p_2=1}^{n-p_1} \left[\binom{n-1}{p_1} \binom{n-p_1}{p_2-1} 2^{n-p_1-p_2} 2^{p_2-1} (-1)^{n-p_1-p_2} 20^{p_2} 90^{p_1} \right] f^*(x_{1j}, x_{2j}) \\
&\quad + \sum_{p_1=1}^{n} \sum_{p_2=0}^{n-p_1} \left[\binom{n-1}{p_1-1} \binom{n-p_1}{p_2} 2^{n-p_1-p_2} 2^{p_2} (-1)^{n-p_1-p_2} 20^{p_2} 90^{p_1} \right] f^*(x_{1j}) \\
&\quad + 90 \sum_{p_1=0}^{n-1} \sum_{p_2=0}^{n-1-p_1} \left[\binom{n-1}{p_1} \binom{n-1-p_1}{p_2} 2^{n-1-p_1-p_2} 2^{p_2} (-1)^{n-1-p_1-p_2} 20^{p_2} 90^{p_1} \right] f^*(x_{2j}) \\
&= -\sum_{p_1=0}^{n-1} \sum_{p_2=0}^{n-1-p_1} \left[\binom{n-1}{p_1} \binom{n-1-p_1}{p_2} (-2)^{n-1-p_1-p_2} 40^{p_2} 90^{p_1} \right] f^*(x_{1j}, 2x_{2j}) \\
&\quad + \sum_{p_1=0}^{n-1} \sum_{p_2=0}^{n-1-p_1} \left[\binom{n-1}{p_1} \binom{n-p_1}{p_2} 2^{n-1-p_1-p_2} 2^{p_2} (-1)^{n-1-p_1-p_2} 20^{p_2+1} 90^{p_1} \right] f^*(x_{1j}, x_{2j}) \\
&\quad + \sum_{p_1=0}^{n-1} \sum_{p_2=0}^{n-1-p_1} \left[\binom{n-1}{p_1} \binom{n-1-p_1}{p_2} 2^{n-1-p_1-p_2} 2^{p_2} (-1)^{n-1-p_1-p_2} 20^{p_2} 90^{p_1+1} \right] f^*(x_{1j})
\end{aligned}$$

$$+90\sum_{p_1=0}^{n-1}\sum_{p_2=0}^{n-1-p_1}\left[\binom{n-1}{p_1}\binom{n-1-p_1}{p_2}2^{n-1-p_1-p_2}2^{p_2}(-1)^{n-1-p_1-p_2}20^{p_2}90^{p_1}\right]f^*(x_{2j})$$

$$= -\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}38^{n-1-p_1}90^{p_1}\right]f^*(x_{1j},2x_{2j})$$

$$+20\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}38^{n-1-p_1}90^{p_1}\right]f^*(x_{1j},x_{2j})$$

$$+90\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}38^{n-1-p_1-p_2}90^{p_1}\right]f^*(x_{1j})$$

$$+90\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}38^{n-1-p_1-p_2}90^{p_1}\right]f^*(x_{2j})$$

$$= -128^{n-1}f^*(x_{1j},2x_{2j})+20\times 128^{n-1}f^*(x_{1j},x_{2j})$$
$$+90\times 128^{n-1}[f^*(x_{1j})+f^*(x_{2j})].$$

Relation (2.3) implies that

$$2f^*(2x_{1j},x_{2j})=-f^*(x_{1j},2x_{2j})+20f^*(x_{1j},x_{2j})+90[f^*(x_{1j})+f^*(x_{2j})].$$

It follows from equality above that f is sextic in the jth variable. $\square$

By means of Theorem 2.4, it is easily seen that the mapping $f(z_1,\dots,z_n)=c\prod_{j=1}^{n}z_j^6$ satisfies (2.2) and so this equation is called *multi-sextic* functional equation.

3. Stability results for (2.2) in non-Archimedean normed spaces

In this section, we prove the Hyers-Ulam stability of the multi-sextic functional equation (2.2) in non-Archimedean normed by applying a fixed point theorem. We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1+\cdots+1}^{n\text{-times}}=0$. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. The next theorem which is a key tool in obtaining our aim in this paper, taken from [9, Theorem 1].

Theorem 3.1. *Let the following hypotheses hold.*

(H1) *E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j\in\mathbb{N}$, $g_1,\dots,g_j:E\longrightarrow E$ and $L_1,\dots,L_j:E\longrightarrow\mathbb{R}_+$;*

(H2) *$\mathcal{T}:Y^E\longrightarrow Y^E$ is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\|\le\max_{i\in\{1,\dots,j\}}L_i(x)\,\|\lambda(g_i(x))-\mu(g_i(x))\|$$

for all $\lambda,\mu\in Y^E$, $x\in E$;

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ *is an operator defined through*

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)), \quad \delta \in \mathbb{R}_+^E,\ x \in E.$$

Moreover, a function $\theta : E \longrightarrow \mathbb{R}_+$ and a mapping $\varphi : E \longrightarrow Y$ fulfill the next two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \quad \lim_{l\to\infty} \Lambda^l\theta(x) = 0, \quad (x \in E)\,.$$

Then, for every $x \in E$, the limit $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ exists and the mapping $\psi \in Y^E$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \sup\nolimits_{l\in\mathbb{N}_0}\Lambda^l\theta(x) \quad (x \in E)\,.$$

For the rest of this section, given a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\Gamma f : V^n \times V^n \longrightarrow W$ defined via

$$\begin{aligned}\Gamma f(x_1, x_2) :=& \sum_{q\in\{-1,1\}^n} f(2x_1 + qx_2)\\ &- \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}(-1)^{n-p_1-p_2-p_3}20^{p_3}90^{p_1+p_2}f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)\end{aligned}$$

for all $x_1, x_2 \in V^n$, where $f\left(\mathbb{A}^n_{(p_1,p_2,p_3)}\right)$ is defined in (2.1).

In the sequel, it is assumed that all mappings as $f : V^n \longrightarrow W$ satisfying zero condition. In the upcoming theorem, we establish the stability of functional equation (2.2) from linear spaces to complete non-Archimedean normed spaces.

Theorem 3.2. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2*. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty}\left(\frac{1}{|2|^{6n\beta}}\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) = 0 \tag{3.1}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\Gamma f(x_1, x_2)\| \leq \varphi\,(x_1, x_2) \tag{3.2}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{S} : V^n \longrightarrow W$ of (2.2) *such that*

$$\|f(x) - \mathcal{S}(x)\| \leq \sup\nolimits_{l\in\mathbb{N}_0}\frac{1}{|2|^n|2|^{6n\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{6n\beta}}\right)^l\varphi\left(2^{l\beta+\frac{\beta-1}{2}}, 0\right) \tag{3.3}$$

for all $x \in V^n$. Moreover, if $\mathcal{S}$ satisfies the sextic condition in each variable, then it is a unique multi-sextic mapping.

Proof. Putting $x_2 = 0$ in (3.2) and using our assumptions, we have

$$\|2^n f(2x) - Tf(x)\| \leq \varphi(x, 0) \tag{3.4}$$

for all $x := x_1 \in V^n$ (and for the rest of this proof, all the equations and inequalities are valid for all $x \in V^n$), where

$$T = \sum_{p_1=0}^{n} \sum_{p_2=1}^{n-p_1} \left(\begin{array}{c} n \\ p_1 \end{array} \right) \left(\begin{array}{c} n-p_1 \\ p_2 \end{array} \right) 2^{n-p_1-p_2} 2^{p_2} (-1)^{n-p_1-p_2} 20^{p_2} 90^{p_1}.$$

On the other hand, we have

$$\begin{aligned} T &= \sum_{p_1=0}^{n} \sum_{p_2=1}^{n-p_1} \left(\begin{array}{c} n \\ p_1 \end{array} \right) \left(\begin{array}{c} n-p_1 \\ p_2 \end{array} \right) 2^{n-p_1-p_2} 2^{p_2} (-1)^{n-p_1-p_2} 20^{p_2} 90^{p_1} \\ &= \sum_{p_1=0}^{n} \left(\begin{array}{c} n \\ p_1 \end{array} \right) 38^{p_2} 90^{p_1} = (38 + 90)^n = 2^{7n}. \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\left\| f(2x) - 2^{6n} f(x) \right\| \leq \frac{1}{|2|^n} \varphi(x, 0). \tag{3.6}$$

Relation (3.6) can be rewritten as

$$\|f(x) - \mathcal{T} f(x)\| \leq \theta(x), \tag{3.7}$$

where

$$\theta(x) := \frac{1}{|2|^n |2|^{6n\frac{\beta+1}{2}}} \varphi\left(2^{\frac{\beta-1}{2}} x, 0\right), \quad \mathcal{T}\xi(x) := \frac{1}{2^{6n\beta}} \xi\left(2^\beta x\right)$$

for all $\xi \in W^{V^n}$. Define $\Lambda\eta(x) := \frac{1}{|2|^{6n\beta}} \eta\left(2^\beta x\right)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$. It is easily seen that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^\beta x$ for $L_1(x) = \frac{1}{|2|^{6n\beta}}$. On the other hand, we have

$$\begin{aligned} \|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\| \frac{1}{2^{6n\beta}} \lambda(2^\beta x) - \frac{1}{2^{6n\beta}} \mu(2^\beta x) \right\| \\ &\leq L_1(x) \left\| \lambda(g_1(x)) - \mu(g_1(x)) \right\| \end{aligned}$$

for all $\lambda, \mu \in W^{V^n}$. It follows from the above relation that the hypothesis (H2) is true. Moreover, one can check by induction on l that for any $l \in \mathbb{N}$, we find

$$\Lambda^l \theta(x) := \left(\frac{1}{|2|^{6n\beta}} \right)^l \theta\left(2^{l\beta} x\right) = \frac{1}{|2|^n |2|^{6n\frac{\beta+1}{2}}} \left(\frac{1}{|2|^{6n\beta}} \right)^l \varphi\left(2^{l\beta + \frac{\beta-1}{2}}, 0\right). \tag{3.8}$$

It concludes from (3.7) and (3.8) that all assumptions of Theorem 3.1 are satisfied and so there exists a unique solution $\mathcal{S} : V^n \longrightarrow W$ of (2.2) such that

$\mathcal{S}(x) = \lim_{l\to\infty} \left(\mathcal{T}^l f\right)(x)$, and (3.3) holds as well. We also can checked by induction on l that

$$\left\|\Gamma\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \leq \left(\frac{1}{|2|^{6n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right) \tag{3.9}$$

for all $x_1, x_2 \in V^n$. Taking $l \to \infty$ in (3.9) and using (3.1), we obtain $\Gamma\mathcal{S}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$ and hence equation (2.2) is valid for $\mathcal{S}$. The last assertion follows from Theorem 2.4. This finishes the proof. □

From here to the rest of this section, V is a non-Archimedean normed space and W is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. In addition, we assume that $|2| < 1$. The following corollaries are taken from Theorem 3.2 regarding the stability of (2.2).

Corollary 3.3. *Given $\delta > 0$. Let $f : V^n \longrightarrow W$ be a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \delta$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{S} : V^n \longrightarrow W$ of (2.2) *such that*

$$\|f(x) - \mathcal{S}(x)\| \leq \frac{1}{|2|^n}\delta$$

for all $x \in V^n$. In addition, if $\mathcal{S}$ satisfies the sextic condition in each variable, then it is a multi-sextic mapping.

Proof. Note that $|2| < 1$. Choosing $\varphi(x_1, x_2) = \delta$ for the case $\beta = -1$ of Theorem 3.2, we get $\lim_{l\to\infty} |2|^{6nl}\delta = 0$, and hence (3.1) is true in Theorem 3.2. The last result follows from Theorem 2.4. □

Corollary 3.4. *Let $p \in \mathbb{R}$ fulfills $p \neq 6n$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^p$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{S} : V^n \longrightarrow W$ of (2.2) *such that*

$$\|f(x) - \mathcal{S}(x)\| \leq \begin{cases} \frac{1}{|2|^n|2|^{6n}} \sum_{j=1}^{n} \|x_{1j}\|^p, & p > 6n, \\ \frac{1}{|2|^n|2|^p} \sum_{j=1}^{n} \|x_{1j}\|^p, & p < 6n \end{cases}$$

for all $x = x_1 \in V^n$. Moreover, if $\mathcal{S}$ has the sextic condition in all components, then it is a multi-sextic mapping.

Proof. Set $\varphi(x_1, x_2) = \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^p$. Then, $\varphi\left(2^l x_1, 2^l x_2\right) = |2|^{lp}\varphi(x_1, x_2)$. Now, Theorem 3.2 and Theorem 2.4 can be applied to arrive the result. □

Under some conditions the functional equation (2.2) can be hyperstable as follows.

Corollary 3.5. *Let $p_{kj} > 0$ for $k \in \{1,2\}$ and $j \in \{1,\dots,n\}$ such that*

$$\sum_{k=1}^{2}\sum_{j=1}^{n} p_{kj} \neq 6n.$$

If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\Gamma f\,(x_1,x_2)\| \leq \prod_{k=1}^{2}\prod_{j=1}^{n} \|x_{kj}\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, then f satisfies (2.2). In particular, if f has the sextic condition in each variable, then it is multi-sextic.

Proof. Defining $\varphi\,(x_1,x_2) = \prod_{k=1}^{2}\prod_{j=1}^{n} \|x_{kj}\|^{p_{kj}}$ in Theorem 3.2, and applying Theorem 2.4, we reach the desired result. □

4. Stability Results for (2.2) in quasi-β-normed spaces

Here, we recall some basic facts regarding the setting of quasi-β-normed space.

Definition 4.1. Let β be a fix real number with $0 < \beta < 1$, and $\mathbb{K}$ denote either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers). Suppose that X is a linear space over $\mathbb{K}$. A quasi-β-norm is a real-valued function on X fulfilling the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|tx\| = |t|^{\beta}\|x\|$ for all $x \in X$ and $t \in \mathbb{K}$;
(iii) There is a constant $M \geq 1$ such that $\|x+y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

When $\beta = 1$, the norm above is a quasinorm. Recall that M is the *modulus of concavity* of the norm $\|\cdot\|$. Moreover, if $\|\cdot\|$ is a quasi-β-norm on X, the pair $(X, \|\cdot\|)$ is said to be a *quasi-β-normed space*. Similar to normed spaces, a complete quasi-β-normed space is called a *quasi-β-Banach space*. For $0 < p \leq 1$, if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$, then the quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm. In this case, every quasi-β-Banach space is said to be a (β,p)-Banach space. By the Aoki-Rolewicz Theorem [22], each quasi-norm is equivalent to some p-norm.

Next, by using an idea of Găvruta [23], we prove the stability of (2.2) in quasi-β-normed spaces by applying the following fixed point lemma which was proved in [25, Lemma 3.1].

Lemma 4.2. *Let $j \in \{-1,1\}$ be fixed, $\mathbf{a}, s \in \mathbb{N}$ with $\mathbf{a} \geq 2$. Suppose that X is a linear space, Y is a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. If $\psi : X \longrightarrow$*

$[0, \infty)$ *is a function such that there exists an* $L < 1$ *with* $\psi(\mathbf{a}^j x) < L\mathbf{a}^{js\beta}\psi(x)$ *for all* $x \in X$ *and* $f : X \longrightarrow Y$ *is a mapping satisfying*

$$\|f(\mathbf{a}x) - \mathbf{a}^s f(x)\|_Y \leq \psi(x)$$

for all $x \in X$*, then there exists a uniquely determined mapping* $F : X \longrightarrow Y$ *such that* $F(\mathbf{a}x) = \mathbf{a}^s F(x)$ *and*

$$\|f(x) - F(x)\|_Y \leq \frac{1}{\mathbf{a}^{s\beta}|1 - L^j|}\psi(x)$$

for all $x \in X$*. Moreover,* $F(x) = \lim_{l\to\infty} \frac{f(\mathbf{a}^{jl}x)}{\mathbf{a}^{jls}}$ *for all* $x \in X$*.*

Theorem 4.3. *Let* $j \in \{-1, 1\}$ *be fixed,* V *be a linear space and* W *be a* (β, p)*-Banach space and* $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *be a function such that there exists an* $0 < L < 1$ *with* $\varphi(2^j x_1, 2^j x_2) \leq 2^{6nj\beta} L\varphi(x_1, x_2)$ *for all* $x_1, x_2 \in V^n$*. Suppose that a mapping* $f : V^n \longrightarrow W$ *fulfilling the inequality*

$$\|\Gamma f(x_1, x_2)\|_W \leq \varphi(x_1, x_2) \tag{4.1}$$

for all $x_1, x_2 \in V^n$*. Then, there exists a unique solution* $\mathcal{S} : V^n \longrightarrow W$ *of* (2.2) *such that*

$$\|f(x) - \mathcal{S}(x)\|_W \leq \frac{1}{|1 - L^j|}\frac{1}{2^{7n\beta}}\varphi(x, 0) \tag{4.2}$$

for all $x \in V^n$*. Moreover, if* $\mathcal{S}$ *satisfies the sextic condition in each variable, then it is a unique multi-sextic mapping.*

Proof. Similar to the proof of Theorem 3.2, by putting $x_2 = 0$ in (4.1) and using our assumptions, we have

$$\left\|f(2x) - 2^{6n} f(x)\right\|_W \leq \frac{1}{2^{n\beta}}\varphi(x, 0)$$

for all $x := x_1 \in V^n$. By Lemma 4.2, there exists a unique mapping $\mathcal{S} : V^n \longrightarrow W$ such that $\mathcal{S}(2x) = 2^{6n}\mathcal{S}(x)$ and

$$\|f(x) - \mathcal{S}(x)\|_W \leq \frac{1}{|1 - L^j|}\frac{1}{2^{7n\beta}}\varphi(x, 0)$$

for all $x \in V^n$. It remains to show that $\mathcal{S}$ is a multi-sextic map. Here, we note from Lemma 4.2 that for all $x \in V^n$, $\mathcal{S}(x) = \lim_{l\to\infty} \frac{f(2^{jl}x)}{2^{6njl}}$. Now, by (4.1), we have

$$\begin{aligned}\left\|\frac{\Gamma f(2^{jl}x_1, 2^{jl}x_2)}{2^{6njl}}\right\|_W &\leq 2^{-6njl\beta}\varphi(2^{jl}x_1, 2^{jl}x_2)\\ &\leq 2^{-6njl\beta}(2^{6nj\beta}L)^l\varphi(x_1, x_2)\\ &= L^l\varphi(x_1, x_2)\end{aligned}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. Letting $l \to \infty$ in the above inequality, we observe that $\Gamma\mathcal{S}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that $\mathcal{S}$ satisfies (2.2). The last assertion follows from Theorem 2.4. □

The next corollary is a direct consequences of Theorem 3.2 concerning the stability of (2.2) when the norm of $\Gamma f(x_1,x_2)$ is controlled by sum of variables norms of x_1 and x_2 with positive powers.

Corollary 4.4. *Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$, and W be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_W$. Let θ and λ be positive numbers with $\lambda\neq 6n\frac{\beta}{\alpha}$. If a mapping $f:V^n\longrightarrow W$ fulfilling the inequality*

$$\|\Gamma f(x_1,x_2)\|_W\leq \theta\sum_{k=1}^{2}\sum_{j=1}^{n}\|x_{kj}\|_V^{\lambda}$$

for all $x_1,x_2\in V^n$, then there exists a unique solution $\mathcal{S}:V^n\longrightarrow W$ of (2.2) such that

$$\|f(x)-\mathcal{S}(x)\|_W\leq\begin{cases}\frac{\theta}{2^{n\beta}(2^{6n\beta}-2^{\alpha\lambda})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda}, & \lambda\in\left(0,6n\frac{\beta}{\alpha}\right),\\ \frac{2^{\alpha\lambda}\theta}{2^{7n\beta}(2^{\alpha\lambda}-2^{6n\beta})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda}, & \lambda\in\left(6n\frac{\beta}{\alpha},\infty\right)\end{cases}$$

for all $x=x_1\in V^n$. In particular, if $\mathcal{S}$ satisfies the sextic condition in all variables, then it is a unique multi-sextic mapping.

Here, we present an elementary lemma without proof as follows.

Lemma 4.5. *If a function $g:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous and satisfies* (1.1), *then it has the form $g(x)=cx^6$ for all $x\in\mathbb{R}$, where $c=f(1)$.*

In the following result, we extend Lemma 4.5 for several variables functions. For doing this, we use an idea taken from the proof of [15, Theorem 13.4.3].

Proposition 4.6. *Let $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ be a continuous n-sextic function. Then, there exists a constant $c\in\mathbb{R}$ such that*

$$f(x_1,\dots,x_n)=c\prod_{j=1}^{n}x_j^6 \tag{4.3}$$

for all $x_1,\dots,x_n\in\mathbb{R}$.

Proof. We argue the proof by induction on n. For $n=1$, (4.3) is true in view of Lemma 4.5. Let (4.3) hold for $n\in\mathbb{N}$. Assume that $f:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ is a continuous $(n+1)$-sextic function. Fix the n variables $x_1,\dots,x_n$. Then, the function $y\mapsto f(x_1,\dots,x_n,y)$ as a function of y is sextic and continuous, and so there exists a constant $c\in\mathbb{R}$ such that

$$f(x_1,\dots,x_n,y)=cy^6,\qquad (y\in\mathbb{R}). \tag{4.4}$$

Note that c depends on $x_1,\dots,x_n$, and indeed

$$c=c(x_1,\dots,x_n). \tag{4.5}$$

Letting $y=1$ in (4.4) and applying (4.5), we get

$$c=c(x_1,\dots,x_n)=f(x_1,\dots,x_n,1).$$

Since f is $(n+1)$-sextic, it follows that c is an n-sextic function and hence by the induction hypothesis there exists a real number c' such that

$$c = c(x_1, \dots, x_n) = c' \prod_{j=1}^{n} x_j^6. \tag{4.6}$$

Now, the result follows from (4.4) and (4.6). □

We end the paper by the following counterexample for multi-sextic mappings on $\mathbb{R}^n$ that its idea is taken from [5] (see also [13]). In fact, we show the hypothesis $\lambda \neq 6n$ can not be removed in Corollary 4.4 when $V = W = \mathbb{R}$ in the case that $\alpha = \beta = 1$.

Example 4.7. Let $\delta > 0$ and $n \in \mathbb{N}$. Put $\mu = \frac{2^{6n}-1}{2^{12n} S}\delta$, where

$$S \geq 2^{2n} + \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2-p_3} 20^{p_3} 90^{p_1+p_2}.$$

Define the function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ through

$$\psi(r_1, \dots, r_n) = \begin{cases} \mu \prod_{j=1}^{n} r_j^6 & \text{for all } r_j \text{ with } |r_j| < 1, \\ \mu & \text{otherwise.} \end{cases}$$

Hence, $\psi(r_1, \dots, r_n) \leq \mu$ for all $(r_1, \dots, r_n) \in \mathbb{R}^n$. Using the function ψ, consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined via

$$f(r_1, \dots, r_n) = \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \dots, 2^l r_n)}{2^{6nl}}, \qquad (r_j \in \mathbb{R}).$$

It is obvious that f is an even function in each variable and non-negative. Moreover, ψ is continuous and bounded by μ. It is known that f is a uniformly convergent series of continuous functions and thus it is continuous and bounded. In other words, for each $(r_1, \dots, r_n) \in \mathbb{R}^n$, we have $f(r_1, \dots, r_n) \leq \frac{2^{6n}}{2^{6n}-1}\mu$. Put $x_i = (x_{i1}, \dots, x_{in})$, where $i \in \{1, 2\}$. We claim that

$$|\Gamma f\,(x_1, x_2)| \leq \delta \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{6n} \tag{4.7}$$

for all $x_1, x_2 \in \mathbb{R}^n$. It is clear that (4.7) is valid for $x_1 = x_2 = 0$. Assume that $x_1, x_2 \in \mathbb{R}^n$ with $\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{6n} < \frac{1}{2^{6n}}$. Thus, there exists a positive integer N such that

$$\frac{1}{2^{6n(N+1)}} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{6n} < \frac{1}{2^{6nN}}. \tag{4.8}$$

Hence, $x_{ij}^{6n} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{6n} < \frac{1}{2^{6nN}}$ and so $2^{N-1}|x_{ij}| < 1$ for all $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$. If $y_1, y_2 \in \{x_{ij} \mid i \in \{1, 2\},\ j \in \{1, \dots, n\}\}$, then

$$\{2^{N-1}|\, y_1 \pm y_2|,\ 2^{N-1}|y_1 \pm 2y_2|,\ 2^{N-1}|2y_1 \pm y_2|\} \subseteq (-1, 1).$$

Since ψ is a multi-sextic function on $(-1,1)^n$, $\Gamma\psi\left(2^lx_1,2^lx_2\right)=0$ for all $l\in\{0,1,2,\dots,N-1\}$. It follows from the last equality and relation (4.8) that

$$\begin{aligned}\frac{\left|\Gamma f\left(2^lx_1,2^lx_2\right)\right|}{\sum_{i=1}^2\sum_{j=1}^n x_{ij}^{6n}} &\leq \sum_{l=N}^{\infty}\frac{\left|\Gamma\psi\left(2^lx_1,2^lx_2\right)\right|}{2^{6nl}\sum_{i=1}^2\sum_{j=1}^n x_{ij}^{6n}}\\ &\leq \sum_{l=0}^{\infty}\frac{\mu S}{2^{6n(l+N)}\sum_{i=1}^2\sum_{j=1}^n x_{ij}^{6n}}\\ &\leq \mu S 2^{6n}\sum_{l=0}^{\infty}\frac{1}{2^{6nl}}\\ &=\mu S\frac{2^{12n}}{2^{6n}-1}=\delta\end{aligned}$$

for all $x_1,x_2\in\mathbb{R}^n$. If $\sum_{i=1}^2\sum_{j=1}^n x_{ij}^{6n}\geq\frac{1}{2^{6n}}$, then

$$\frac{\left|\Gamma f\left(2^lx_1,2^lx_2\right)\right|}{\sum_{i=1}^2\sum_{j=1}^n x_{ij}^{6n}}\leq\frac{2^{12n}}{2^{6n}-1}\mu S=\delta.$$

Therefore, f fulfills (4.7) for all $x_1,x_2\in\mathbb{R}^n$. Now, suppose contrary to our claim, that there are a number $\gamma\in[0,\infty)$ and a multi-sextic function $\mathcal{S}:\mathbb{R}^n\longrightarrow\mathbb{R}$ such that

$$|f(r_1,\dots,r_n)-\mathcal{S}(r_1,\dots,r_n)|<\gamma\sum_{j=1}^n r_j^6.$$

Without loss of generality, one can take a number $b\in[0,\infty)$ so that $\gamma\sum_{j=1}^n r_j^6\leq b\prod_{j=1}^n r_j^6$. Hence, $|f(r_1,\dots,r_n)-\mathcal{S}(r_1,\dots,r_n)|<b\prod_{j=1}^n r_j^6$ for all $(r_1,\dots,r_n)\in\mathbb{R}^n$. By Proposition 4.6, there exists a constant $c\in\mathbb{R}$ such that $\mathcal{S}(r_1,\dots,r_n)=c\prod_{j=1}^n r_j^6$, and thus

$$f(r_1,\dots,r_n)\leq(|c|+b)\prod_{j=1}^n r_j^6 \tag{4.9}$$

for all $(r_1,\dots,r_n)\in\mathbb{R}^n$. On the other hand, consider $p\in\mathbb{N}$ such that $(p+1)\mu>|c|+b$. If $r=(r_1,\dots,r_n)$ belongs to $\mathbb{R}^n$ such that $r_j\in\left(0,\frac{1}{2^p}\right)$ for all $j\in\{1,\dots,n\}$, then $2^lr_j\in(0,1)$ for all $l=0,1,\dots,p$. Thus, we get

$$\begin{aligned}f(r_1,\dots,r_n)&=\sum_{l=0}^{\infty}\frac{\psi\left(2^lr_1,\dots,2^lr_2\right)}{2^{6nl}}\\ &=\sum_{l=0}^{p}\frac{\mu2^{6nl}\prod_{j=1}^n r_j^6}{2^{6nl}}\\ &=(p+1)\mu\prod_{j=1}^n r_j^6>(|c|+b)\prod_{j=1}^n r_j^6.\end{aligned}$$

The relation above leads us to a contradiction with (4.9).

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