Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 389–400
https://doi.org/10.4134/CKMS.c210315
pISSN: 1225-1763 / eISSN: 2234-3024

RESULTS ASSOCIATED WITH THE SCHWARZ LEMMA ON THE BOUNDARY

Bülent Nafi Örnek

Abstract. In this paper, some estimations will be given for the analytic functions belonging to the class $\mathcal{R}(\alpha)$. In these estimations, an upper bound and a lower bound will be determined for the first coefficient of the expansion of the analytic function $h(z)$ and the modulus of the angular derivative of the function $\frac{zh'(z)}{h(z)}$, respectively. Also, the relationship between the coefficients of the analytical function $h(z)$ and the derivative mentioned above will be shown.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $h(z) = z + \sum_{p=2}^{\infty} d_p z^p$ that are analytic in D. Also, let $\mathcal{R}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $h(z)$ satisfying

$$\sum_{p=2}^{\infty} \{(p-1) + |(a-b)\alpha - b(p-1)|\} |d_p| \leq (a-b)|\alpha|, \tag{1.1}$$

where $-1 \leq b < a \leq 1$ and $0 \neq \alpha \in \mathbb{C}$. In this paper, we study some of the properties of the classes $\mathcal{R}(\alpha)$. Namely, an upper bound will be found for the modulus of $d_2 = \frac{h''(0)}{2!}$, which is one of the coefficients forming the analytic function $h(z)$ belonging to this class.

Let $h \in \mathcal{R}(\alpha)$ and consider the following function

$$r(z) = \frac{\frac{zh'(z)}{h(z)} - 1}{b\left(\frac{zh'(z)}{h(z)} - 1\right) - (a-b)\alpha}. \tag{1.2}$$

Here, $r(z)$ is an analytic function in D and $r(0) = 0$. Now, let us show that $|r(z)| < 1$ in D. At this step, we will consider the difference between the modules of the numerator and the denominator of $r(z)$ in (1.2). So, we take

$$|zh'(z) - h(z)| - |b(zh'(z) - h(z)) - (a-b)\alpha h(z)|$$

Received September 24, 2021; Revised June 2, 2022; Accepted August 17, 2022.
2020 *Mathematics Subject Classification.* Primary 30C80.
Key words and phrases. Analytic function, Schwarz lemma on the boundary, angular limit.

$$= \left| z\left(1+\sum_{p=2}^{\infty} p d_p z^{p-1}\right) - \left(z+\sum_{p=2}^{\infty} d_p z^p\right)\right|$$

$$- \left| b\left(z\left(1+\sum_{p=2}^{\infty} p d_p z^{p-1}\right) - \left(z+\sum_{p=2}^{\infty} d_p z^p\right)\right) - (a-b)\,\alpha\left(z+\sum_{p=2}^{\infty} d_p z^p\right)\right|$$

$$= \left|\sum_{p=2}^{\infty} (p-1)\, d_p z^p\right| - \left|(a-b)\,\alpha\left(z+\sum_{p=2}^{\infty} d_p z^p\right) - b\sum_{p=2}^{\infty}(p-1)\, d_p z^p\right|$$

$$\le \sum_{p=2}^{\infty}\left\{(p-1)+\left|(a-b)\,\alpha - b\,(p-1)\right|\right\}|d_p|\,|z|^p - (a-b)\,|\alpha|\,|z|\,.$$

If we pass to limit in the last expression as $|z| \to 1^-$, we take

$$\left|zh'(z)-h(z)\right| - \left|b\left(zh'(z)-h(z)\right)-(a-b)\,\alpha h(z)\right|$$
$$\le \sum_{p=2}^{\infty}\left\{(p-1)+\left|(a-b)\,\alpha - b\,(p-1)\right|\right\}|d_p| - (a-b)\,|\alpha|\,.$$

Also, since $\sum_{p=2}^{\infty}\left\{(p-1)+\left|(a-b)\,\alpha-b\,(p-1)\right|\right\}|d_p| \le (a-b)\,|\alpha|$, we obtain

$$\left|zh'(z)-h(z)\right| - \left|b\left(zh'(z)-h(z)\right)-(a-b)\,\alpha h(z)\right| \le 0$$

and

$$|r(z)| < 1.$$

Therefore, the function $r(z)$ satisfies the conditions of the Schwarz lemma ([6], p. 329). That is, $r(z)$ is an analytic function in D, $r(0)=0$ and $|r(z)|<1$ for $z\in D$. From the Schwarz lemma, we obtain

$$r(z) = \frac{\frac{zh'(z)}{h(z)}-1}{b\left(\frac{zh'(z)}{h(z)}-1\right)-(a-b)\,\alpha} = \frac{d_2 z+\left(2d_3-d_2^2\right)z^2+\cdots}{b\left(d_2 z+\left(2d_3-d_2^2\right)z^2+\cdots\right)-(a-b)\,\alpha}$$

and

$$\frac{r(z)}{z} = \frac{d_2+\left(2d_3-d_2^2\right)z+\cdots}{b\left(d_2 z+\left(2d_3-d_2^2\right)z^2+\cdots\right)-(a-b)\,\alpha}.$$

If we pass to limit in the last expression as $|z|\to 0$, we have

$$|r'(0)| = \left|\frac{d_2}{(a-b)\,\alpha}\right| \le 1$$

and

$$|d_2| \le (a-b)\,|\alpha|\,.$$

We thus obtain the following lemma.

Lemma 1.1. *If $h\in\mathcal{R}(\alpha)$, then we have the inequality*

$$|d_2| \le (a-b)\,|\alpha|\,. \tag{1.3}$$

At this point, we will investigate the following function assuming that zeros of $h(z) - z$ are different from zero,

$$\phi(z) = \frac{r(z)}{\prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z}}.$$

Since $\phi(z)$ satisfies the conditions of the Schwarz lemma, we obtain

$$\phi(z) = \frac{\frac{zh'(z)}{h(z)} - 1}{b\left(\frac{zh'(z)}{h(z)} - 1\right) - (a-b)\,\alpha} \frac{1}{\prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z}}$$

$$= \frac{d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\,\alpha} \frac{1}{\prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z}},$$

$$\frac{\phi(z)}{z} = \frac{d_2 + \left(2d_3 - d_2^2\right) z + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\,\alpha} \frac{1}{\prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z}},$$

$$|\phi'(0)| = \frac{|d_2|}{(a-b)\,|\alpha| \prod_{i=1}^{n} |a_i|} \leq 1$$

and

$$|d_2| \leq (a-b)\,|\alpha| \prod_{i=1}^{n} |a_i|.$$

We thus obtain the following lemma.

Lemma 1.2. *Let $h \in \mathcal{R}(\alpha)$ and $a_1, a_2, \ldots, a_n$ be zeros of the function $h(z) - z$ in D that are different from zero. Then we have the inequality*

$$|d_2| \leq (a-b)\,|\alpha| \prod_{i=1}^{n} |a_i|.$$

Schwarz lemma is a hot topic and it is possible to encounter several studies in the literature. The majority of these studies consider the boundary version of Schwarz lemma, which actually corresponds to the estimation of the modulus of the derivative of the function from below at some boundary point of the unit disc. The boundary version of Schwarz lemma is defined as follows [12, 14]:

Lemma 1.3. *If $m(z)$ extends continuously to some boundary point $\varsigma \in \partial D = \{z : |z| = 1\}$ with $|\varsigma| = 1$, and if $|m(\varsigma)| = 1$ and $m'(\varsigma)$ exists, then*

$$|m'(\varsigma)| \geq \frac{2}{1 + |m'(0)|} \tag{1.4}$$

and

$$|m'(\varsigma)| \geq 1. \tag{1.5}$$

Moreover, the equality in (1.4) *holds if and only if*

$$m(z) = z\frac{z-a}{1-az}$$

for some $a \in (-1, 0]$. *Also, the equality in* (1.5) *holds if and only if* $m(z) = ze^{i\theta}$.

Inequality (1.5) is frequently encountered in literature as it has important applications in the geometric theory of functions and also, its generalizations are still popular in mathematics community [1–5, 7–12].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [13]).

Lemma 1.4 (Julia-Wolff lemma)**.** *Let* m *be an analytic function in* D, $m(0) = 0$ *and* $m(D) \subset D$. *If, in addition, the function* m *has an angular limit* $m(\varsigma)$ *at* $\varsigma \in \partial D$, $|m(\varsigma)| = 1$, *then the angular derivative* $m'(\varsigma)$ *exists and* $1 \leq |m'(\varsigma)| \leq \infty$.

Corollary 1.5. *The analytic function* m *has a finite angular derivative* $m'(\varsigma)$ *if and only if* m' *has the finite angular limit* $m'(\varsigma)$ *at* $\varsigma \in \partial D$.

2. Main results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{R}(\alpha)$ class.

Theorem 2.1. *Let* $h \in \mathcal{R}(\alpha)$. *Assume that, for* $1 \in \partial D$, h *has an angular limit* $h(1)$ *at the point* 1, $h'(1) = \left(1 - \frac{a-b}{1-b}\alpha\right) h(1)$. *Then we have the inequality*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right| \geq \frac{a-b}{(1-b)^2}\,|\alpha|\,. \tag{2.1}$$

The inequality (2.1) *is sharp for* $b = -1$ *with extremal functions* $h(z) = \frac{z}{(1+z)^{(1+a)\alpha}}$.

Proof. Let

$$r(z) = \frac{p(z)-1}{b\,(p(z)-1) - (a-b)\,\alpha},\ p(z) = \frac{zh'(z)}{h(z)}.$$

For $h'(1) = \left(1 - \frac{a-b}{1-b}\alpha\right) h(1)$, we have

$$r(1) = \frac{p(1)-1}{b\,(p(1)-1) - (a-b)\,\alpha} = \frac{\frac{a-b}{b-1}\alpha}{b\frac{a-b}{b-1}\alpha - (a-b)\,\alpha} = 1.$$

Also, with the simple calculations, we take

$$r'(z) = \frac{-\,(a-b)\,\alpha p'(z)}{\left(b\,(p(z)-1) - (a-b)\,\alpha\right)^2}.$$

Therefore, from (1.5), we obtain

$$1 \leq |r'(1)| = \left| \frac{-(a-b)\,\alpha p'(1)}{\left(b\,(p(1)-1)-(a-b)\,\alpha\right)^2} \right| = \frac{(1-b)^2\,|p'(1)|}{(a-b)\,|\alpha|}$$

and

$$|p'(1)| \geq \frac{(a-b)\,|\alpha|}{(1-b)^2}.$$

Now, weak-sharpness of an inequality (2.1) can be given as follows:

Let

$$h(z) = \frac{z}{(1+z)^{(1+a)\alpha}}.$$

By taking $\ln h(z)$ and differentiating it, $p(z) = \frac{zh'(z)}{h(z)} = 1 - \frac{(a+1)\alpha z}{1+z}$ and $p'(z) = -\frac{(1+a)\alpha}{(1+z)^2}$. So, $p(1) = \frac{h'(1)}{h(1)} = 1 - \frac{(a+1)\alpha}{2}$ and $(p'(z))\,|_{z=1} = -\frac{(a+1)\alpha}{4}$. □

The inequality (2.1) can be strengthened from below by taking into account, $d_2 = \frac{h''(0)}{2!}$, the first coefficient of the expansion of the function $h(z) = z + d_2 z^2 + \cdots$.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left| \left(\frac{zh'(z)}{h(z)} \right)'_{z=1} \right| \geq \frac{4\,(a-b)\,|\alpha|^2}{2\,(a-b)\,|\alpha| + |h''(0)|}. \tag{2.2}$$

Proof. From the expression of $r(z)$ and from (1.4), we obtain

$$\frac{2}{1+|r'(0)|} \leq |r'(1)| = \frac{(1-b)^2\,|p'(1)|}{(a-b)\,|\alpha|}.$$

Since

$$|r'(0)| = \left| \frac{d_2}{(a-b)\,\alpha} \right|,$$

we take

$$\frac{2}{1+\left|\frac{d_2}{(a-b)\alpha}\right|} \leq \frac{(1-b)^2\,|p'(1)|}{(a-b)\,|\alpha|},$$

and

$$|p'(1)| \geq \frac{2\,(a-b)\,|\alpha|^2}{(a-b)\,|\alpha| + |d_2|}.$$

□

The inequality (2.2) can be strengthened as below by taking into account $d_3 = \frac{h'''(0)}{3!}$ which is the coefficient in the expansion of the function $h(z) = z + d_2 z^2 + d_3 z^3 + \cdots$.

Theorem 2.3. *Let $h \in \mathcal{R}(\alpha)$. Assume that, for $1 \in \partial D$, h has an angular limit $h(1)$ at the point 1, $h'(1) = \left(1 - \frac{a-b}{1-b}\alpha\right) h(1)$. Then we have the inequality*

$$\left| \left(\frac{zh'(z)}{h(z)} \right)'_{z=1} \right| \tag{2.3}$$
$$\geq \frac{(a-b)|\alpha|}{(1-b)^2} \left(1 + \frac{2\left((a-b)|\alpha| - |d_2|\right)^2}{(a-b)^2|\alpha|^2 - |d_2|^2 + \left|\left(2d_3 - d_2^2\right)(a-b)\alpha + bd_2^2\right|} \right).$$

Proof. Let $r(z)$ be the same as in the proof of Theorem 2.1 and $t(z) = z$. By the maximum principle, for each $z \in D$, we have the inequality $|r(z)| \leq |t(z)|$. So,

$$\begin{aligned} u(z) = \frac{r(z)}{t(z)} &= \frac{1}{z}\left(\frac{p(z)-1}{b(p(z)-1) - (a-b)\alpha} \right) \\ &= \frac{1}{z}\left(\frac{d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\alpha} \right) \\ &= \frac{d_2 + \left(2d_3 - d_2^2\right) z + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\alpha} \end{aligned}$$

is an analytic function in D and $|u(z)| \leq 1$ for $z \in D$. In particular, we have

$$|u(0)| = \frac{|d_2|}{(a-b)|\alpha|} \leq 1 \tag{2.4}$$

and

$$|u'(0)| = \frac{\left|\left(2d_3 - d_2^2\right)(a-b)\alpha + bd_2^2\right|}{(a-b)^2|\alpha|^2}.$$

The auxiliary function

$$s(z) = \frac{u(z) - u(0)}{1 - \overline{u(0)}u(z)}$$

is analytic in D, $s(0) = 0$, $|s(z)| < 1$ for $|z| < 1$ and $|s(1)| = 1$ for $1 \in \partial D$. From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |s'(0)|} \leq |s'(1)| &= \frac{1 - |u(0)|^2}{\left|1 - \overline{u(0)}u(1)\right|^2} |u'(1)| \\ &\leq \frac{1 + |u(0)|}{1 - |u(0)|} \left\{ |r'(1)| - |t'(1)| \right\} \\ &= \frac{(a-b)|\alpha| + |d_2|}{(a-b)|\alpha| - |d_2|} \left(\frac{(1-b)^2 |p'(1)|}{(a-b)|\alpha|} - 1 \right). \end{aligned}$$

Since

$$s'(z) = \frac{1 - |u(0)|^2}{\left(1 - \overline{u(0)}u(z)\right)^2} u'(z)$$

and

$$|s'(0)| = \frac{|u'(0)|}{1-|u(0)|^2} = \frac{\frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}{(a-b)^2|\alpha|^2}}{1-\left(\frac{|d_2|}{(a-b)|\alpha|}\right)^2} = \frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}{(a-b)^2|\alpha|^2-|d_2|^2},$$

we obtain

$$\frac{2}{1+\frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}{(a-b)^2|\alpha|^2-|d_2|^2}} \leq \frac{(a-b)|\alpha|+|d_2|}{(a-b)|\alpha|-|d_2|}\left(\frac{(1-b)^2|p'(1)|}{(a-b)|\alpha|}-1\right),$$

$$\frac{2\left((a-b)|\alpha|-|d_2|\right)^2}{(a-b)^2|\alpha|^2-|d_2|^2+\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|} \leq \frac{(1-b)^2|p'(1)|}{(a-b)|\alpha|}-1$$

and

$$|p'(1)| \geq \frac{(a-b)|\alpha|}{(1-b)^2}\left(1+\frac{2\left((a-b)|\alpha|-|d_2|\right)^2}{(a-b)^2|\alpha|^2-|d_2|^2+\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}\right).$$

□

If $h(z)-z$ has zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following theorem.

Theorem 2.4. *Let $h \in \mathcal{R}(\alpha)$ and $a_1, a_2, \ldots, a_n$ be zeros of the function $h(z)-z$ in D that are different from zero. Assume that, for $1 \in \partial D$, h has an angular limit $h(1)$ at the point 1, $h'(1) = \left(1-\frac{a-b}{1-b}\alpha\right)h(1)$. Then we have the inequality*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right| \geq \frac{(a-b)|\alpha|}{(1-b)^2}\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right. \tag{2.5}$$
$$\left.+\frac{2\left((a-b)|\alpha|\prod_{i=1}^{n}|a_i|-|d_2|\right)^2}{\left((a-b)|\alpha|\prod_{i=1}^{n}|a_i|\right)^2-|d_2|^2+\prod_{i=1}^{n}|a_i|\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2+(a-b)\alpha d_2\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\right).$$

Proof. Let $r(z)$ be as in (1.2) and $a_1, a_2, \ldots, a_n$ be zeros of the function $h(z)-z$ in D that are different from zero. Also, consider the function

$$B(z) = z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

By the maximum principle for each $z \in D$, we have

$$|r(z)| \leq |B(z)|.$$

Consider the function

$$\vartheta(z) = \frac{r(z)}{B(z)} = \left(\frac{p(z)-1}{b(p(z)-1)-(a-b)\alpha}\right)\frac{1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}$$

$$= \frac{d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\,\alpha} \frac{1}{z \prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i} z}}$$

$$= \frac{d_2 + \left(2d_3 - d_2^2\right) z + \cdots}{b\left(d_2 z + \left(2d_3 - d_2^2\right) z^2 + \cdots\right) - (a-b)\,\alpha} \frac{1}{\prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i} z}}.$$

The function $\vartheta(z)$ is analytic in D and $|\vartheta(z)| < 1$ for $|z| < 1$. In particular, we have

$$|\vartheta(0)| = \frac{|d_2|}{(a-b)\,|\alpha| \prod_{i=1}^{n} |a_i|}$$

and

$$|\vartheta'(0)| = \frac{\left| \left(2d_3 - d_2^2\right)(a-b)\,\alpha + b d_2^2 + (a-b)\,\alpha d_2 \sum_{i=1}^{n} \frac{1-|a_i|^2}{a_i} \right|}{(a-b)^2\,|\alpha|^2 \prod_{i=1}^{n} |a_i|}.$$

The composite function

$$\psi(z) = \frac{\vartheta(z) - \vartheta(0)}{1 - \overline{\vartheta(0)}\vartheta(z)}$$

is analytic in D, $|\psi(z)| < 1$ for $|z| < 1$ and $\psi(0) = 0$. For $1 \in \partial D$ and $h'(1) = \left(1 - \frac{a-b}{1-b}\alpha\right) h(1)$, we take $|\psi(1)| = 1$.

From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1 + |\psi'(0)|} \leq |\psi'(1)| &= \frac{1 - |\vartheta(0)|^2}{\left|1 - \overline{\vartheta(0)}\vartheta(1)\right|} |\vartheta'(1)| \\
&\leq \frac{1 + |\vartheta(0)|}{1 - |\vartheta(0)|} \left(|r'(1)| - |B'(1)|\right) \\
&= \frac{(a-b)\,|\alpha| \prod_{i=1}^{n} |a_i| + |d_2|}{(a-b)\,|\alpha| \prod_{i=1}^{n} |a_i| - |d_2|} \left(|r'(1)| - |B'(1)|\right).
\end{aligned}$$

It can be seen that

$$|\psi'(0)| = \frac{|\vartheta'(0)|}{1 - |\vartheta(0)|^2}$$

and

$$|\psi'(0)| = \frac{\frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2+(a-b)\alpha d_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{(a-b)^2|\alpha|^2\prod\limits_{i=1}^{n}|a_i|}}{1-\left(\frac{|d_2|}{(a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|}\right)^2}$$

$$= \prod_{i=1}^{n}|a_i| \frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2+(a-b)\alpha d_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\left((a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|\right)^2-|d_2|^2\right)}.$$

Also, we have

$$|B'(1)| = 1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2},\ 1\in\partial U.$$

Therefore, we obtain

$$\frac{2}{1+\prod\limits_{i=1}^{n}|a_i|\frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2+(a-b)\alpha d_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\left((a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|\right)^2-|d_2|^2\right)}}$$

$$\leq \frac{(a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|+|d_2|}{(a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|-|d_2|}\left(\frac{(1-b)^2|p'(1)|}{(a-b)|\alpha|}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right),$$

$$\frac{2\left((a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|-|d_2|\right)^2}{\left((a-b)|\alpha|\prod\limits_{i=1}^{n}|a_i|\right)^2-|d_2|^2+\prod\limits_{i=1}^{n}|a_i|\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2+(a-b)\alpha d_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}$$

$$\leq \frac{(1-b)^2|p'(1)|}{(a-b)|\alpha|}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2},$$

and so, we get inequality (2.5). □

If $h(z)-z$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.5. *Let $h\in\mathcal{R}(\alpha)$, $h(z)-z$ has no zeros in D except $z=0$ and $d_2>0$. Assume that, for $1\in\partial D$, h has an angular limit $h(1)$ at the point 1, $h'(1)=\left(1-\frac{a-b}{1-b}\alpha\right)h(1)$. Then we have the inequality*

$$\left|\left(\frac{zh'(z)}{h(z)}\right)'_{z=1}\right| \geq \frac{(a-b)|\alpha|}{(1-b)^2}\left(1-\frac{2(a-b)|\alpha|d_2\ln^2\left(\frac{d_2}{(a-b)|\alpha|}\right)}{2(a-b)|\alpha|d_2\ln\left(\frac{d_2}{(a-b)|\alpha|}\right)-\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}\right). \tag{2.6}$$

Proof. Let $d_2 > 0$ in the expression of the function $h(z)$. Having in mind the inequality (2.4) and the function $h(z) - z$ has no zeros in D except $z = 0$, we denote by $\ln u(z)$ the analytic branch of the logarithm normed by the condition

$$\ln u(0) = \ln\left(\frac{d_2}{(a-b)\,|\alpha|}\right) < 0.$$

The auxiliary function

$$\Theta(z) = \frac{\ln u(z) - \ln u(0)}{\ln u(z) + \ln u(0)}$$

is analytic in the unit disc D, $|\Theta(z)| < 1$, $\Theta(0) = 0$ and $|\Theta(1)| = 1$ for $1 \in \partial D$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Theta'(0)|} \le |\Theta'(1)| &= \frac{|2\ln u(0)|}{|\ln u(1) + \ln u(0)|^2}\left|\frac{u'(1)}{u(1)}\right| \\ &= \frac{-2\ln u(0)}{\ln^2 u(0) + \arg^2 u(1)}\left\{|r'(1)| - 1\right\}.\end{aligned}$$

Replacing $\arg^2 u(1)$ by zero, then

$$\frac{1}{1 - \frac{\frac{|(2d_3 - d_2^2)(a-b)\alpha + bd_2^2|}{(a-b)^2|\alpha|^2}}{\frac{2d_2}{(a-b)|\alpha|}\ln\left(\frac{d_2}{(a-b)|\alpha|}\right)}} \le \frac{-1}{\ln\left(\frac{d_2}{(a-b)|\alpha|}\right)}\left\{\frac{(1-b)^2\,|p'(1)|}{(a-b)\,|\alpha|} - 1\right\}$$

and

$$1 - \frac{2\,(a-b)\,|\alpha|\,d_2\ln^2\left(\frac{d_2}{(a-b)|\alpha|}\right)}{2\,(a-b)\,|\alpha|\,d_2\ln\left(\frac{d_2}{(a-b)|\alpha|}\right) - |(2d_3 - d_2^2)\,(a-b)\,\alpha + bd_2^2|} \le \frac{(1-b)^2\,|p'(1)|}{(a-b)\,|\alpha|}.$$

Thus, we obtain the inequality (2.6). □

The following theorem shows the relationship between the coefficients $d_2 = \frac{h''(0)}{2!}$ and $d_3 = \frac{h'''(0)}{3!}$ in the Maclaurin expansion of the function $h(z) = z + d_2 z^2 + d_3 z^3 + \cdots$.

Theorem 2.6. *Let $h \in \mathcal{R}(\alpha)$, $h(z) - z$ has no zeros in D except $z = 0$ and $d_2 > 0$. Then we have the inequality*

$$\left|\left(2d_3 - d_2^2\right)(a-b)\,\alpha + bd_2^2\right| \le 2\,(a-b)\left|\alpha d_2 \ln\left(\frac{d_2}{(a-b)\,|\alpha|}\right)\right|.$$

Proof. The function $\Theta(z)$ we have expressed in Theorem 2.5 satisfies the conditions for the Schwarz lemma. Thus, if we apply the Schwarz lemma to the function $\Theta(z)$, we obtain

$$1 \ge |\Theta'(0)| = \frac{|2\ln u(0)|}{|\ln u(0) + \ln u(0)|^2}\left|\frac{u'(0)}{u(0)}\right|$$

$$= \frac{-1}{2\ln u(0)}\left|\frac{u'(0)}{u(0)}\right|$$

$$= -\frac{\frac{\left|\left(2d_3-d_2^2\right)(a-b)\alpha+bd_2^2\right|}{(a-b)^2|\alpha|^2}}{\frac{2d_2}{(a-b)|\alpha|}\ln\left(\frac{d_2}{(a-b)|\alpha|}\right)}$$

and

$$\left|\left(2d_3-d_2^2\right)(a-b)\,\alpha+bd_2^2\right| \le 2\,(a-b)\left|\alpha d_2\ln\left(\frac{d_2}{(a-b)\,|\alpha|}\right)\right|. \qquad \square$$

Acknowledgement. The author would like to thank the reviewer for their constructive comments and suggestions on the earlier version of this paper.

References

[1] T. Akyel and B. N. Örnek, *Some remarks on Schwarz lemma at the boundary*, Filomat **31** (2017), no. 13, 4139–4151. https://doi.org/10.2298/fil1713139a

[2] T. Aliyev Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Var. Elliptic Equ. **58** (2013), no. 4, 571–577. https://doi.org/10.1080/17476933.2012.718338

[3] H. P. Boas, *Julius and Julia: mastering the art of the Schwarz lemma*, Amer. Math. Monthly **117** (2010), no. 9, 770–785. https://doi.org/10.4169/000298910X521643

[4] B. Çatal and B. N. Örnek, *Applications of Jack's lemma for certain subclasses of holomorphic functions on the unit disc*, Commun. Korean Math. Soc. **34** (2019), no. 2, 543–555. https://doi.org/10.4134/CKMS.c180157

[5] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci. (N.Y.) **122** (2004), no. 6, 3623–3629; translated from Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **286** (2002), Anal. Teor. Chisel i Teor. Funkts. 18, 74–84, 228–229. https://doi.org/10.1023/B:JOTH.0000035237.43977.39

[6] G. M. Goluzin, *Geometrical Theory of Functions of Complex Variable* (Russian), second edition, Izdat. "Nauka", Moscow, 1966.

[7] M. Mateljević, *Schwarz lemma, and distortion for harmonic functions via length and area*, Potential Anal. **53** (2020), no. 4, 1165–1190. https://doi.org/10.1007/s11118-019-09802-x

[8] M. Mateljević, N. Mutavdžić, and B. N. Örnek, *Note on some classes of holomorphic functions related to Jack's and Schwarz's lemma*, Appl. Anal. Discrete Math. **16** (2022), no. 1, 111–131. https://doi.org/10.2298/aadm200319006m

[9] P. R. Mercer, *Boundary Schwarz inequalities arising from Rogosinski's lemma*, J. Class. Anal. **12** (2018), no. 2, 93–97. https://doi.org/10.7153/jca-2018-12-08

[10] P. R. Mercer, *An improved Schwarz lemma at the boundary*, Open Math. **16** (2018), no. 1, 1140–1144. https://doi.org/10.1515/math-2018-0096

[11] B. N. Örnek, *Some results of the Carathéodory's inequality at the boundary*, Commun. Korean Math. Soc. **33** (2018), no. 4, 1205–1215. https://doi.org/10.4134/CKMS.c170372

[12] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3513–3517. https://doi.org/10.1090/S0002-9939-00-05463-0

[13] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der mathematischen Wissenschaften, 299, Springer-Verlag, Berlin, 1992. https://doi.org/10.1007/978-3-662-02770-7

[14] H. Unkelbach, *Über die Randverzerrung bei konformer Abbildung*, Math. Z. **43** (1938), no. 1, 739–742. https://doi.org/10.1007/BF01181115

Bülent Nafi Örnek
Department of Computer Engineering
Amasya University
Merkez-Amasya 05100, Türkiye
Email address: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr