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UNITARY ANALOGUES OF A GENERALIZED NUMBER-THEORETIC SUM

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Abstract. In this paper, we investigate the sums of the elements in the finite set $\{x^k : 1 \leq x \leq \frac{n}{m}, \gcd_u(x,n) = 1\}$, where k, m and n are positive integers and $\gcd_u(x,n)$ is the unitary greatest common divisor of x and n. Moreover, for some cases of k and m, we can give the explicit formulae for the sums involving some well-known arithmetic functions.

1. Introduction

For a finite set X of positive integers, let $\sum X$ denote the sum of all elements in X. For positive integers k, m and n, define the set

$$R_k^m(n) = \left\{x^k : 1 \leq x \leq \frac{n}{m},\ \gcd(x,n) = 1\right\} \tag{1.1}$$

and let

$$S_k^m(n) = \sum R_k^m(n).$$

The formula for the case $m = 1$ and $k = 1$ can be found in [2] and the formula for the case $m = 1$ and $k = 2$ can be found in [1, 7]. For the cases $m = 1$ and 2 with any positive integer k, the explicit formulae for $S_k^m(n)$ were given in [4] by N. R. Kanasri, P. Pornsurat, and Y. Tongron. Later in [5], N. R. Kanasri and Y. Tongron proved in general that for $n > m$,

$$S_k^m(n) = \sum_{d|n} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right), \tag{1.2}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x, $g_k(n) = 1^k + 2^k + \cdots + n^k$ and $g_k(0) = 0$.

In this paper, we work on unitary analogues of the set in (1.1), that is, the classical greatest common divisor is replaced by the unitary greatest common divisor. Next, we will define some terminologies involving unitary analogues. A divisor d of n is called a unitary divisor of n if $\gcd(d, \frac{n}{d}) = 1$, denoted by $d \parallel n$. This was defined by E. Cohen [3] in 1960. Let a, b be integers with

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$b > 0$. The divisor of a which is the largest unitary divisor of b is called the unitary greatest common divisor of a and b [3], denoted by $\gcd_u(a, b)$, i.e.,

$$\gcd_u(a, b) = \max\{d : d \mid a, \text{ and } d \parallel b\}.$$

It is clear that if b is a square-free integer, then $\gcd_u(a, b) = \gcd(a, b)$. There are several interesting arithmetic functions defined via a unitary divisor, for example, the unitary Euler's totient function,

$$\bar{\phi}(n) = \sum_{\substack{x \leq n \\ \gcd_u(x,n)=1}} 1,$$

and the unitary Möbius function,

$$\bar{\mu}(n) = (-1)^{\omega(n)},$$

where $\omega(n)$ is the number of distinct prime factors of n with $\omega(1) = 0$. A unitary analogue of the Möbius inversion formula [3] states that for an arithmetic functions f,

$$F(n) = \sum_{d \parallel n} f(d) \Longleftrightarrow f(n) = \sum_{d \parallel n} \bar{\mu}(d) F(n/d). \tag{1.3}$$

Moreover, both $\bar{\phi}$ and $\bar{\mu}$ are multiplicative, in other words, if $\gcd(n, m) = 1$, then

$$\bar{\phi}(mn) = \bar{\phi}(m)\bar{\phi}(n) \quad \text{and} \quad \bar{\mu}(mn) = \bar{\mu}(m)\bar{\mu}(n).$$

We also have [8]

$$\bar{\phi}(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{a_i}}\right),$$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization and we have [3]

$$\sum_{d \parallel n} \bar{\mu}(d) = I(n),$$

where $I(1) = 1$ and $I(n) = 0$ for all positive integers $n \geq 2$. For positive integers k, m and n, define the following finite sets of positive integers

$$Ru_k^m(n) = \{x^k : 1 \leq x \leq \frac{n}{m}, \ \gcd_u(x, n) = 1\}$$

and we let

$$Su_k^m(n) = \sum Ru_k^m(n).$$

For our convenience, write $Ru_k(n)$ and $Su_k(n)$ for $Ru_k^1(n)$ and $Su_k^1(n)$, respectively. In the next section, we establish the formulae for $Su_k^m(n)$.

2. Main results

Lemma 2.1 ([3]). *If f is a multiplicative function and F is defined by*

$$F(n) = \sum_{d \| n} f(d),$$

then F is also multiplicative.

Theorem 2.2. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, then*

$$\sum_{d \| n} \bar{\mu}(d) f(d) = \prod_{i=1}^{k} \left(1 - f(p_i^{a_i})\right).$$

Proof. Since f is multiplicative, we have $f(1) = 1$.

By Lemma 2.1, the function F defined by $F(n) = \sum_{d \| n} \bar{\mu}(d) f(d)$ is multiplicative; hence, $F(n)$ is the product of the values $F(p_i^{a_i})$.

Consider

$$\begin{aligned}\sum_{d \| n} \bar{\mu}(d) f(d) &= \prod_{i=1}^{k} \sum_{d \| p_i^{a_i}} \bar{\mu}(d) f(d) \\ &= \prod_{i=1}^{k} (\bar{\mu}(1) f(1) + \bar{\mu}(p_i^{a_i}) f(p_i^{a_i})) \\ &= \prod_{i=1}^{k} (1 - f(p_i^{a_i})).\end{aligned}$$

□

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of the integer $n > 1$. For a positive integer m, we define $\bar{\psi}_m(n) = \prod_{i=1}^{k} (1 - p_i^{m a_i})$ and $\bar{\psi}_m(1) = 1$.

Note that if $m = 1$, we write $\bar{\psi}$ instead $\bar{\psi}_1$.

Corollary 2.3. *For any positive integers n and m, we have*

$$\sum_{d \| n} \bar{\mu}(d) d^m = \bar{\psi}_m(n),$$

$$\sum_{d \| n} \frac{\bar{\mu}(d)}{d} = \frac{\bar{\phi}(n)}{n},$$

$$\sum_{d \| n} \bar{\mu}(d) = I(n).$$

Proof. The results follow immediately from Theorem 2.2. □

Another proof of Corollary 2.3 can be found in [6].

Next, we establish the formula for $Su_k^m(n)$ in the following theorem.

Theorem 2.4. *Let k, m, and n be positive integers with $n \geq m$. Then*

$$Su_k^m(n) = \sum_{d \| n} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right).$$

Proof. For a positive divisor d of n, define

$$Bu_d^m = \{x^k : 1 \leq x \leq \frac{n}{m},\ \gcd_u(x, n) = d\}.$$

It is clear that

$$\bigcup_{d \| n} Bu_d^m = \left\{1^k, 2^k, \ldots, \left\lfloor \frac{n}{m} \right\rfloor^k\right\} \text{ and } Bu_{d_1}^m \cap Bu_{d_2}^m = \emptyset \text{ for } d_1 \neq d_2,$$

which imply that

$$g_k\left(\left\lfloor \frac{n}{m} \right\rfloor\right) = \sum_{i=1}^{\lfloor n/m \rfloor} i^k = \sum_{d \| n} \sum Bu_d^k. \tag{2.1}$$

Next, we show that

$$Bu_d^m = d^k Ru_k^m\left(\frac{n}{d}\right). \tag{2.2}$$

If $x^k \in Bu_d^m$, then $1 \leq x \leq n/m$ and $\gcd_u(x, n) = d$, so $1 \leq x/d \leq n/dm$ and $\gcd_u(x/d, n/d) = 1$. Consequently, $(x/d)^k \in Ru_k^m(n/d)$ and so $x^k \in d^k Ru_k^m(n/d)$. On the other hand, if $y^k \in Ru_k^m(n/d)$, then $1 \leq y \leq n/dm$ and $\gcd_u(y, n/d) = 1$. It follows that $d \leq dy \leq n/m$ and $\gcd_u(dy, n) = d$. This shows that $(dy)^k \in Bu_d^m$ and the desired result follows.

For $d \| n$, we obtain by using (2.2) that

$$\begin{aligned} \sum Bu_d^m &= d^k \sum Ru_k^m\left(\frac{n}{d}\right) \\ &= d^k Su_k^m\left(\frac{n}{d}\right). \end{aligned}$$

It follows by (2.1) that

$$\begin{aligned} g_k\left(\left\lfloor \frac{n}{m} \right\rfloor\right) &= \sum_{d \| n} d^k Su_k^m\left(\frac{n}{d}\right) \\ &= \sum_{d \| n} \left(\frac{n}{d}\right)^k Su_k^m(d). \end{aligned}$$

By unitary analogue of the Möbius inversion formula with $f(n) = Su_k^m(n)/n^k$ and $F(n) = g_k(\lfloor n/m \rfloor)/n^k$, we get

$$\frac{Su_k^m(n)}{n^k} = \sum_{d \| n} \bar{\mu}(d) \frac{d^k}{n^k} g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right).$$

After multiplying by n^k, we get the desired result. □

To establish the explicit formulae for $Su_k(n)$ where $k = 1, 2, 3$, we utilize the well known fact of $g_k(n)$ as follows:

$$g_1(n) = \frac{n(n+1)}{2}, \quad g_2(n) = \frac{n(n+1)(2n+1)}{6}, \quad g_3(n) = \frac{n^2(n+1)^2}{4}.$$

Corollary 2.5. *We have*

$$\begin{aligned}
Su_1(n) &= \frac{n}{2}\left(\bar{\phi}(n) + I(n)\right), \\
Su_2(n) &= \frac{n^2}{3}\bar{\phi}(n) + \frac{n^2}{2}I(n) + \frac{n}{6}\bar{\psi}(n), \\
Su_3(n) &= \frac{n^3}{4}\bar{\phi}(n) + \frac{n^3}{2}I(n) + \frac{n^2}{4}\bar{\psi}(n).
\end{aligned}$$

Proof. By using Theorem 2.4 to calculate $Su_k(n)$ where $k = 1, 2, 3$, we get

$$\begin{aligned}
Su_1(n) &= \sum_{d\|n} \bar{\mu}(d) d g_1\left(\frac{n}{d}\right) \\
&= \sum_{d\|n} \bar{\mu}(d) d \left(\frac{n}{2d}\right)\left(\frac{n}{d}+1\right) \\
&= \frac{n^2}{2}\sum_{d\|n}\frac{\bar{\mu}(d)}{d} + \frac{n}{2}\sum_{d\|n}\bar{\mu}(d) \\
&= \frac{n}{2}\bar{\phi}(n) + \frac{n}{2}I(n), \\
Su_2(n) &= \sum_{d\|n} \bar{\mu}(d) d^2 g_2\left(\frac{n}{d}\right) \\
&= \sum_{d\|n} \bar{\mu}(d) d^2 \left(\frac{n}{6d}\right)\left(\frac{n}{d}+1\right)\left(\frac{2n}{d}+1\right) \\
&= \frac{n^3}{3}\sum_{d\|n}\frac{\bar{\mu}(d)}{d} + \frac{n^2}{2}\sum_{d\|n}\bar{\mu}(d) + \frac{n}{6}\sum_{d\|n}\bar{\mu}(d)d \\
&= \frac{n^3}{3}\bar{\phi}(n) + \frac{n^2}{2}I(n) + \frac{n}{6}\bar{\psi}(n), \\
Su_3(n) &= \sum_{d\|n} d^3 \bar{\mu}(d) g_3\left(\frac{n}{d}\right) \\
&= \sum_{d\|n} d^3 \bar{\mu}(d) \left(\frac{n}{2d}\right)^2\left(\frac{n}{d}+1\right)^2 \\
&= \frac{n^4}{4}\sum_{d\|n}\frac{\bar{\mu}(d)}{d} + \frac{n^3}{2}\sum_{d\|n}\bar{\mu}(d) + \frac{n^2}{4}\sum_{d\|n}\bar{\mu}(d)d \\
&= \frac{n^3}{4}\bar{\phi}(n) + \frac{n^3}{2}I(n) + \frac{n^2}{4}\bar{\psi}(n).
\end{aligned}$$

□

Theorem 2.6. *Let a, k and n be positive integers. If $p^b \parallel n$, where p is prime and $a \leq b$, then*

$$Su_k^{p^a}(n) = \sum_{d \parallel (n/p^b)} \bar{\mu}(d) d^k \left(g_k\left(\frac{n}{p^a d}\right) - p^{bk} g_k\left(\left\lfloor \frac{n}{p^{a+b} d} \right\rfloor\right)\right).$$

Proof. Let $n = p^b t$, where $p \nmid t$.

By Theorem 2.4, we have

$$\begin{aligned}
Su_k^{p^a}(n) &= \sum_{d \parallel n} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{n}{dp^a} \right\rfloor\right) \\
&= \sum_{\substack{d \parallel n \\ d \mid t}} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{p^{b-a} t}{d} \right\rfloor\right) + \sum_{\substack{d \parallel n \\ d \nmid t}} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{p^{b-a} t}{d} \right\rfloor\right) \\
&= \sum_{d \parallel t} \bar{\mu}(d) d^k g_k\left(\frac{p^{b-a} t}{d}\right) + \sum_{\substack{d \parallel n \\ d \nmid t}} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{p^{b-a} t}{d} \right\rfloor\right).
\end{aligned}$$

For $d \parallel n$, it is clearly that $d \nmid t$ if and only if $p^b \parallel d$, that is, $d = p^b e$ for some positive integer e with $p \nmid e$. Then

$$\begin{aligned}
\sum_{\substack{d \parallel n \\ d \nmid t}} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{p^{b-a} t}{d} \right\rfloor\right) &= \sum_{\substack{d \parallel n \\ p^b \parallel d}} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{p^{b-a} t}{d} \right\rfloor\right) \\
&= \sum_{p^b e \parallel p^b t} \bar{\mu}(p^b e)(p^b e)^k g_k\left(\left\lfloor \frac{p^{b-a} t}{p^b e} \right\rfloor\right) \\
&= -p^{bk} \sum_{e \parallel t} \bar{\mu}(e) e^k g_k\left(\left\lfloor \frac{t}{p^a e} \right\rfloor\right).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
Su_k^{p^a}(n) &= \sum_{d \parallel t} \bar{\mu}(d) d^k g_k\left(\frac{p^{b-a} t}{d}\right) - p^{bk} \sum_{d \parallel t} \bar{\mu}(d) d^k g_k\left(\left\lfloor \frac{t}{p^a d} \right\rfloor\right) \\
&= \sum_{d \parallel t} \bar{\mu}(d) d^k \left(g_k\left(\frac{p^{b-a} t}{d}\right) - p^{bk} g_k\left(\left\lfloor \frac{t}{p^a d} \right\rfloor\right)\right) \\
&= \sum_{d \parallel (n/p^b)} \bar{\mu}(d) d^k \left(g_k\left(\frac{n}{p^a d}\right) - p^{bk} g_k\left(\left\lfloor \frac{n}{p^{a+b} d} \right\rfloor\right)\right).
\end{aligned}$$

□

Next, we will calculate $Su_k^2(n)$, where $k = 1, 2, 3$. First, we use Theorem 2.4 and Theorem 2.6 to calculate general form of $Su_k^2(n)$ by consider two cases as

follows:
Case 1: n is odd. This implies that for any divisor d of n, $\frac{n}{d}$ is odd and $\left\lfloor \frac{n}{2d} \right\rfloor = \left\lfloor \frac{n/d}{2} \right\rfloor = \frac{n/d-1}{2}$. From Theorem 2.4, we have

$$Su_k^2(n) = \sum_{d \| n} \bar{\mu}(d) d^k g_k \left(\left\lfloor \frac{n}{2d} \right\rfloor \right) = \sum_{d \| n} \bar{\mu}(d) d^k g_k \left(\frac{n/d-1}{2} \right).$$

Case 2: n is even. There is a positive integer b such that $2^b \| n$. This implies that for any divisor d of $n/2^b$, $\frac{n}{2^b d}$ is odd and

$$\left\lfloor \frac{n}{2^{b+1} d} \right\rfloor = \left\lfloor \frac{n/2^b d}{2} \right\rfloor = \frac{n/2^b d - 1}{2}.$$

From Theorem 2.6 with $a = 1$ and $p = 2$, we have

$$Su_k^2(n) = \sum_{d \| (n/2^b)} \bar{\mu}(d) d^k \left(g_k \left(\frac{n}{2d} \right) - 2^{bk} g_k \left(\frac{n/2^b d - 1}{2} \right) \right).$$

Let p be a prime number and let n, m be positive integers such that $p^a \| n$ for some integer a. Then we can write $n = p^a t$ for some positive integer t such that $p \nmid t$ and we obtain

$$\bar{\phi}\left(\frac{n}{p^a} \right) = \bar{\phi}(t) = \frac{\bar{\phi}(p^a)\bar{\phi}(t)}{p^a - 1} = \frac{\bar{\phi}(n)}{p^a - 1},$$
$$\bar{\psi}_m\left(\frac{n}{p^a} \right) = \bar{\psi}_m(t) = \frac{\bar{\psi}_m(p^a)\bar{\psi}_m(t)}{1 - p^{ma}} = \frac{\bar{\psi}_m(n)}{1 - p^{ma}}.$$

Corollary 2.7. *For $n \geq 2$, we have*

$$Su_1^2(n) = \begin{cases} \frac{1}{8}\left(n\bar{\phi}(n) - \bar{\psi}(n) \right) & \text{if } n \text{ is odd,} \\ \frac{1}{8}\left(n\bar{\phi}(n) + 2nI\left(\frac{n}{2^b}\right) - \frac{2^b \bar{\psi}(n)}{2^b - 1} \right) & \text{if } n \text{ is even,} \end{cases}$$

$$Su_2^2(n) = \begin{cases} \frac{1}{24}\left(n^2\bar{\phi}(n) - n\bar{\psi}(n) \right) & \text{if } n \text{ is odd,} \\ \frac{1}{24}\left(n^2\bar{\phi}(n) + 3n^2 I\left(\frac{n}{2^b}\right) - \left(\frac{2^b+2}{2^b-1}\right) n\bar{\psi}(n) \right) & \text{if } n \text{ is even,} \end{cases}$$

$$Su_3^2(n) = \begin{cases} \frac{1}{64}\left(n^3\bar{\phi}(n) - 2n^2\bar{\psi}(n) + \bar{\psi}_3(n) \right) & \text{if } n \text{ is odd,} \\ \frac{1}{64}\left(n^3\bar{\phi}(n) + 4n^3 I\left(\frac{n}{2^b}\right) - \left(\frac{2^{b+1}+4}{2^b-1}\right) n^2\bar{\psi}(n) + \frac{8^b\bar{\psi}_3(n)}{8^b-1} \right) & \text{if } n \text{ is even,} \end{cases}$$

where b is a positive integer such that $2^b \| n$.

Proof. If n is odd, then by using Case 1, we have

$$\begin{aligned} Su_1^2(n) &= \sum_{d \| n} \bar{\mu}(d) d g_1 \left(\frac{n/d - 1}{2} \right) \\ &= \sum_{d \| n} \bar{\mu}(d) d \left(\frac{1}{2} \right) \left(\frac{n/d - 1}{2} \right) \left(\frac{n/d - 1}{2} + 1 \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\sum_{d\|n}\bar{\mu}(d)\left(\frac{n^2}{d}-d\right)\\
&= \frac{1}{8}\left(n^2\sum_{d\|n}\frac{\bar{\mu}(d)}{d}-\sum_{d\|n}\bar{\mu}(d)d\right)\\
&= \frac{1}{8}\left(n\bar{\phi}(n)-\bar{\psi}(n)\right),\\
Su_2^2(n) &= \sum_{d\|n}\bar{\mu}(d)d^2 g_2\left(\frac{n/d-1}{2}\right)\\
&= \sum_{d\|n}\bar{\mu}(d)d^2\left(\frac{1}{6}\right)\left(\frac{n/d-1}{2}\right)\left(\frac{n/d-1}{2}+1\right)\left(\frac{2(n/d-1)}{2}+1\right)\\
&= \frac{1}{24}\sum_{d\|n}\bar{\mu}(d)\left(\frac{n^3}{d}-nd\right)\\
&= \frac{1}{24}\left(n^3\sum_{d\|n}\frac{\bar{\mu}(d)}{d}-n\sum_{d\|n}\bar{\mu}(d)d\right)\\
&= \frac{1}{24}\left(n^2\bar{\phi}(n)-n\bar{\psi}(n)\right),\\
Su_3^2(n) &= \sum_{d\|n}\bar{\mu}(d)d^3 g_3\left(\frac{n/d-1}{2}\right)\\
&= \sum_{d\|n}\bar{\mu}(d)d^3\left(\frac{1}{4}\right)\left(\frac{n/d-1}{2}\right)^2\left(\frac{n/d-1}{2}+1\right)^2\\
&= \frac{1}{64}\sum_{d\|n}\bar{\mu}(d)\left(\frac{n^4}{d}-2n^2d+d^3\right)\\
&= \frac{1}{64}\left(n^4\sum_{d\|n}\frac{\bar{\mu}(d)}{d}-2n^2\sum_{d\|n}\bar{\mu}(d)d+\sum_{d\|n}\bar{\mu}(d)d^3\right)\\
&= \frac{1}{64}\left(n^3\bar{\phi}(n)-2n^2\bar{\psi}(n)+\bar{\psi}_3(n)\right).
\end{aligned}
$$

If n is even, then by using Case 2, we have

$$
\begin{aligned}
Su_1^2(n) &= \sum_{d\|(n/2^b)}\bar{\mu}(d)d\left(g_1\left(\frac{n}{2d}\right)-2^b g_1\left(\frac{n/2^b d-1}{2}\right)\right)\\
&= \sum_{d\|(n/2^b)}\bar{\mu}(d)d\left(\frac{1}{8}\left(\frac{n^2}{d^2}+\frac{2n}{d}\right)-\frac{2^b}{8}\left(\frac{n^2}{4^b d^2}-1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^b}{8}\left((2^b-1)\left(\frac{n}{2^b}\right)\bar{\phi}\left(\frac{n}{2^b}\right) + \frac{2n}{2^b} I\left(\frac{n}{2^b}\right) + \bar{\psi}\left(\frac{n}{2^b}\right)\right)\\
&= \frac{1}{8}\left(n\bar{\phi}(n) + 2nI\left(\frac{n}{2^b}\right) - \frac{2^b\bar{\psi}(n)}{2^b-1}\right),\\
Su_2^2(n) &= \sum_{d\|(n/2^b)} \bar{\mu}(d)d^2\left(g_2\left(\frac{n}{2d}\right) - 4^b g_2\left(\frac{n/2^b d - 1}{2}\right)\right)\\
&= \sum_{d\|(n/2^b)} \bar{\mu}(d)d^2\left(\frac{1}{24}\left(\frac{n^3}{d^3} + \frac{3n^2}{d^2} + \frac{2n}{d}\right) - \frac{4^b}{24}\left(\frac{n^3}{8^b d^3} - \frac{n}{2^b d}\right)\right)\\
&= \frac{4^b}{24}\left((2^b-1)\left(\frac{n}{2^b}\right)^2\bar{\phi}\left(\frac{n}{2^b}\right) + \frac{3n^2}{4^b} I\left(\frac{n}{2^b}\right)\right.\\
&\qquad \left. +(1+2^{1-b})\left(\frac{n}{2^b}\right)\bar{\psi}\left(\frac{n}{2^b}\right)\right)\\
&= \frac{1}{24}\left(n^2\bar{\phi}(n) + 3n^2 I\left(\frac{n}{2^b}\right) - \left(\frac{2^b+2}{2^b-1}\right)n\bar{\psi}(n)\right),\\
Su_3^2(n) &= \sum_{d\|(n/2^b)} \bar{\mu}(d)d^3\left(g_3\left(\frac{n}{2d}\right) - 8^b g_3\left(\frac{n/2^b d - 1}{2}\right)\right)\\
&= \sum_{d\|(n/2^b)} \bar{\mu}(d)d^3\left(\frac{1}{64}\left(\frac{n^4}{d^4} + \frac{4n^3}{d^3} + \frac{4n^2}{d^2}\right) - \frac{8^b}{64}\left(\frac{n^4}{16^b d^4} - \frac{2n^2}{4^b d^2} + 1\right)\right)\\
&= \frac{8^b}{64}\left((2^b-1)\left(\frac{n}{2^b}\right)^3\bar{\phi}\left(\frac{n}{2^b}\right) + \frac{4n^3}{8^b} I\left(\frac{n}{2^b}\right)\right.\\
&\qquad \left. +(2+2^{2-b})\left(\frac{n}{2^b}\right)^2\bar{\psi}\left(\frac{n}{2^b}\right) - \bar{\psi}_3\left(\frac{n}{2^b}\right)\right)\\
&= \frac{1}{64}\left(n^3\bar{\phi}(n) + 4n^3 I\left(\frac{n}{2^b}\right) - \left(\frac{2^{b+1}+4}{2^b-1}\right)n^2\bar{\psi}(n) + \frac{8^b\bar{\psi}_3(n)}{8^b-1}\right). \qquad \square
\end{aligned}
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