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SIMPLE FORMULATIONS ON CIRCULANT MATRICES WITH ALTERNATING FIBONACCI

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Abstract. In this article, an alternating Fibonacci sequence is defined from a second-order linear homogeneous recurrence relation with constant coefficients. Then, the determinant, inverse, and eigenvalues of the circulant matrices with entries in the first row having the formation of the sequence are formulated explicitly in a simple way. In this study, the method for deriving the formulation of the determinant and inverse is simply using traditional elementary row or column operations. For the eigenvalues, the known formulation from the case of general circulant matrices is simplified by considering the specialty of the sequence and using cyclic group properties. We also propose algorithms for the formulation to show how efficient the computations are.

1. Introduction

Circulant matrices have a wide range of applications in many areas of mathematical problems: numerical analysis, linear differential equations, operator theory, lightweight cryptography, and many others; hence also connected to computer science and engineering. All of those take advantage of the nice structure of the circulant matrix that the calculation of eigenvalues, eigenvectors, determinants, and inverse of the matrices can be formulated explicitly and computed efficiently.

So many papers recently studied the above problems with various specializations. Without intending to exclude any other articles whose similar topics to the topic of this paper but missed from our consideration, in the following we refer to some of those. Bueno [4] formulated the determinants and inverse of circulant matrices with geometric progression. Shen et al. [18] gave conditions for the invertibility of circulant matrices with special entries of the Fibonacci number and the Lucas number, the formulations of the determinant, and inverse of these kinds of matrices are derived as well. Jiang et al. [9] generalized those works by defining circulant matrices with the k-Fibonacci and k-Lucas

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numbers. With a similar problem, Jiang and Li [11] continued those results by applying to the left circulant and $\mathcal{G}$-circulant matrices. In the same year, the explicit determinants of circulant and left circulant matrices involving Tribonacci numbers or generalized Lucas numbers have been investigated by Li et al. see in [13].

Further study about the explicit inverse matrices continued but now of entries Tribonacci and the matrix structure is skew circulant, performed by Jiang and Hong in [10]. Besides, a computational approach using a symbolic algorithm for computing the inverse and determinant of general bordered tridiagonal matrices is presented by Jia and Li in [8]. Then, Radicic [16] followed the study of k-circulant matrices with geometric sequence, while Bozkurt and Tam [3] were interested in r-circulant matrices associated with a number sequence. Most recently, similar problems can be seen in [2, 5, 14, 15, 17, 19].

Inspired by all the above beautiful references, in this article, an alternating Fibonacci sequence is defined from a second-order linear homogenous recurrence relation with constant coefficients. Then, the determinant, inverse, and eigenvalues of circulant matrices with entries in the first-row having formation of that sequence are formulated explicitly in a much simpler way than in the formulation for the general case. In this study, the method for determining the determinant and the inverse of the formulation is simply based on an elementary row or column operations to get a simpler equivalent matrix. Note that this kind of method is different from all the above methods in the references. For the eigenvalues, the previous formulation from the case of general circulant matrices is simplified by considering the specialty of the alternating Fibonacci sequence and using cyclic group properties. Below is the outline of this article.

In Section 2, we review the general circulant matrix notion and the previous results associated with its eigenvalues, determinant, and inverse; also defining the alternating Fibonacci sequence and associated with the definition of its circulant matrix. In Section 3 we propose a theorem containing a simple formulation of the determinant and inverse of the matrix as defined in Section 2. Section 4 presents a theorem with its proof containing a simplified formulation for the eigenvalues of the matrix. We also present algorithms of those results to show how efficient the computations are, and we close the paper with a concluding remark, described in Sections 5 and 6, respectively.

2. Circulant matrix with alternating Fibonacci numbers

The first subsection of this section talks about the notion of the general circulant matrix and the previous results associated with the formulation of the eigenvalues, determinants, and inverse. For the last subsection, the alternating Fibonacci sequence is defined from a second-order linear homogenous recurrence relation with constant coefficients, and some of its properties are derived.

2.1. General circulant matrix

For a sequence of numbers $c_0, c_1, \ldots, c_{n-2}, c_{n-1}$, the $n \times n$ *circulant matrix*, with the usual notation from the references, is defined as

$$\operatorname{Circ}(c_0, c_1, \ldots, c_{n-2}, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & a_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Let C be the $\operatorname{Circ}(c_0, c_1, \ldots, c_{n-2}, c_{n-1})$, λ_k be the eigenvalues, and $\boldsymbol{u}_k$ be the corresponding eigenvectors of C for $k = 0, 1, 2, \ldots, n-1$. Then, λ_k and $\boldsymbol{u}_k$ are well-known formulated (see for examples in [6] and [1]) as

$$\lambda_k = \sum_{j=0}^{n-1} c_j \omega^{jk} \quad \text{and} \quad \boldsymbol{u}_k = \left(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-2)k}, \omega^{(n-1)k}\right), \tag{2.1}$$

where $\omega = e^{\frac{2\pi}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$ and $i = \sqrt{-1}$. In fact, the set $\mathcal{S} = \left\{1, \omega, \omega^2, \ldots, \omega^{n-1}\right\}$ is a cyclic subgroup in the multiplication group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, ω is one of the generators of $\mathcal{S}$, and all the elements in $\mathcal{S}$ are nth roots of unity over $\mathbb{C}$ which mean as the solutions of $x^n - 1 = 0$. Besides, that Equation (2.1) can also be written as

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}. \tag{2.2}$$

These notions will be used in the last section.

Direct consequences of Equation (2.1) is the formulation:

$$\det(C) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j \omega^{jk} \quad \text{and} \quad C^{-1} = \operatorname{Circ}(u_0, u_1, \ldots, u_{n-2}, u_{n-1}), \tag{2.3}$$

where $u_j = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k \omega^{-jk}$ for $j = 0, 1, \ldots, n-1$. When n is getting larger, those determinant and inverse formulas are computationally not very efficient to be implemented, especially because of involving the complex number arithmetics even though the entries of the matrix are real numbers. However, if the sequence of $c_0, c_1, \ldots, c_{n-2}, c_{n-1}$ has a nice formation or structure, such as coming from recurrence relation, then there is a possibility to simplify to get more explicit forms of the eigenvalues, determinant, and inverse of C. These kinds of studies become more interesting research topics over last decades which mostly focus on the determinant and inverse. Thus, one of the main results of this paper is also observing simplification of the eigenvalues formula. Also, in this time we use new special formation of $c_0, c_1, \ldots, c_{n-2}, c_{n-1}$ that call it as *alternating Fibonacci numbers* (sequence).

2.2. Alternating Fibonacci numbers

The basic theory of this subsection can be seen in [7]. When a second-order linear homogenous recurrence relation with constant coefficients is defined recursively as

$$a_n + a_{n-1} - a_{n-2} = 0,\ n \geq 2,\ \text{with initial condition } a_0 = 0,\ a_1 = 1,$$

then it will get a sequence on the form: $0, 1, -1, 2, -3, 5, -8, 13, -21, 34, \ldots$ which is in the subsequent of this paper called as *alternating Fibonacci sequence.* It is easy to derive that the solution of the recurrence relation is the explicit formula of the nth term $a_n = \frac{(-1)^n\left[\left(1-\sqrt{5}\right)^n - \left(1+\sqrt{5}\right)^n\right]}{2^n\sqrt{5}}$.

The relationship between the sequence $\{a_n\}_{n=1}^{\infty}$ and the Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ is simply

$$a_n = (-1)^{n-1} f_n \quad \text{or} \quad f_n = (-1)^{n-1} a_n.$$

The following proposition will be used later and can be proved by induction.

Proposition 1. *For any positive integer n, sum of the first $(n+1)$ terms in the alternating Finonacci sequence is formulated as $A_n = \sum_{i=0}^{n} a_i = 1 - a_{n-1}$ and sum of the first $(n+1)$ terms in the Finonacci sequence is formulated as $F_n = \sum_{i=0}^{n} f_i = f_{n+2} - 1$.*

We close this section by defining the matrix that will become the main object of this topic in this paper.

Definition 1. For any integer $n \geq 2$, the $n \times n$ circulant matrix with entry the alternating Finonacci sequence $\{a_i\}_{i=1}^{n}$ is the matrix $\mathrm{Circ}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

3. Determinant and inverse formulation

In this section, for the basic theory, we refer to [12] especially for the proof of the following theorem.

Theorem 1. *For integer $n \geq 3$, let $A = \mathrm{Circ}\,(a_1, a_2, \ldots, a_{n-1}, a_n)$ be the matrix defined in Definition 1 and let $x = 1 + a_n - a_{n-1}$. Then*

$$\det(A) = x^{n-1} + \sum_{j=0}^{n-2} a_{n-1-j} a_n^j x^{n-2-j}.$$

If $\delta = \det(A) \neq 0$, then $A^{-1} = \frac{1}{\delta}\,\mathrm{Circ}\,(e_1, e_2, e_3, e_4, \ldots, e_{n-1}, e_n)$, where

$$e_1 = \frac{\delta - a_n^{n-2}}{x},\ e_2 = \frac{\delta - x^{n-2}}{a_n},\ e_j = -x^{n-j} a_n^{j-3} \quad \text{for } j = 3, 4, \ldots, n.$$

Proof. The proof of this theorem is described step by step in 6 steps as follows.

(1) Applying E_1 as a series of elementary row operations on A: by substituting the ith row with the resulting operation of the ith row is added to the $(i+1)$th row and subtracted by the $(i+1)$th row, for $i = 2, 3, \ldots, n-2$; then, by substituting the $(n-1)$th row with the $(n-1)$th row is added to the nth row and subtracted by the first row; the last is by substituting the nth row with the first row is added to the nth row. The result is $A \sim D_1$, that is $D_1 = E_1(A)$ having entry structure

$$D_1 = \begin{pmatrix} 1 & -1 & 2 & -3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 0 & x & -a_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & -a_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x & -a_n & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & x & -a_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & x & -a_n \\ 0 & 1 & -1 & 2 & \cdots & a_{n-3} & a_{n-2} & x + a_{n-1} \end{pmatrix},$$

$$x = 1 + a_n - a_{n-1} \Leftrightarrow 1 + a_n = x + a_{n-1}. \tag{3.1}$$

Then, there exists the matrix $L_1 = E_1(I_n)$ such that $D_1 = L_1 A$, where

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(2) Applying K_1 as a series of elementary column operations on D_1 by substituting the jth column with the jth column is added to the result operation of the first column multiplied by $(-a_j)$ for $j = 2, 3, \ldots, n$. The result is $A \sim D_2 = K_1(D_1)$, where

$$D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x & -a_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & -a_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x & -a_n & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & x & -a_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & x & -a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-3} & a_{n-2} & x + a_{n-1} \end{pmatrix}.$$

Then, there exists the matrix $R_1 = K_1(I_n)$ such that $D_2 = L_1 A R_1$, where

$$R_1 = \begin{pmatrix} 1 & -a_2 & -a_3 & \cdots & -a_{n-3} & -a_{n-2} & -a_{n-1} & -a_n \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(3) Applying E_2 as a series of elementary row operations on D_2 by substituting the ith row with the ith row is multiplied by $\frac{1}{x}$ for $i = 2, 3, \ldots, n-1$. The result is $A \sim D_3 = E_2(D_2)$, where

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & q & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & q & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & q & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & q & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & q \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-3} & a_{n-2} & x + a_{n-1} \end{pmatrix},$$

$$q = \frac{-a_n}{x} \Leftrightarrow x = \frac{-a_n}{q}. \tag{3.2}$$

Then, there exists the matrix $L_2 = E_2(L_1)$ such that $D_3 = L_2 A R_1$, where

$$L_2 = \frac{1}{x} \begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ x & 0 & 0 & 0 & 0 & \cdots & 0 & x \end{pmatrix}.$$

(4) Applying K_2 as a series of elementary column operations on D_3 by substituting the $(j+1)$th column with the jth column is multiplied by $-q$ and added to the $(j+1)$th column for $j = 2, 3, \ldots, n-1$. The result is $A \sim D_4 =$

$K_2(D_3)$, where

$$D_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & u_1 & u_2 & u_3 & \cdots & u_{n-3} & u_{n-2} & d \end{pmatrix}.$$

In this case, $u_1 = a_1 = 1$, and for $j = 2, 3, \dots, n-2$,

$$u_j = -qu_{j-1} + a_j \Leftrightarrow a_j = qu_{j-1} + u_j \Leftrightarrow (a_j - u_j) = qu_{j-1}, \tag{3.3}$$

$$d = -qu_{n-2} + x + a_{n-1} = -qu_{n-2} + a_n + 1 \text{ and } \det(A) = x^{n-2}d. \tag{3.4}$$

Now, we will prove the formula of $\det(A)$ by formulating d. Consider Equation (3.3). Then we have the sequence

$$\begin{aligned} & u_1 = 1,\ u_2 = -qu_1 + a_2 = -qa_1 + a_2, \\ & u_3 = -q(-qa_1 + a_2) + a_3 = a_1(-q)^2 + a_2(-q) + a_3, \\ & u_4 = a_1(-q)^3 + a_2(-q)^2 + a_3(-q) + a_4, \\ & \quad \vdots \\ & u_{n-2} = \sum_{j=1}^{n-2} a_j (-q)^{n-2-j} \text{ and so that} \\ & \det(A) = x^{n-2}\left(x + a_{n-1} + \sum_{j=1}^{n-2} a_j(-q)^{n-1-j}\right). \end{aligned}$$

Then, using Equation (3.2) and changing the counter variable,

$$\det(A) = x^{n-1} + \sum_{j=0}^{n-2} a_{n-1-j} a_n^j x^{n-j-2}.$$

In this step, there exists $R = K_2(R_1)$ such that $D_4 = L_2 A R$ with

$$R = \begin{pmatrix} 1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_n \\ 0 & 1 & -q & (-q)^2 & \cdots & (-q)^{n-3} & (-q)^{n-2} \\ 0 & 0 & 1 & -q & (-q)^2 & \cdots & (-q)^{n-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -q & (-q)^2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -q \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $v_2 = 1$, and for $j = 3, 4, \ldots, n$, $v_j = -qv_{j-1} - a_j$.

(5) Applying E_3 as a series of elementary row operations on D_4 by substituting the nth row with the ith row is multiplied by $-u_i$ and added to the nth row, for $i = 2, 3, \ldots, n-2$. We have $A \sim D = E_3(D_4)$ is a diagonal matrix $D = \left(\begin{array}{c|c} I_{n-1} & O \\ \hline O & d \end{array}\right)$, and there exists $L = E_3(L_2)$ such that $D = LAR$ and

$$L = \frac{1}{x}\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & \cdots & z_{n-2} & z_{n-1} & z_n \end{pmatrix}, \text{ where}$$

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_n \end{pmatrix} = \begin{pmatrix} 0 & -u_1 & \cdots & -u_{n-2} & 1 \end{pmatrix}(xL_2). \tag{3.5}$$

For the purpose to formulate A^{-1}, we will derive the formulation of z_j, for $j = 1, 2, \ldots, n$. Based on Equations: (3.1), (3.2), (3.3), (3.4), and (3.5), notice that

$$\begin{aligned} &z_1 = u_{n-2} + x = u_{n-2} + \frac{-a_n}{q} = \frac{1-d}{q} = \frac{x(d-1)}{a_n},\ z_2 = -1, \\ &z_3 = -u_1 - u_2 = -1 - (-q(1) + (-1)) = q = (-q)\, z_2, \\ &z_4 = u_1 - u_2 - u_3 = -q^2 = (-q)\, z_3, \\ &z_5 = u_2 - u_3 - u_4 = q^3 = (-q)\, z_4 \end{aligned} \tag{3.6}$$

and so on, by the initial value $z_2 = 1$ for $j = 3, 5, \ldots, n-1$, we have

$$z_j = u_{j-3} - u_{j-2} - u_{j-1} = (-q)\, z_{j-1} = (-1)^{j-1} q^{j-2}, \tag{3.7}$$

then we also obtain that $z_n = u_{n-3} - u_{n-2} + x \Leftrightarrow$

$$z_n = -qz_{n-1} + d = (-1)^{n-1} q^{n-2} + d. \tag{3.8}$$

(6) From $D = LAR$ in Step 5, then $D^{-1} = (LAR)^{-1} \Leftrightarrow$

$$A^{-1} = RD^{-1}L = \begin{pmatrix} 1 & v_2 & v_3 & \cdots & v_{n-1} & \frac{v_n}{d} \\ 0 & 1 & -q & (-q)^2 & \cdots & \frac{(-q)^{n-2}}{d} \\ 0 & 0 & 1 & -q & \ddots & \frac{(-q)^{n-3}}{d} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \frac{-q}{d} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{d} \end{pmatrix} L. \tag{3.9}$$

Based on Equation (3.9), and since A^{-1} is also circulant (see Equation (2.3)), we may write $A^{-1} = \frac{1}{xd}\operatorname{Circ}(z_n, z_1, z_2, z_3, \ldots, z_{n-2}, z_{n-1})$. Since $d = \frac{\delta}{x^{n-2}}$, then $A^{-1} = \frac{x^{n-3}}{\delta}\operatorname{Circ}(z_n, z_1, z_2, z_3, \ldots, z_{n-2}, z_{n-1})$. To simplify, we rewrite

$$A^{-1} = \frac{1}{\delta}\operatorname{Circ}(e_1, e_2, e_3, e_4, \ldots, e_{n-1}, e_n),$$

where e_j can be formulated based on the formulation of z_j and substituting back that $q = \frac{-a_n}{x}$. Consider Equations: (3.2), (3.6), (3.7), (3.8), (3.9) to get

$$\begin{aligned} e_1 &= z_n x^{n-3} = \frac{\delta - (a_n)^{n-2}}{x}, \\ e_2 &= z_1 x^{n-3} = \frac{x^{n-2}(d-1)}{a_n} = \frac{\delta - x^{n-2}}{a_n}, \\ e_3 &= x^{n-3} z_2 = -x^{n-3}, \end{aligned}$$

and for $j = 4, 5, \ldots, n$, we have

$$e_j = x^{n-3} z_{j-1} = x^{n-3}\left((-1)^{j-2} q^{j-3}\right) \Leftrightarrow e_j = -x^{n-j} a_n^{j-3}. \qquad \square$$

4. Eigenvalues formulation

Recall the cyclic group $\mathcal{S} = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ from Section 2. All n elements of S geometrically occupy the unit circle in the complex plane and divide the circle into n equal parts, then it is very clear from the definition of S that for $l = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, we have

$$\omega^l + \omega^{n-l} = \omega^l + \omega^{-l} = 2\cos(l\theta) \text{ and } \omega^l - \omega^{n-l} = \omega^l - \omega^{-l} = 2i\sin(l\theta), \tag{4.1}$$

where $\theta = \frac{2\pi}{n}$. These equations will be used as an important part in the proof of the following theorem.

Theorem 2. *For integer $n \geq 3$, let $A = \operatorname{Circ}(a_1, a_2, \ldots, a_{n-1}, a_n)$ be the matrix defined in Definition 1 and for $j = 0, 1, 2, \ldots, n-1$, let λ_j be the eigenvalues of A. If $\theta = \frac{2\pi}{n}$ and $m = \lfloor \frac{n-1}{2} \rfloor$, then $\lambda_0 = 1 - a_{n-1}$, and for $k = 1, 2, \ldots, m$, we have $\lambda_k = R_k + C_k i$ and $\lambda_{n-k} = \overline{\lambda_k}$, where*

$$\begin{aligned} R_k &= 1 + \sum_{s=1}^{m} (a_{s+1} + a_{n-s+1}) \cos(sk\theta) \quad \text{and} \\ C_k &= \sum_{s=1}^{m} (a_{s+1} - a_{n-s+1}) \sin(sk\theta). \end{aligned}$$

For the case of n is even, we also include $\lambda_{\frac{n}{2}} = a_{n-1} - 2a_n - 1$ and R_k becomes

$$R_k = 1 + (-1)^k a_{\frac{n}{2}+1} + \sum_{s=1}^{m} (a_{s+1} + a_{n-s+1}) \cos(sk\theta).$$

Proof. Based on Equation (2.2), in the context of matrix A here we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{1-n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}$$

and so by Proposition 1, it is clear that $\lambda_0 = \sum_{t=0}^{n} a_t = 1 - a_{n-1}$, and for the case of n is even, it is also very simple that

$$\begin{aligned} \lambda_{\frac{n}{2}} &= \sum_{t=0}^{n} a_t \omega^{\frac{n}{2}t} = \sum_{t=0}^{n} (-1)^t a_t = \sum_{t=0}^{n} f_t = f_{n+2} - 1 = f_{n-1} + 2f_n - 1 \\ &= (-1)^{n-2} a_{n-1} + (-1)^{n-1} 2a_n - 1 = a_{n-1} - 2a_n - 1. \end{aligned}$$

Next, for $k = 1, 2, \ldots, m = \left\lfloor \frac{n-1}{2} \right\rfloor$, consider that

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= \sum_{t=0}^{n-1} a_{t+1} \left(\omega^{tk} + \omega^{t(n-k)}\right) = 2a_1 + \sum_{t=1}^{n-1} a_{t+1} \left(\omega^{tk} + \omega^{-tk}\right) \\ &= 2 + \sum_{t=1}^{m} a_{t+1} \left(\omega^{tk} + \omega^{-tk}\right) + \sum_{t=n-m}^{n-1} a_{t+1} \left(\omega^{tk} + \omega^{-tk}\right) \end{aligned}$$

but for the case of n is even,

$$\lambda_k + \lambda_{n-k} = 2 + \sum_{t=1}^{m} a_{t+1} \left(\omega^{tk} + \omega^{-tk}\right) + \sum_{t=n-m}^{n-1} a_{t+1} \left(\omega^{tk} + \omega^{-tk}\right) + 2(-1)^k a_{\frac{n}{2}+1}.$$

Transforming the counter variable: $s = t$ when $t = 1, \ldots, m$ and $s = n - t$ when $t = n - m, \ldots, n - 1$, we have

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= 2 + \sum_{s=1}^{m} a_{s+1} \left(\omega^{sk} + \omega^{-sk}\right) + \sum_{s=1}^{m} a_{n-s+1} \left(\omega^{(n-s)k} + \omega^{-(n-s)k}\right) \\ &= 2 + \sum_{s=1}^{m} \left(a_{s+1} + a_{n-s+1}\right) \left(\omega^{sk} + \omega^{-sk}\right) \end{aligned}$$

and for the case of n is even,

$$\lambda_k + \lambda_{n-k} = 2 + 2(-1)^k a_{\frac{n}{2}+1} + \sum_{s=1}^{m} \left(a_{s+1} + a_{n-s+1}\right) \left(\omega^{sk} + \omega^{-sk}\right).$$

By applying Equation (4.1), now we have

(i) $$\lambda_k + \lambda_{n-k} = 2 \left(1 + \sum_{s=1}^{m} \left(a_{s+1} + a_{n-s+1}\right) \cos(sk\theta)\right)$$

and when n is even,

$$\text{(ii)} \qquad \lambda_k + \lambda_{n-k} = 2\left(1 + (-1)^k a_{\frac{n}{2}+1} + \sum_{s=1}^{m} (a_{s+1} + a_{n-s+1}) \cos(sk\theta)\right).$$

Analogously, consider that

$$\begin{aligned} \lambda_k - \lambda_{n-k} &= \sum_{t=0}^{n-1} a_{t+1} \left(\omega^{tk} - \omega^{t(n-k)}\right) = \sum_{t=1}^{n-1} a_{t+1} \left(\omega^{tk} - \omega^{-tk}\right) \\ &= \sum_{s=1}^{m} (a_{s+1} - a_{n-s+1}) \left(\omega^{sk} - \omega^{-sk}\right), \end{aligned}$$

and applying Equation (4.1) to get

$$\text{(iii)} \qquad \lambda_k - \lambda_{n-k} = 2i \sum_{s=1}^{m} (a_{s+1} - a_{n-s+1}) \sin(sk\theta).$$

Finally, by adding and subtracting Equations: (i) with (iii), and when n is even: (ii) with (iii), we have $\lambda_k = R_k + iC_k$ and $\lambda_{n-k} = R_k - iC_k$, where

$$\begin{aligned} R_k &= 1 + \sum_{s=1}^{m} (a_{s+1} + a_{n-s+1}) \cos(sk\theta) \ \text{ and} \\ C_k &= \sum_{s=1}^{m} (a_{s+1} - a_{n-s+1}) \sin(sk\theta) \end{aligned}$$

and for the case of n is even, R_k becomes

$$R_k = 1 + (-1)^k a_{\frac{n}{2}+1} + \sum_{s=1}^{m} (a_{s+1} + a_{n-s+1}) \cos(sk\theta).$$

□

5. Computation remark

In this subsection, we present a simple illustration to figure out how to apply the formulations to compute the determinant and inverse based on Theorem 1 and the eigenvalues based on Theorem 2. Then, by considering that illustration, we could construct efficient algorithms.

Example 1 (Simple illustration)**.** For $n = 5$, we have $A = \mathrm{Circ}(1, -1, 2, -3, 5)$. Then, $x = 1 + 5 + 3 = 9$, and the determinant and inverse are

$$\begin{aligned} \delta &= 9^4 - 3(5)^0(9)^3 + 2(5)^1(9)^2 - (5)^2(9)^1 + (5)^3(9)^0 = 5084, \\ A^{-1} &= \frac{1}{\delta} \mathrm{Circ}\left(\frac{\delta - 5^3}{9}, \frac{\delta - 9^3}{5}, -\left(5^0\right)\left(9^2\right), -\left(5^1\right)\left(9^1\right), -(5)^2\left(9^0\right)\right) \\ &= \frac{1}{5084} \mathrm{Circ}\left(\ 551 \quad 871 \quad -81 \quad -45 \quad -25\ \right). \end{aligned}$$

For the eigenvalues, $\lambda_0 = 1 - a_4 = 4$, then $\theta = \frac{2\pi}{5}$ and

$$\begin{aligned}
R_1 &= 1 + (-1+5)\cos\frac{2\pi}{5} + (2-3)\cos\frac{4\pi}{5} = \frac{5}{4}\sqrt{5} + \frac{1}{4} \approx 3.0451,\\
R_2 &= 1 + 4\cos\frac{4\pi}{5} - \cos\frac{8\pi}{5} = 1 + 4\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5} \approx -2.5451,\\
C_1 &= (-1-5)\sin\frac{2\pi}{5} + (2+3)\sin\frac{4\pi}{5} \approx -2.7674,\\
C_2 &= -6\sin\frac{4\pi}{5} + 5\sin\frac{8\pi}{5} = -6\sin\frac{4\pi}{5} - 5\sin\frac{2\pi}{5} = -8.2820,
\end{aligned}$$

so that

$$\lambda_1 = 3.0451 - 2.7674i \text{ and } \lambda_4 = 3.0451 + 2.7674i,$$
$$\lambda_2 = -2.5451 - 8.2820i \text{ and } \lambda_3 = -2.5451 + 8.2820i.$$

From the above illustration, it is easy to see that in the iterative process of computing the determinant, some data can be stored for the next process of computing the inverse. So, the computation process can be done in one function in a parallel way to get very fast and efficient performance.

Algorithm 1. *INPUT*: $A = \mathrm{Circ}\,(a_1, a_2, \ldots, a_{n-1}, a_n)$ *with the entries of alternating Fibonacci numbers.*
OUTPUT: $\delta = \det(A)$ *and* $A^{-1} = \frac{1}{\delta}\,\mathrm{Circ}\,(e_1, e_2, \ldots, e_{n-1}, e_n)$.

(1) $x \leftarrow a_n - a_{n-1} + 1$; $s \leftarrow 1$; $r \leftarrow x^{n-2}$;
(2) $\delta \leftarrow r(x + a_{n-1})$; $e_1 \leftarrow -1$; $e_1 \leftarrow -r$;
(3) ***for*** $j = 1$ ***to*** $n-2$ ***do***
  $r \leftarrow \frac{r}{x}$; $e_{j+2} \leftarrow -sr$; $s \leftarrow sa_n$;
  $t \leftarrow sra_{n-1-j}$; $\delta \leftarrow \delta + t$;
 end do;
(4) $e_1 \leftarrow \frac{\delta + se_1}{x}$; $e_2 \leftarrow \frac{\delta + e_2}{a_n}$;
(5) ***return***(δ, A^{-1}).

For the eigenvalues, it is also very easy to see that only $\left\lfloor \frac{n-1}{2} \right\rfloor$ eigenvalues are computed iteratively and all without any complex number arithmetic used. So, it must be much faster than applying the general formula as mentioned in Section 2.

Algorithm 2. *INPUT*: $A = \mathrm{Circ}\,(a_1, a_2, \ldots, a_{n-1}, a_n)$ *with the entries of alternating Fibonacci numbers.*
OUTPUT: $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1}$; *the eigenvalues of* A.

(1) $x \leftarrow n \bmod 2$;
(2) ***if*** $x = 0$ ***then*** $\lambda_{\frac{n}{2}} \leftarrow (a_{n-1} - a_n - 1)$ ***endif***;
(3) $\lambda_0 \leftarrow (1 - a_{n-1})$; $m \leftarrow \left\lfloor \frac{n-1}{2} \right\rfloor$; $\theta \leftarrow \frac{2\pi}{n}$;
(4) ***for*** $k = 1$ ***to*** m ***do***
  $R \leftarrow 1$; $C \leftarrow 0$; $S \leftarrow 0$; $T \leftarrow k\theta$;
  for $s = 1$ ***to*** m ***do***
   $S \leftarrow S + T$; $x \leftarrow (a_{s+1} + a_{n-s+1})\cos S$; $R \leftarrow R + x$;

$y \leftarrow (a_{s+1} - a_{n-s+1}) \sin S$; $C \leftarrow C + y$;
end do;
if $x = 0$ ***then*** $R \leftarrow \left(R + (-1)^k a_{\frac{n}{2}+1}\right)$ ***endif***;
$\lambda_k \leftarrow R + C.\mathbf{i}$; $\lambda_{n-k} \leftarrow R - C.\mathbf{i}$;
end do;
(5) ***return***$(\lambda_0, \lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1})$.

6. Concluding remark

The formulation for the determinant and inverse of the matrices involving the alternating Fibonacci sequence can be presented in one theorem and in a simple way, so an efficient algorithm can be constructed for its computation. The method of deriving the formulation is simply using elementary row or column operations. For the eigenvalues, the previous formulation from the case of general circulant matrices can be simplified by considering the specialty of the alternating Fibonacci numbers and using cyclic group properties, so the computation can be done efficiently without involving any complex number arithmetic, i.e., all complex number eigenvalues are constructed.

The methods in this article should be applicable for any variant of circulant matrices (such as skew or more general r-circulant) with any specific formation of numbers (such as Fibonacci, Lucas, Pell, etc.). These would become the nearly future works.

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