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ESTIMATES FOR CERTAIN SHIFTED CONVOLUTION SUMS INVOLVING HECKE EIGENVALUES

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Abstract. In this paper, we obtain certain estimates for averages of shifted convolution sums involving Hecke eigenvalues of classical holomorphic cusp forms. This generalizes some results of Lü and Wang in this direction.

1. Introduction

The Fourier coefficients of automorphic forms are interesting and important research objects in modern number theory. Let H_k be the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, which consists of the eigenfunctions for the all Hecke operators T_n. The Fourier coefficients of $f \in H_k$ at the cusp infinity admit the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where we normalize $\lambda_f(1) = 1$ and $\lambda_f(n) \in \mathbb{R}$ is the nth normalized Fourier coefficient (Hecke eigenvalue) of f. It is well-known that the Hecke eigenvalue $\lambda_f(n)$ satisfies the Hecke relation

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers $m, n \geq 1$. In 1974, P. Deligne [4] proved the celebrated Ramanujan-Petersson conjecture which asserts that

$$(1) \qquad |\lambda_f(n)| \leq d(n),$$

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where $d(n)$ is the classical divisor function. Then the result (1) implies that for any prime number p, there exist two complex numbers $\alpha_f(p)$, $\beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2}$$

The Hecke L-function associated to $f(z)$ is defined by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \ \Re(s) > 1,$$

where the local parameters $\alpha_f(p)$, $\beta_f(p)$ are given by (2).

The nontrivial bounds of various shifted convolution sums played important roles in the modern analytic number theory (see e.g. [14, 16, 23]). Define

$$\lambda_{j,f}(n) := \sum_{n_1 \cdots n_j = n} \lambda_f(n_1) \cdots \lambda_f(n_j)$$

for any fixed integer $j \geq 1$. Then it is not hard to find the associated L-function of $\lambda_{j,f}(n)$ that

$$L(f,s)^j = \sum_{n=1}^{\infty} \frac{\lambda_{j,f}(n)}{n^s}, \quad \Re(s) > 1.$$

In 2019, Lü and Wang [15] considered the average estimates of shifted convolution sums involving $\lambda_{j,f}(n)$ and proved that

$$\sum_{h \leq H} \sum_{N < n \leq 2N} \lambda_{2,f}(n)\lambda_{2,f}(n+h) \ll N^{\frac{6}{5}+\varepsilon} H^{\frac{2}{5}}, \ N^{\frac{1}{3}} \leq H \leq N^{1-\varepsilon},$$

$$\sum_{h \leq H} \sum_{N < n \leq 2N} \lambda_{3,f}(n)\lambda_{3,f}(n+h) \ll N^{\frac{4}{3}+\varepsilon} H^{\frac{1}{3}}, \ N^{\frac{1}{2}} \leq H \leq N^{1-\varepsilon},$$

and for $j \geq 4$

$$\sum_{h \leq H} \sum_{N < n \leq 2N} \lambda_{j,f}(n)\lambda_{j,f}(n+h) \ll N^{\frac{4j+1}{2j+4}+\varepsilon} H^{\frac{2}{j+2}}, \ N^{\frac{2j-3}{2j}} \leq H \leq N^{1-\varepsilon}$$

for any given $\varepsilon > 0$. In fact, as remarked below Theorem 1.1 in Lü and Wang [15], Corollary 1.2 in Lü [14] can give the corresponding weaker results of the above results.

We define

$$\lambda_{j,f}^{i_1,\ldots,i_j}(n) := \sum_{n_1 \cdots n_j = n} \lambda_f(n_1^{i_1}) \cdots \lambda_f(n_j^{i_j}) \tag{3}$$

for the integers $j \geq 2$ and $i_1, \ldots, i_j \geq 1$. In this paper, we are interested in the average estimates for shifted convolution sums of the type

$$\sum_{h \leq H} \sum_{N < n \leq 2N} \lambda_{j,f}^{i_1,\ldots,i_j}(n)\lambda_{j,f}^{i_1,\ldots,i_j}(n+h).$$

More precisely, we will be able to prove the following results.

Theorem 1.1. *Assume that* $1 \le H \le N$, *and* $\lambda_{j,f}^{i_1,\ldots,i_j}(n)$ *is defined by* (3). *Then for any* $\varepsilon > 0$ *we have*

$$\sum_{h\le H}\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\ldots,i_j}(n)\lambda_{j,f}^{i_1,\ldots,i_j}(n+h)$$
$$\ll N^{2-\frac{4}{A_{j,i_1,\ldots,i_j}+2}}H^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}},\ N^{1-\frac{2}{A_{j,i_1,\ldots,i_j}}}\le H\le N^{1-\varepsilon},$$

where $A_{j,i_1,\ldots,i_j}$ *is given by*

$$A_{j,i_1,\ldots,i_j}=\sum_{l=1}^{j}(i_l+1).$$

Let $g \in H_r$ be another distinct Hecke eigenform. Define

$$\lambda_{j,f,g}^{i_1,\ldots,i_j}(n) := \sum_{n_1\cdots n_j=n}\lambda_{f\times g}(n_1^{i_1})\cdots\lambda_{f\times g}(n_j^{i_j}) \tag{4}$$

for $j \ge 2$ and $i_1, \ldots, i_j \ge 1$, where $\lambda_{f\times g}(n)$ is the nth normalized coefficient of the Dirichlet expansion of the associated Rankin-Selberg L-function $L(f \times g, s)$ attached to f and g. Using the similar method as in the proof of Theorem 1.1, we can also establish the following theorem.

Theorem 1.2. *Let* $f \in H_k$ *and* $g \in H_r$ *be two distinct Hecke eigenforms. Assume that* $1 \le H \le N$, *and* $\lambda_{j,f,g}^{i_1,\ldots,i_j}(n)$ *is defined by* (4). *Then for any* $\varepsilon > 0$ *we have*

$$\sum_{h\le H}\sum_{N<n\le 2N}\lambda_{j,f\times g}^{i_1,\ldots,i_j}(n)\lambda_{j,f\times g}^{i_1,\ldots,i_j}(n+h)$$
$$\ll N^{2-\frac{4}{B_{j,i_1,\ldots,i_j}+2}}H^{\frac{2}{B_{j,i_1,\ldots,i_j}+2}},\ N^{1-\frac{2}{B_{j,i_1,\ldots,i_j}}}\le H\le N^{1-\varepsilon},$$

where $B_{j,i_1,\ldots,i_j}$ *is given by*

$$B_{j,i_1,\ldots,i_j}=\sum_{l=1}^{j}(i_l+1)^2.$$

The proofs of Theorem 1.1 and Theorem 1.2 are based on a series of vital works of Gelbart and Jacquet [6], Kim and Shahidi [10–12], Dieulefait [5], Clozel and Thorne [1–3], and Newton and Thorne [17, 18] that the automorphy of all symmetric powers for cuspidal Hecke eigenforms of level 1 and weight $k \ge 2$. And we will also use the well-known Perron's formula and the individual or average convexity bounds of the associated L-function in the proof of the main results in this paper.

2. Auxiliary results

In this section, we review some relevant facts of the symmetric power L-functions and collect some important lemmas which play an important role in the proof of the main results in this paper.

Let $f \in H_k$ be a Hecke eigenform. Then we can define the jth symmetric power L-function attached to f by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\right)^{-1} \tag{5}$$

for $\Re(s) > 1$. We can rewrite it as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right) \\ &=: \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \ \Re(s) > 1. \end{aligned} \tag{6}$$

It is well-known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. And from (2), (5), (6) and the Hecke operator theory, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \ j \geq 1. \tag{7}$$

Let $g \in H_r$ be another distinct Hecke eigenform. Then the Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ &:= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} (1 - \alpha_f(p)^{i-2m}\alpha_g(p)^{j-2m'}p^{-s})^{-1} \end{aligned} \tag{8}$$

for $\Re(s) > 1$. Similarly, it can also be expressed as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) &= \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^k)}{p^{ks}}\right) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} \end{aligned} \tag{9}$$

for $\Re(s) > 1$, where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is a real multiplicative function. From (7)-(16), we have

$$\begin{aligned} \lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) &= \sum_{m=0}^{i} \sum_{m'=0}^{j} \alpha_f(p)^{i-2m}\alpha_g(p)^{j-2m'} \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p). \end{aligned} \tag{10}$$

Lemma 2.1. *Let $f(s) = \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series which converges absolutely for $\Re(s) > 1$. And $|a_n| \leq A(n)$, where $A(n)$ is a positive monotonically increasing function of n and*

$$\sum_{n \geq 1} |a_n| n^{-\sigma} = O((\sigma - 1)^{-\alpha})$$

for some $\alpha > 0$ as $\sigma \to 1^+$. Then we have

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\bigg(\frac{x^b}{T(b-1)^\alpha}\bigg) + O\bigg(\frac{xA(2x)\log x}{T}\bigg)$$

for any $b > 1$, $1 < b \le b_0$, $T \ge 2$, the constants in O-symbols depend on b_0.

Proof. See Karatsuba and Voronin [9, pp. 334–336]. □

Let π_f and π_g be two automorphic cuspidal automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$. It is well-known that an automorphic cuspidal representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ is associated to a primitive form f, and hence an automorphic function $L(\pi_f, s)$ coincides with $L(f, s)$. Denote by $\mathrm{sym}^j\pi_f$ the jth symmetric power lift of π_f. For $2 \le j \le 8$, the automorphy of $\mathrm{sym}^j\pi_f$ was proved by a series of important works of Gelbart and Jacquet [6], Kim and Shahidi [10–12], Dieulefait [5], and Clozel and Thorne [1–3]. Very recently, Newton and Thorne [17, Theorem B], [18, Theorem A] showed that there exists a cuspidal automorphy representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function equals $L(\mathrm{sym}^j f, s)$ for all $j \ge 1$. Hence for $j \ge 1$, the L-function $L(\mathrm{sym}^j f, s)$ is an entire function and satisfies a functional equation of certain Riemann zeta-type with degree $j+1$. Furthermore, based on the works of Jacquet-Shalika [7,8], Shahidi [21,22], Rudnick-Sarnak [20], Lau-Wu [13], the L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ is an entire function (except possibly for simple poles at $s = 0, 1$ when $\mathrm{sym}^i\pi_f \cong \mathrm{sym}^j\pi_g$) and satisfies a certain Riemann zeta-type function equation with degree $(i+1)(j+1)$.

Lemma 2.2. *Let $f \in H_k$ be a Hecke eigenform. Let $\lambda_{j,f}^{i_1,\dots,i_j}(n)$ defined by (3) for $j \ge 2, i_1, \dots, i_j \ge 1$ and define*

$$L_{j,i_1,i_2,\dots,i_j}(s) := \sum_{n=1}^{\infty} \frac{\lambda_{j,f}^{i_1,\dots,i_j}(n)}{n^s},\ \Re(s) > 1.$$

Then we have

$$L_{j,i_1,i_2,\dots,i_j}(s) = L(sym^{i_1} f, s)L(sym^{i_2} f, s)\cdots L(sym^{i_j} f, s)U_{j,i_1,i_2,\dots,i_j}(s),$$

where the function $U_{j,i_1,i_2,\dots,i_j}(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \ge \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof. By (7), it is not hard to find for $\Re(s) > 1$ we have

$$L_i(s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n^i)}{n^s} = L(\mathrm{sym}^i f, s) \times \prod_p \bigg(1 + \frac{\lambda_f(p^{2i}) - \lambda_{\mathrm{sym}^i}(p^2)}{p^{2s}} + \cdots\bigg).$$

By Deligne's bound (2), we know that

$$U_i(s) := \prod_p \bigg(1 + \frac{\lambda_f(p^{2i}) - \lambda_{\mathrm{sym}^i}(p^2)}{p^{2s}} + \cdots\bigg)$$

admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. Hence the results follow from the Dirichlet convolution of the Dirichlet series of L-functions $L_{i_l}(s)$ with $l = 1, 2, \ldots, j$. $\square$

Lemma 2.3. *Let $f \in H_k$ be a Hecke eigenform. Let $\lambda_{j,f}^{i_1,\ldots,i_j}(n)$ defined by (4) for $j \geq 2, i_1, \ldots, i_j \geq 1$ and define*

$$L^*_{j,i_1,i_2,\ldots,i_j}(s) := \sum_{n=1}^{\infty} \frac{\lambda_{j,f\times g}^{i_1,\ldots,i_j}(n)}{n^s}, \ \Re(s) > 1.$$

Then we have

$$L^*_{j,i_1,i_2,\ldots,i_j}(s) = L(sym^{i_1} f \times sym^{i_1} g, s)L(sym^{i_2} f \times sym^{i_2} g, s) \times \cdots \times L(sym^{i_j} f \times sym^{i_j} g, s)U^*_{j,i_1,i_2,\ldots,i_j}(s),$$

where the function $U^*_{j,i_1,i_2,\ldots,i_j}(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof. The proof is analogue to the proof of Lemma 2.2 by noting the relation (10). $\square$

From above we observe that $L(\mathrm{sym}^j f, s)$, $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$, $(i, j \geq 1)$ are general L-functions in the sense of Perelli [19]. For general L-functions, we have the following average or individual convexity bounds.

Lemma 2.4. *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma + it)|^2 dt \ll T^{m(1-\sigma)+\varepsilon} \tag{11}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$, and

$$\mathfrak{L}(\sigma + it) \ll (1 + |t|)^{\max\{\frac{m}{2}(1-\sigma),0\}+\varepsilon} \tag{12}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 1$.

Proof. The results follow from Perelli's mean value theorem and convexity bound for the general L-function in [19]. $\square$

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we shall give the proof of Theorem 1.1 in details, and Theorem 1.2 can be handled by the similar approaches.

Define

$$S_{j,i_1,\ldots,i_j}(N, H) = \sum_{h \leq H} \sum_{N < n \leq 2N} \lambda_{j,f}^{i_1,\ldots,i_j}(n) \lambda_{j,f}^{i_1,\ldots,i_j}(n + h).$$

By interchanging the order of summation, we get

$$S_{j,i_1,\ldots,i_j}(N, H) = \sum_{N < n \leq 2N} \lambda_{j,f}^{i_1,\ldots,i_j}(n) \sum_{h \leq H} \lambda_{j,f}^{i_1,\ldots,i_j}(n + h). \tag{13}$$

By Lemma 2.1, the inner sum in (13) can be expressed in the form

$$\sum_{h\le H}\lambda_{j,f}^{i_1,\dots,i_j}(n+h)$$
$$= \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} L_{j,i_1,\dots,i_j}(s)((n+H)^s-n^s)\frac{ds}{s}+O\bigg(\frac{N^{1+\varepsilon}}{T}\bigg), \tag{14}$$

where $\eta = 1+\varepsilon$ and $2\le T\le N$ is a parameter to be chosen later. Next we shift the line of integration to the parallel segment with $\Re(s)=\frac{1}{2}+\varepsilon$ and apply Cauchy residue theorem to get

$$\sum_{h\le H}\lambda_{j,f}^{i_1,\dots,i_j}(n+h)$$
$$= \frac{1}{2\pi i}\bigg\{\int_{\kappa-iT}^{\kappa+iT}+\int_{\kappa+iT}^{\eta+iT}+\int_{\eta-iT}^{\kappa-iT}\bigg\}L_{j,i_1,\dots,i_j}(s)((n+H)^s-n^s)\frac{ds}{s}$$
$$+O\bigg(\frac{N^{1+\varepsilon}}{T}\bigg)$$
$$:= J_1+J_2+J_3+O\bigg(\frac{N^{1+\varepsilon}}{T}\bigg), \tag{15}$$

where $\kappa=\frac{1}{2}+\varepsilon$.

By applying the convexity bound (12), the integrals over the horizontal segments J_2, J_3 can be estimated as

$$J_2+J_3 \ll \int_{\kappa}^{\eta}\big|L_{j,i_1,\dots,i_j}(s)\big|\frac{N^{\sigma}}{T}d\sigma$$
$$\ll \max_{\kappa\le\sigma\le\eta} T^{\frac{A_{j,i_1,\dots,i_j}}{2}(1-\sigma)+\varepsilon}N^{\sigma}T^{-1}$$
$$\ll \frac{N^{1+\varepsilon}}{T}+N^{\frac{1}{2}}T^{\frac{A_{j,i_1,\dots,i_j}}{4}-1}$$
$$\ll \frac{N^{1+\varepsilon}}{T} \tag{16}$$

provided $N^{\frac{1}{2}}T^{\frac{A_{j,i_1,\dots,i_j}}{4}-1}\le\frac{N^{1+\varepsilon}}{T}$, that is to say the first term dominated, i.e., $T\le N^{\frac{2}{A_{j,i_1,\dots,i_j}}}$. Here we note $L_{j,i_1,\dots,i_j}(s)$ is an L-function of degree $A_{j,i_1,\dots,i_j}$ given by

$$A_{j,i_1,\dots,i_j} := \sum_{l=1}^{j}(i_l+1).$$

Combining (13)-(16), we obtain

$$S_{j,i_1,\dots,i_j}(N,H)$$
$$= \sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)\bigg\{\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}L_{j,i_1,\dots,i_j}(s)((n+H)^s-n^s)\frac{ds}{s}\bigg\}$$

$$+O\biggl(\sum_{N<n\le 2N}|\lambda_{j,f}^{i_1,\dots,i_j}(n)|\frac{N^{1+\varepsilon}}{T}\biggr)$$

$$:= R_{j,i_1,\dots,i_j}(N,H)+O\biggl(\frac{N^{2+\varepsilon}}{T}\biggr), \tag{17}$$

where

$$R_{j,i_1,\dots,i_j}(N,H)$$
$$=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}L_{j,i_1,\dots,i_j}(s)\biggl(\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)((n+H)^s-n^s)\biggr)\frac{ds}{s}. \tag{18}$$

Applying partial summation formula (see e.g. [9, Appendix,Theorem 1]) to the inner sum in (18), we get

$$\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)((n+H)^s-n^s)$$
$$=\sum_{N<n\le 2N}\biggl(\Bigl(1+\frac{H}{n}\Bigr)^s-1\biggr)\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s$$
$$=\biggl(\Bigl(1+\frac{H}{2N}\Bigr)^s-1\biggr)\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s$$
$$+sH\int_N^{2N}\Bigl(1+\frac{H}{x}\Bigr)^{s-1}\biggl(\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s\biggr)\frac{dx}{x^2}.$$

Hence we have

$$R_{j,i_1,\dots,i_j}(N,H)=R^1_{j,i_1,\dots,i_j}(N,H)+R^2_{j,i_1,\dots,i_j}(N,H), \tag{19}$$

where

$$R^1_{j,i_1,\dots,i_j}(N,H)$$
$$=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}L_{j,i_1,\dots,i_j}(s)\biggl(\Bigl(1+\frac{H}{2N}\Bigr)^s-1\biggr)\biggl(\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s\biggr)\frac{ds}{s}$$
$$=\frac{1}{4\pi iN}\int_0^H\int_{\kappa-iT}^{\kappa+iT}L_{j,i_1,\dots,i_j}(s)\Bigl(1+\frac{\theta}{2N}\Bigr)^{s-1}$$
$$\times\biggl(\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s\biggr)ds d\theta, \tag{20}$$

and

$$R^2_{j,i_1,\dots,i_j}(N,H)=\frac{H}{2\pi i}\int_N^{2N}\int_{\kappa-iT}^{\kappa+iT}L_{j,i_1,\dots,i_j}(s)\Bigl(1+\frac{H}{x}\Bigr)^{s-1}$$
$$\times\biggl(\sum_{N<n\le 2N}\lambda_{j,f}^{i_1,\dots,i_j}(n)n^s\biggr)\frac{ds dx}{x^2}. \tag{21}$$

For $i = 1, 2$, we get

$$R^{i}_{j,i_1,\ldots,i_j}(N,H) \ll \frac{H}{N}\sup_{N<x\le 2N}\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|\left|\sum_{N<n\le x}\lambda^{i_1,\ldots,i_j}_{j,f}(n)n^{\frac{1}{2}+it}\right|dt. \tag{22}$$

By Lemma 2.1, we obtain

$$\begin{aligned}&\sum_{N<n\le x}\lambda^{i_1,\ldots,i_j}_{j,f}(n)n^{\frac{1}{2}+it}\\ &= \frac{2}{2\pi i}\int_{\frac{3}{2}+\varepsilon+iT}^{\frac{3}{2}+\varepsilon-iT} L_{j,i_1,\ldots,i_j}\left(w-\left(\frac{1}{2}+it\right)\right)(x^w-N^w)\frac{dw}{w}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right),\end{aligned}$$

where $|t| \le T$ and $2 \le T \le N$ is a parameter to be chosen later.

By shifting the line of integration to the parallel segment with $\Re(w) = 1$ and applying the Cauchy residue theorem, we get

$$\begin{aligned}&\sum_{N<n\le x}\lambda^{i_1,\ldots,i_j}_{j,f}(n)n^{\frac{1}{2}+it}\\ &= \frac{1}{2\pi i}\left\{\int_{1-2iT}^{1+2iT}+\int_{1+2iT}^{\frac{3}{2}+\varepsilon+2iT}+\int_{\frac{3}{2}+\varepsilon-2iT}^{1-2iT}\right\}L_{j,i_1,\ldots,i_j}\left(w-\left(\frac{1}{2}+it\right)\right)\\ &\quad\times(x^w-N^w)\frac{dw}{w}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right)\\ &:= I_1+I_2+I_3+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right).\end{aligned}$$

For the integral over the horizontal segments, by the subconvexity bound (12), we obtain

$$\begin{aligned}I_2+I_3 &\ll \int_{1}^{\frac{3}{2}+\varepsilon}\left|L_{j,i_1,\ldots,i_j}\left(w-\left(\frac{1}{2}+it\right)\right)\right|\frac{N^{\sigma}}{T}d\sigma\\ &\ll \max_{1\le\sigma\le\frac{3}{2}+\varepsilon} T^{\frac{A_{j,i_1,\ldots,i_j}}{2}(\frac{3}{2}-\sigma)}N^{\sigma}T^{-1}\\ &\ll T^{\frac{A_{j,i_1,\ldots,i_l}}{4}-1}N+N^{\frac{3}{2}+\varepsilon}T^{-1} \ll N^{\frac{3}{2}+\varepsilon}T^{-1}\end{aligned} \tag{23}$$

provided $T^{\frac{A_{j,i_1,\ldots,i_j}}{4}-1}N \le N^{\frac{3}{2}+\varepsilon}T^{-1}$, i.e., $T \le N^{\frac{2}{A_{j,i_1,\ldots,i_j}}}$.

Hence we have

$$\begin{aligned}&\sum_{N<n\le x}\lambda^{i_1,\ldots,i_j}_{j,f}(n)n^{\frac{1}{2}+it}\\ &= \frac{1}{2\pi i}\int_{1-2iT}^{1+2iT} L_{j,i_1,\ldots,i_j}\left(w-\left(\frac{1}{2}+it\right)\right)(x^w-N^w)\frac{dw}{w}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right).\end{aligned} \tag{24}$$

Substituting (24) to (22), we have

$$\begin{aligned}
&R^i_{j,i_1,\ldots,i_j}(N,H)\\
&\ll \frac{H}{N}\sup_{N<x\le 2N}\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|\\
&\quad\times\left|\left\{\frac{1}{2\pi i}\int_{1-2iT}^{1+2iT}L_{j,i_1,\ldots,i_j}\left(w-\left(\frac{1}{2}+it\right)\right)(x^w-N^w)\frac{dw}{w}\right\}\right|dt\\
&\quad+\frac{H}{N}\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|\frac{N^{\frac{3}{2}+\varepsilon}}{T}dt\\
&:= A_{j,i_1,\ldots,i_j}(N,H)+B_{j,i_1,\ldots,i_j}(N,H). \qquad (25)
\end{aligned}$$

Using (11) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&A_{j,i_1,\ldots,i_j}(N,H)\\
&\ll H\int_{-2T}^{2T}\left(\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\frac{dt_1}{1+|t_1|}\\
&\ll H\int_{-2T}^{2T}T^{\frac{A_{j,i_1,\ldots,i_j}}{2}}\frac{dt_1}{1+|t_1|}\ll HT^{\frac{A_{j,i_1,\ldots,i_j}}{2}+\varepsilon}. \qquad (26)
\end{aligned}$$

For $B_{j,i_1,\ldots,i_j}(N,H)$, by the Cauchy-Schwarz inequality and (11), we obtain

$$\begin{aligned}
B_{j,i_1,\ldots,i_j}(N,H)&\ll\frac{HN^{\frac{1}{2}+\varepsilon}}{T}T^{\frac{1}{2}}\left(\int_{-T}^{T}\left|L_{j,i_1,\ldots,i_j}\left(\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\\
&\ll HN^{\frac{1}{2}+\varepsilon}T^{-\frac{1}{2}}T^{\frac{A_{j,i_1,\ldots,i_j}}{4}}. \qquad (27)
\end{aligned}$$

Combining (17), (19), (22), (25), (26) and (27), we obtain

$$S_{j,i_1,\ldots,i_j}(N,H)\ll\frac{N^{2+\varepsilon}}{T}+HT^{\frac{A_{j,i_1,\ldots,i_j}}{2}+\varepsilon}+HN^{\frac{1}{2}+\varepsilon}T^{\frac{A_{j,i_1,\ldots,i_j}}{4}-\frac{1}{2}+\varepsilon}.$$

Now we choose T such that $HN^{\frac{1}{2}+\varepsilon}T^{\frac{A_{j,i_1,\ldots,i_j}}{4}-\frac{1}{2}+\varepsilon}\le HT^{\frac{A_{j,i_1,\ldots,i_j}}{2}+\varepsilon}$, i.e., $T\ge N^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}}$, then

$$S_{j,i_1,\ldots,i_j}(N,H)\ll\frac{N^{2+\varepsilon}}{T}+HT^{\frac{A_{j,i_1,\ldots,i_j}}{2}+\varepsilon}.$$

On taking $T=(\frac{N^2}{H})^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}}$, then

$$S_{j,i_1,\ldots,i_j}(N,H)\ll N^{2-\frac{4}{A_{j,i_1,\ldots,i_j}+2}}H^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}}.$$

For $N^{1-\frac{2}{A_{j,i_1,\ldots,i_j}}} \leq H \leq N^{1-\varepsilon}$, $T = (\frac{N^2}{H})^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}}$ satisfies the restriction $N^{\frac{2}{A_{j,i_1,\ldots,i_j}+2}} \leq T \leq N^{\frac{2}{A_{j,i_1,\ldots,i_j}}}$, and obviously it improves the trivial bound $N^{1+\varepsilon}H$.

This completes the proof of Theorem 1.1.

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