J. Korean Math. Soc. **60** (2023), No. 3, pp. 683–694
https://doi.org/10.4134/JKMS.j220479
pISSN: 0304-9914 / eISSN: 2234-3008

ON RELATIVE COHEN-MACAULAY MODULES

Zhongkui Liu, Pengju Ma, and Xiaoyan Yang

Abstract. Let $\mathfrak{a}$ be an ideal of a commutative noetherian ring R. We give some descriptions of the $\mathfrak{a}$-depth of $\mathfrak{a}$-relative Cohen-Macaulay modules by cohomological dimensions, and study how relative Cohen-Macaulayness behaves under flat extensions. As applications, the perseverance of relative Cohen-Macaulayness in a polynomial ring, formal power series ring and completion are given.

1. Introduction

The theory of Cohen-Macaulay rings and modules is among the most deep influential parts of commutative algebra, with numerous applications in commutative algebra, algebraic geometry and combinatorics and so on; more details see [6]. In the words of Hochster, 'Life is really worth living in a Cohen-Macaulay ring' (see [8, p. 887]).

Let $(R, \mathfrak{m})$ be a local ring. A finitely generated R-module M is said to be *Cohen-Macaulay* if $\mathrm{depth}_R M = \dim_R M$. These notions have been extended to non-local rings. Let $\mathfrak{a}$ be a proper ideal of an arbitrary noetherian ring R. A finitely generated R-module M with $M \neq \mathfrak{a}M$ is said to be $\mathfrak{a}$-*relative Cohen-Macaulay*, $\mathfrak{a}$-RCM, if $\mathrm{depth}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, M)$. This notion as a ganaralization of classical Cohen-Macaulay modules was introduced by Zargar and Zakeri in [10] and its study was continued in [2, 7, 9, 11].

It is well-known that, for a Cohen-Macaulay R-module M over a local ring $(R, \mathfrak{m})$, $\mathrm{depth}_R M = \dim R/\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R M$ and the depth with respect to an arbitrary ideal $\mathfrak{a} \subseteq \mathfrak{m}$ is given by its codimension, that is, $\mathrm{depth}(\mathfrak{a}, M) = \dim_R M - \dim_R M/\mathfrak{a}M$. The first aim of this paper is to consider the the following question:

Received September 21, 2022; Revised January 29, 2023; Accepted February 14, 2023.
2020 *Mathematics Subject Classification.* Primary 13C15, 13H10.
Key words and phrases. Relative Cohen-Macaulay module, $\mathfrak{a}$-depth, cohomological dimension.

The authors thank the referee for important comments and suggestions on improving this paper. This research was partially supported by National Natural Science Foundation of China (11901463), graduate Research Fund project of Northwest Normal University (2021KYZZ01031) and Gansu Province outstanding graduate student "Innovation Star" project (2022CXZX-238).

Question 1. *Can we use* $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p})$ *of an* $\mathfrak{a}$*-RCM module* M *to calculate* $\operatorname{depth}(\mathfrak{a}, M)$*?*

In Section 1, we show that, for an $\mathfrak{a}$-RCM module M, $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}_R M$. As an applications of this equality, we show that, for two ideals $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{b} \subseteq \mathfrak{a}$, if $\operatorname{cd}(\mathfrak{b}, M) = \operatorname{ara}(\mathfrak{b}, M)$, then $\operatorname{depth}(\mathfrak{b}, M) = \operatorname{cd}(\mathfrak{a}, M) - \operatorname{cd}(\mathfrak{a}, M/\mathfrak{b}M)$.

Bruns and Herzog [6, Theorem 2.1.7] showed that the Cohen-Macaulay property is stable under flat local extensions: Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of local rings. Suppose that M is a finitely generated R-module and N is an R-flat finitely generated S-module. Then $M \otimes_R N$ is a Cohen-Macaulay S-module if and only if M is a Cohen-Macaulay R-module and $N/\mathfrak{m}N$ is a Cohen-Macaulay S-module. The second aim of this paper is to consider the following question:

Question 2. *Is there an analogous theorem for the* $\mathfrak{a}$*-relative Cohen-Macaulayness?*

In Section 2, we give a positive answer for Question 2 under some conditions and study the perseverance of relative Cohen-Macaulayness under flat extensions (not necessarily local). It is discovered that relative Cohen-Macaulay modules with respect to the Jacosbson radical enjoy many interesting properties which are analogous to those of Cohen-Macaulay modules over local rings.

Unless stated to the contrary we assume throughout this paper that R is a commutative Noetherian ring which is not necessarily local. Next, we recall some notions and preliminaries which we will need later.

Regular sequence. Let M be a finitely generated R-module. An element x of R is a *nonzerodivisor* on M if $xm = 0$ implies $m = 0$; if in addition $xM \neq M$, then x is said to be M*-regular*. A sequence $\boldsymbol{x} = x_1, \ldots, x_d$ of elements in R is an M*-regular sequence* if x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for $1 \leq i \leq d$ and $\boldsymbol{x}M \neq M$.

Associated prime and support. We write $\operatorname{Spec}R$ for the set of prime ideals of R. For an ideal $\mathfrak{a}$ of R, set

$$\operatorname{V}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}R \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is said to be an *associated prime* of M if it is the annihilator of an element in M. This is equivalent to M containing the cyclic submodule $R/\mathfrak{p}$. The set of all associated prime ideals of M is denoted by $\operatorname{Ass}_R M$. Fix $\mathfrak{p} \in \operatorname{Spec}R$, let $M_\mathfrak{p}$ denote the localization of M at $\mathfrak{p}$. The *support* of M is the set

$$\operatorname{Supp}_R M := \{\mathfrak{p} \in \operatorname{Spec}R \mid M_\mathfrak{p} \neq 0\}.$$

It is well known that $\operatorname{Ass}_R M \subseteq \operatorname{Supp}_R M$.

Dimension. Let $\mathfrak{a}$ be a proper ideal of R and M a finitely generated R-module. The *dimension* of M, denoted by $\dim_R M$, is

$$\dim_R M := \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R M\}.$$

The *height* of M with respect to $\mathfrak{a}$, denoted by $\mathrm{ht}_M(\mathfrak{a})$, is

$$\mathrm{ht}_M(\mathfrak{a}) := \inf\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{V}(\mathfrak{a})\}.$$

The *n-th local cohomology module* of M is defined as

$$\mathrm{H}^n_\mathfrak{a}(M) := \varinjlim_t \mathrm{Ext}^n_R(R/\mathfrak{a}^t, M).$$

The reader can refer to [5] for more details about local cohomology.

The *cohomological dimension* of M with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}(\mathfrak{a}, M)$, is

$$\mathrm{cd}(\mathfrak{a}, M) := \sup\{n \in \mathbb{Z} \mid \mathrm{H}^n_\mathfrak{a}(M) \neq 0\}.$$

The cohomological dimension of the zero module is $-\infty$. One easily sees that $\mathrm{cd}(\mathfrak{a}, M) = -\infty$ if and only if $M = \mathfrak{a}M$.

The *finiteness dimension* of M with respect to $\mathfrak{a}$, denoted by $\mathrm{f}_\mathfrak{a}(M)$, is

$$\begin{aligned}\mathrm{f}_\mathfrak{a}(M) : &= \inf\{n \in \mathbb{Z} \mid \mathrm{H}^n_\mathfrak{a}(M) \text{ is not finitely generated}\} \\ &= \inf\{n \in \mathbb{Z} \mid \mathfrak{a} \nsubseteq \sqrt{(0 : \mathrm{H}^n_\mathfrak{a}(M))}\}.\end{aligned}$$

Note that $\mathrm{f}_\mathfrak{a}(M)$ is either a positive integer or ∞ since $\mathrm{H}^0_\mathfrak{a}(M)$ is finitely generated.

Depth. Let $\mathfrak{a}$ be a proper ideal of R and M a finitely generated R-module. The *depth* of M with respect to $\mathfrak{a}$, denoted by $\mathrm{depth}(\mathfrak{a}, M)$, is

$$\begin{aligned}\mathrm{depth}(\mathfrak{a}, M) : &= \inf\{n \in \mathbb{Z} \mid \mathrm{Ext}^n_R(R/\mathfrak{a}, M) \neq 0\} \\ &= \inf\{n \in \mathbb{Z} \mid \mathrm{H}^n_\mathfrak{a}(M) \neq 0\}.\end{aligned}$$

In particular, if $(R, \mathfrak{m})$ is local, the $\mathrm{depth}(\mathfrak{m}, M)$ is denoted by $\mathrm{depth}_R M$.

The *minimum adjusted depth* of M with respect to $\mathfrak{a}$, denoted by $\lambda_\mathfrak{a}(M)$, is

$$\lambda_\mathfrak{a}(M) := \inf\{\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \mathrm{ht}(\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec} R \setminus \mathrm{V}(\mathfrak{a})\}.$$

It follows from [5, Theorem 9.3.5] that $\mathrm{f}_\mathfrak{a}(M) \leq \lambda_\mathfrak{a}(M)$.

2. Characterizations of the $\mathfrak{a}$-depth of $\mathfrak{a}$-RCM modules

In this section, we provide some descriptions of the $\mathfrak{a}$-depth of $\mathfrak{a}$-relative Cohen-Macaulay modules by cohomological dimensions.

Definition ([2])**.** Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module with $M \neq \mathfrak{a}M$. The module M is said to be $\mathfrak{a}$-*relative Cohen-Macaulay*, $\mathfrak{a}$-RCM, if

$$\mathrm{depth}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, M).$$

Let $c = \mathrm{cd}(\mathfrak{a}, M)$. We call a sequence $x_1, \ldots, x_c \in \mathfrak{a}$ an $\mathfrak{a}$-*relative system of parameters*, $\mathfrak{a}$-Rs.o.p, of M if

$$\sqrt{\langle x_1, \ldots, x_c\rangle + \mathrm{Ann}_R M} = \sqrt{\mathfrak{a} + \mathrm{Ann}_R M}.$$

The *arithmetic rank* of an ideal $\mathfrak{a}$ of R with respect to a module M, denoted by $\mathrm{ara}(\mathfrak{a}, M)$, is defined as the infimum of the integers n such that there exist $x_1, \ldots, x_n \in R$ satisfying

$$\sqrt{\langle x_1, \ldots, x_n\rangle + \mathrm{Ann}_R M} = \sqrt{\mathfrak{a} + \mathrm{Ann}_R M}.$$

Remark 2.1. Let $\mathfrak{a}$ be a proper ideal of R and M a non-zero finitely generated R-module.

(1) If M is an $\mathfrak{a}$-RCM R-module, then $M \neq \mathfrak{a}M$ implies that $\mathrm{cd}(\mathfrak{a}, M) \neq -\infty$. Thus, $\mathrm{depth}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, M) \geq 0$.

(2) If $(R, \mathfrak{m})$ is a local ring, then the class of $\mathfrak{m}$-RCM coincide with the class of Cohen-Macaulay modules. In fact, one has M is a Cohen-Macaulay module if and only if $\mathrm{depth}_R M = \dim_R M$ if and only if $\mathrm{depth}(\mathfrak{m}, M) = \mathrm{cd}(\mathfrak{m}, M)$ if and only if M is $\mathfrak{m}$-RCM.

(3) Suppose that $\mathfrak{a}$ is contained in the Jacosbson radical $J(R)$ of R and $\boldsymbol{x} = x_1, \ldots, x_n \in \mathfrak{a}$ an $\mathfrak{a}$-Rs.o.p of M with $\mathfrak{a} = \langle \boldsymbol{x}\rangle$. If $\mathrm{cd}(\mathfrak{a}, M) = \mathrm{ara}(\mathfrak{a}, M) = n > 0$, then M is $\mathfrak{a}$-RCM if and only if $M/\langle x_1, \ldots, x_i\rangle M$ is $\mathfrak{a}$-RCM for $1 \leq i \leq n$ by [2, Lemma 2.4], [2, Theorem 3.3] and [6, Proposition 1.2.10]. In particalar, if $(R, \mathfrak{m})$ is a local ring and $\mathfrak{a} = \mathfrak{m}$, then M is a Cohen-Macaulay R-module if and only if $M/\langle x_1, \ldots, x_i\rangle M$ is a Cohen-Macaulay R-module for any $1 \leq i \leq n$.

(4) If $\mathrm{cd}(\mathfrak{a}, M) = 0$, then $0 \leq \mathrm{depth}(\mathfrak{a}, M) \leq \mathrm{cd}(\mathfrak{a}, M) = 0$. So M is $\mathfrak{a}$-RCM.

Lemma 2.2 ([4, Theorem 2.2])**.** *Let $\mathfrak{a}$ be an ideal of R, M and N two finitely generated R-modules with* $\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R M$. *Then*

$$\mathrm{cd}(\mathfrak{a}, N) \leq \mathrm{cd}(\mathfrak{a}, M).$$

In particular, if $\mathrm{Supp}_R N = \mathrm{Supp}_R M$*, then* $\mathrm{cd}(\mathfrak{a}, N) = \mathrm{cd}(\mathfrak{a}, M)$*.*

Corollary 2.3. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. For any* $\mathfrak{p} \in \mathrm{Supp}_R M$*, one has*

$$\mathrm{cd}(\mathfrak{a}, M/\mathfrak{p}M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{cd}(\mathfrak{a}, M).$$

Proof. Since $\mathrm{Supp}_R M/\mathfrak{p}M = \mathrm{V}(\mathfrak{p}) \cap \mathrm{Supp}_R M = \mathrm{V}(\mathfrak{p}) \subseteq \mathrm{Supp}_R M$, it follows from Lemma 2.2 that $\mathrm{cd}(\mathfrak{a}, M/\mathfrak{p}M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{cd}(\mathfrak{a}, M)$, as desired. ☐

We have the following useful remark.

Remark 2.4 ([3, Remark 3.1]). If $M \neq \mathfrak{a}M$ and $\mathrm{ht}_M(\mathfrak{a}) > 0$, then

$$\mathrm{depth}(\mathfrak{a}, M) \leq \mathrm{f}_{\mathfrak{a}}(M) \leq \mathrm{ht}_M(\mathfrak{a}) \leq \mathrm{cd}(\mathfrak{a}, M) \leq \mathrm{ara}(\mathfrak{a}, M) \leq \dim_R M.$$

We now present the first main theorem of this section, which is a more general version of [6, Theorem 2.1.2(a)].

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-RCM R-module with* $\mathrm{cd}(\mathfrak{a}, M) = c$.

(1) *If $c = 0$, then*

$$\mathrm{depth}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \mathrm{ht}_M(\mathfrak{a}) = 0 \textit{ for all } \mathfrak{p} \in \mathrm{Ass}_R M \cap \mathrm{V}(\mathfrak{a}).$$

(2) *If $c > 0$, then*

$$\mathrm{depth}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \mathrm{ht}_M(\mathfrak{a}) = \mathrm{f}_{\mathfrak{a}}(M) = \lambda_{\mathfrak{a}}(M) = c \textit{ for all } \mathfrak{p} \in \mathrm{Ass}_R M.$$

Proof. (1) If $c = 0$, then $\mathrm{Hom}_R(R/\mathfrak{a}, M) \neq 0$, it follows that $\mathrm{Ass}_R M \cap \mathrm{V}(\mathfrak{a}) \neq \emptyset$. Let $\mathfrak{p} \in \mathrm{Ass}_R M \cap \mathrm{V}(\mathfrak{a})$. Then $0 \neq M/\mathfrak{p}M$ is $\mathfrak{a}$-torsion, and hence $\mathrm{H}^i_{\mathfrak{a}}(M/\mathfrak{p}M) = 0$ for $i > 0$. Thus $0 = \mathrm{cd}(\mathfrak{a}, M/\mathfrak{p}M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{cd}(\mathfrak{a}, M) = 0$ by Corollary 2.3. Also $0 \leq \mathrm{ht}_M(\mathfrak{a}) \leq \mathrm{cd}(\mathfrak{a}, M) = 0$, as required.

(2) If $c > 0$, then $\mathrm{Hom}_R(R/\mathfrak{a}, M) = 0$, and so $\mathrm{Ass}_R M \cap \mathrm{V}(\mathfrak{a}) = \emptyset$. Let $\mathfrak{p} \in \mathrm{Ass}_R M$. Then $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$. One has the following (in)equalities

$$\begin{aligned}
\mathrm{cd}(\mathfrak{a}, M) &= \mathrm{ht}_M(\mathfrak{a}) \\
&= \mathrm{f}_{\mathfrak{a}}(M) \\
&\leq \lambda_{\mathfrak{a}}(M) \\
&\leq \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \mathrm{ht}(\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}}) \\
&\leq \mathrm{cd}(\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}}, R/\mathfrak{p}) \\
&= \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \\
&\leq \mathrm{cd}(\mathfrak{a}, M),
\end{aligned}$$

where the first, the second and the fifth ones are by Remark 2.4, the third one is by [5, Theorem 9.3.5], the forth one is by the definition, the sixth one is by the isomorphism $\mathrm{H}^i_{\mathfrak{a}+\mathfrak{p}/\mathfrak{p}}(R/\mathfrak{p}) \cong \mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ for any $i \geq 0$, and the seventh one is by Corollary 2.3. □

According to Theorem 2.5, one can obtain the following classical result about Cohen-Macaulay modules (see [6, Theorem 2.1.2(a)]).

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a local ring and M a Cohen-Macaulay R-module. Then*

$$\mathrm{depth}_R M = \mathrm{dim} R/\mathfrak{p} \quad \textit{for all} \quad \mathfrak{p} \in \mathrm{Ass}_R M.$$

The following is the second main theorem of this section, which is a generalization of [6, Theorem 2.1.2(b)].

Theorem 2.7. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R with $\mathfrak{b} \subseteq \mathfrak{a} \subseteq J(R)$ and M an $\mathfrak{a}$-RCM R-module. If $\mathrm{cd}(\mathfrak{b}, M) = \mathrm{ara}(\mathfrak{b}, M)$, then M is $\mathfrak{b}$-RCM and*

$$\mathrm{depth}(\mathfrak{b}, M) = \mathrm{cd}(\mathfrak{a}, M) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M).$$

Proof. First, we show that the equality holds. If $\mathrm{cd}(\mathfrak{a}, M) = 0$, then $0 \leq \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M) \leq \mathrm{cd}(\mathfrak{a}, M) = 0$ by Corollary 2.3, the equality holds. Next suppose that $\mathrm{cd}(\mathfrak{a}, M) > 0$ and do induction on $\mathrm{depth}(\mathfrak{b}, M)$. If $\mathrm{depth}(\mathfrak{b}, M) = 0$, then $\mathrm{Hom}_R(R/\mathfrak{b}, M) \neq 0$, and so

$$\emptyset \neq \mathrm{Ass}_R \mathrm{Hom}_R(R/\mathfrak{b}, M) = \mathrm{Ass}_R(M) \cap \mathrm{V}(\mathfrak{b}) \subseteq \mathrm{Supp}_R M \cap \mathrm{V}(\mathfrak{b}).$$

Set $\mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{V}(\mathfrak{b})$. There exists $\mathfrak{q} \in \mathrm{Ass}_R(M) \cap \mathrm{V}(\mathfrak{b})$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, it follows from Theorem 2.5 that

$$\mathrm{cd}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{q}) = \mathrm{cd}(\mathfrak{a}, M/\mathfrak{q}M).$$

Since $\mathrm{Supp}_R M/\mathfrak{q}M \subseteq \mathrm{Supp}_R M/\mathfrak{b}M$, $\mathrm{cd}(\mathfrak{a}, M/\mathfrak{q}M) \leq \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M)$ by Lemma 2.2. So

$$\mathrm{cd}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, R/\mathfrak{q}) \leq \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M) \leq \mathrm{cd}(\mathfrak{a}, M).$$

Thus the equality holds. If $\mathrm{depth}(\mathfrak{b}, M) > 0$, then we can choose $x \in \mathfrak{b}$ that is M-regular, which implies that M/xM is $\mathfrak{a}$-RCM as $x \in \mathfrak{a}$. It follows from [2, Theorem 2.7] that $\mathrm{cd}(\mathfrak{b}, M/xM) = \mathrm{cd}(\mathfrak{b}, M) - 1$. Note that x is part of a $\mathfrak{b}$-Rs.o.p for M, it follows from the definition that $\mathrm{cd}(\mathfrak{b}, M/xM) = \mathrm{ara}(\mathfrak{b}, M/xM)$. Therefore, by induction,

$$\begin{aligned} \mathrm{depth}(\mathfrak{b}, M) &= \mathrm{depth}(\mathfrak{b}, M/xM) + 1 \\ &= \mathrm{cd}(\mathfrak{a}, M/xM) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M) + 1 \\ &= \mathrm{cd}(\mathfrak{a}, M) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M). \end{aligned}$$

Next, we prove that M is $\mathfrak{b}$-RCM, it suffices to prove that $\mathrm{cd}(\mathfrak{b}, M) = \mathrm{cd}(\mathfrak{a}, M) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M)$. If $\mathrm{cd}(\mathfrak{b}, M) = 0$, then we are done. If $\mathrm{cd}(\mathfrak{b}, M) > 0$, then there is $x \in \mathfrak{b}$ that is a part of $\mathfrak{b}$-Rs.o.p for M. So we can find elements $y_1, \ldots, y_s \in \mathfrak{b}$ such that

$$\sqrt{\langle x, y_1, \ldots, y_s \rangle + \mathrm{Ann}_R M} = \sqrt{\mathfrak{b} + \mathrm{Ann}_R M},$$

that is to say, $x, y_1, \ldots, y_s$ is $\mathfrak{b}$-Rs.o.p for M. Hence [2, Theorem 3.3] implies that $x, y_1, \ldots, y_s$ is M-regular, and so M/xM is $\mathfrak{a}$-RCM as $x \in \mathfrak{a}$ is M-regular. Note that $\mathrm{ara}(\mathfrak{b}, M/xM) = \mathrm{cd}(\mathfrak{b}, M/xM)$, by induction, one has

$$\begin{aligned} \mathrm{cd}(\mathfrak{b}, M) &= \mathrm{cd}(\mathfrak{b}, M/xM) + 1 \\ &= \mathrm{cd}(\mathfrak{a}, M/xM) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M) + 1 \\ &= \mathrm{cd}(\mathfrak{a}, M) - \mathrm{cd}(\mathfrak{a}, M/\mathfrak{b}M), \end{aligned}$$

so the proof is complete. □

The following proposition shows that $\mathfrak{a}$-relative Cohen-Macaulayness is stable under localization, which is a relative version of [6, Corollary 2.1.3(b)].

Proposition 2.8. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-RCM R-module. Then, for every multiplicatively closed set S of R with $S \cap \mathfrak{a} = \emptyset$, the localized module $S^{-1}M$ is an $S^{-1}\mathfrak{a}$-RCM $S^{-1}R$-module. In particular, $M_{\mathfrak{p}}$ is an $\mathfrak{a}R_{\mathfrak{p}}$-RCM $R_{\mathfrak{p}}$-module for $\mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{V}(\mathfrak{a})$.*

Proof. By [5, Corollary 4.3.3], for every $n \in \mathbb{Z}$, one has

$$S^{-1}(\mathrm{H}^n_{\mathfrak{a}}(M)) \cong \mathrm{H}^n_{S^{-1}\mathfrak{a}}(S^{-1}M).$$

which implies that

$$\begin{aligned}\mathrm{cd}(\mathfrak{a}, M) = \mathrm{depth}(\mathfrak{a}, M) &\le \mathrm{depth}(S^{-1}\mathfrak{a}, \mathrm{S}^{-1}M)\\ &\le \mathrm{cd}(S^{-1}\mathfrak{a}, S^{-1}M)\\ &\le \mathrm{cd}(\mathfrak{a}, M).\end{aligned}$$

Thus $\mathrm{depth}(S^{-1}\mathfrak{a}, S^{-1}M) = \mathrm{cd}(S^{-1}\mathfrak{a}, S^{-1}M)$, as required. □

The following proposition gives a behaviours of $\mathfrak{a}$-RCM modules under localization for a special subset of support.

Proposition 2.9. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-RCM R-module. Then for every $\mathfrak{p} \in \mathrm{Supp}_R M$ with $\mathfrak{p} \subseteq \mathfrak{a}$, $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module.*

Proof. If $\{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathfrak{p} \subseteq \mathfrak{a}\} = \emptyset$, then there is nothing to prove. For every $\mathfrak{p} \in \mathrm{Supp}_R M$ with $\mathfrak{p} \subseteq \mathfrak{a}$, we do induction on $\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = t$. If $t = 0$, then $\mathfrak{p} \in \mathrm{Ass}_R M$, and hence $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$ by Theorem 2.5. Next, assumes that $t > 0$ and the result has been proved for smaller values of t. Then $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q} \in \mathrm{Ass}_R M} \mathfrak{q}$, and there is $x \in \mathfrak{p}$ that is an M-regular element. Note that $x \in \mathfrak{a}$, so M/xM is $\mathfrak{a}$-RCM. Also $\mathfrak{p} \in \mathrm{Supp}_R M/xM$ and $\frac{x}{1} \in \mathfrak{p}R_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$-regular. By induction, $M_{\mathfrak{p}}/(\frac{x}{1})M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module. Therefore,

$$\begin{aligned}\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &= \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/(\frac{x}{1})M_{\mathfrak{p}} + 1\\ &= \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/(\frac{x}{1})M_{\mathfrak{p}} + 1\\ &= \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}},\end{aligned}$$

the proof is complete. □

3. The behaviour of relative Cohen-Macaulayness under flat extensions

In this section, we study the relative Cohen-Macaulayness under flat extensions. More precisely, we give the next main theorem, which is a generalization of [6, Theorem 2.1.7].

Theorem 3.1. *Let $f : R \to S$ be a homomorphism of rings, $J(R)$ and $J(S)$ the Jacosbson radicals of R and S, respectively. Suppose that M is a finitely generated R-module, N a finitely generated S-module and N faithfully flat over R with $N \neq J(R)N$. If $R/J(R)$ is semisimple and* $\mathrm{cd}(J(R), R) = \mathrm{ara}(J(R), R)$, *then $M \otimes_R N$ is a $J(S)$-RCM S-module if and only if M is a $J(R)$-RCM R-module and $N/J(R)N$ is a $J(S)$-RCM S-module.*

The proof of this theorem is divided into the following lemmas.

Lemma 3.2. *Let $f : R \to S$ be a homomorphism of rings, M a finitely generated R-module, N a finitely generated S-module and N is faithfully flat over R with $N \neq J(R)N$. If $R/J(R)$ is semisimple and $\boldsymbol{y} \subseteq J(S)$ an $N/J(R)N$-regular sequence, then $\boldsymbol{y}$ is an $(M \otimes_R N)$-regular sequence and $N/\boldsymbol{y}N$ is faithfully flat over R.*

Proof. First, one show that $\boldsymbol{y} \in S$ is an $(M \otimes_R N)$-regular sequence. We use induction on the length n of $\boldsymbol{y}$, and only the case $n = 1$, $\boldsymbol{y} = y$ needs justification. Set $J = J(R)$. By Krull's intersection theorem, one has $\bigcap_{i=0}^{\infty} J^i(M \otimes_R N) = 0$. Suppose that $yz = 0$ for some $z \in M \otimes_R N$. If $z \neq 0$, then there exists i such that $z \in J^i(M \otimes_R N)\backslash J^{i+1}(M \otimes_R N)$ and y would be a zerodivisor on $J^i(M \otimes_R N)/J^{i+1}(M \otimes_R N)$. For any $t \geq 1$, consider the embedding $J^t M \to M$, which induces a monomorphism $J^t M \otimes_R N \to M \otimes_R N$ as N is flat, and its image is $J^t(M \otimes_R N)$, the flatness of N yields an isomorphism

$$J^i(M \otimes_R N)/J^{i+1}(M \otimes_R N) \cong (J^i M/J^{i+1}M) \otimes_R N.$$

Since $J^i M/J^{i+1}M$ is a finitely generated R/J-module and R/J is semisimple, it follows that $(J^i M/J^{i+1}M) \otimes_R N$ is a direct summand of $(R/J)^n \otimes_R N \cong (N/JN)^n$ for some $n \geq 1$. Since y is regular on N/JN, it must be regular on $J^i(M \otimes_R N)/J^{i+1}(M \otimes_R N)$.

Next, we prove that N/yN is faithfully flat. Consider the exact sequence of finitely generated R-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0,$$

which induces the following exact sequence

$$0 \longrightarrow M_1 \otimes_R N \longrightarrow M_2 \otimes_R N \longrightarrow M_3 \otimes_R N \longrightarrow 0.$$

As y is $M_3 \otimes_R N$-regular and $(M_3 \otimes_R N)/y(M_3 \otimes_R N) \cong M_3 \otimes_R N/yN$, it follows from [4, Lemma 1.1.4] that the sequence

$$0 \longrightarrow M_1 \otimes_R N/yN \longrightarrow M_2 \otimes_R N/yN \longrightarrow M_3 \otimes_R N/yN \longrightarrow 0$$

is exact, which implies that N/yN is flat. For every $\mathfrak{m} \in \mathrm{Max}R$, since N is faithfully flat, one has $N \otimes_R R/\mathfrak{m} \neq 0$. Also $y \in J(S)$, the Nakayama's lemma implies that $(N \otimes_R R/\mathfrak{m})/y(N \otimes_R R/\mathfrak{m}) \neq 0$. Thus N/yN is a faithfully flat R-module. □

The following lemma is a more general version of [6, Theorem 1.2.16].

Lemma 3.3. *Let $f : R \to S$ be a homomorphism of rings, M a finitely generated R-module, N a finitely generated S-module and N faithfully flat over R with $N \neq J(R)N$. If $R/J(R)$ is semisimple, then*

$$\mathrm{depth}(J(S), M \otimes_R N) = \mathrm{depth}(J(R), M) + \mathrm{depth}(J(S), N/J(R)N).$$

Proof. Suppose that $\mathrm{depth}(J(R), M) = m$ and $\mathrm{depth}(J(S), N/J(R)N) = n$. We only need to prove that $\mathrm{depth}(J(S), M \otimes_R N) = m+n$. Let $\boldsymbol{x} = x_1, \ldots, x_m \in J(R)$ be a maximal M-regular sequence and $\boldsymbol{y} = y_1, \ldots, y_n \in J(S)$ a maximal

$N/J(R)N$-regular sequence. It follows from [6, Proposition 1.1.2] that $f(\boldsymbol{x}) = f(x_1), \dots, f(x_m) \in J(R)S$ is an $(M \otimes_R N)$-regular sequence. Since $J(R)S \subseteq J(S)$ by [1, Proposition 9.14], it follows from Lemma 3.2 that $\boldsymbol{y}$ is $(\bar{M} \otimes_R N)$-regular, where $\bar{M} = M/\boldsymbol{x}M$. Since $\bar{M} \otimes_R N \cong (M \otimes_R N)/f(\boldsymbol{x})(M \otimes_R N)$, one has $f(\boldsymbol{x}), \boldsymbol{y} \in J(S)$ is an $(M \otimes_R N)$-regular sequence. Hence $\operatorname{depth}(J(S), M \otimes_R N) \geq m + n$. Set $\bar{N} = N/\boldsymbol{y}N$. Then

$$\bar{N}/J(R)\bar{N} \cong (N/J(R)N)/\boldsymbol{y}(N/J(R)N),$$
$$\bar{M} \otimes_R \bar{N} \cong (M \otimes_R N)/(f(\boldsymbol{x}), \boldsymbol{y})(M \otimes_R N),$$

which implies that

$$\begin{aligned}\operatorname{Ext}_S^{m+n}(S/J(S), M \otimes_R N) &\cong \operatorname{Hom}_S(S/J(S), \bar{M} \otimes_R \bar{N}) \\ &\cong \operatorname{Hom}_S(S/J(S), \operatorname{Hom}_S(S/J(R)S, \bar{M} \otimes_R \bar{N})) \\ &\cong \operatorname{Hom}_S(S/J(S), \operatorname{Hom}_R(R/J(R), \bar{M}) \otimes_R \bar{N}),\end{aligned}$$

where the first isomorphism is by [6, Lemma 1.2.4], the second one is by adjointness and the third one is by the flatness of $\bar{N}$. Since $\operatorname{Hom}_R(R/J(R), \bar{M}) \neq 0$ by [6, Proposition 1.2.3] and $\bar{N}$ is faithfully flat by Lemma 3.2, it follows that $\operatorname{Hom}_R(R/J(R), \bar{M}) \otimes_R \bar{N} \neq 0$. Note that $\operatorname{Hom}_R(R/J(R), \bar{M})$ is a finitely generated $R/J(R)$-module and $R/J(R)$ is semisimple, so $\operatorname{Hom}_R(R/J(R), \bar{M}) \otimes_R \bar{N}$ is a direct summand of $(R/J(R))^s \otimes_R \bar{N} \cong (\bar{N}/J(R)\bar{N})^s$ for some $s \geq 1$. By [6, Proposition 1.2.3], one has $\operatorname{Hom}_S(S/J(S), \bar{N}/J(R)\bar{N}) \neq 0$, which implies that $\operatorname{Hom}_S(S/J(S), \operatorname{Hom}_R(R/J(R), \bar{M}) \otimes_R \bar{N}) \neq 0$. Therefore,

$$\operatorname{depth}(J(S), M \otimes_R N) \leq m + n,$$

as desired. □

Lemma 3.4. *Let $f : R \to S$ be a ring homomorphism. Suppose that*

$$\operatorname{cd}(J(R), R) = \operatorname{ara}(J(R), R).$$

Then

$$\operatorname{cd}(J(S), S) = \operatorname{cd}(J(R), R) + \operatorname{cd}(J(S), S/J(R)S).$$

Proof. Set $\operatorname{cd}(J(R), R) = \operatorname{ara}(J(R), R) = n$. By [2, Lemma 2.2], there exist $x_1, \dots, x_n \in J(R)$ which is a $J(R)$-Rs.o.p of R. So $\sqrt{\langle x_1, \dots, x_n \rangle} = \sqrt{J(R)}$ and then $\sqrt{\langle x_1, \dots, x_n \rangle S} = \sqrt{J(R)S}$. Hence [2, Lemma 2.4] implies that

$$\begin{aligned}\operatorname{cd}(J(S), S/J(R)S) &= \operatorname{cd}(J(S), S/\langle x_1, \dots, x_n \rangle S) \\ &= \operatorname{cd}(J(S), S) - n.\end{aligned}$$

We obtain the equality we seek. □

The following lemma is a nice generalization of [6, A.11].

Lemma 3.5. *Let $f : R \to S$ be a ring homomorphism with $\operatorname{cd}(J(R), R) = \operatorname{ara}(J(R), R)$. Suppose that M is a finitely generated R-module, N a finitely generated S-module and N faithfully flat over R with $N \neq J(R)N$. Then*

$$\operatorname{cd}(J(S), M \otimes_R N) = \operatorname{cd}(J(R), M) + \operatorname{cd}(J(S), N/J(R)N).$$

Proof. Set $I = \mathrm{Ann}_R M$ and $\bar{R} = R/I$. Then $M \otimes_R N \cong M \otimes_{\bar{R}} N/IN$ replace R by $\bar{R}$, S by S/IS and N by N/IN, we may assume that $\mathrm{Supp}_R M = \mathrm{Spec} R$. Hence $\mathrm{cd}(J(R), R) = \mathrm{cd}(J(R), M)$. Next, replacing S by $S/\mathrm{Ann}_S N$, we may assume that $\mathrm{Supp}_S N = \mathrm{Spec} S$. Since $N/J(R)N \cong S/J(R)S \otimes_S N$, we have $\mathrm{Supp}_S S/J(R)S = \mathrm{V}(J(R)S) = \mathrm{Supp}_S N/J(R)N$. Thus $\mathrm{cd}(J(S), S/J(R)S) = \mathrm{cd}(J(S), N/J(R)N)$. Take $\mathfrak{q} \in \mathrm{Spec} S$ and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then

$$\mathfrak{q} \in \mathrm{Supp}_S(R/\mathfrak{p} \otimes_R N) \text{ and so } (R/\mathfrak{p} \otimes_R N)_\mathfrak{q} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} \neq 0,$$

which implies that $N_\mathfrak{q}$ is a faithfully flat $R_\mathfrak{p}$-module. Hence $(M \otimes_R N)_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} \neq 0$ and $\mathrm{Supp}_S(M \otimes_R N) = \mathrm{Spec} S$. So by Lemma 2.2, one has $\mathrm{cd}(J(S), S) = \mathrm{cd}(J(S), M \otimes_R N)$. Hence Lemma 3.4 yields the desired equality. □

Proof of Theorem 3.1. Since N is faithfully flat over R with $N \neq J(R)N$ and $R/J(R)$ is semisimple, it follows from Lemma 3.3 and Lemma 3.5 that

$$\mathrm{depth}(J(S), M \otimes_R N) = \mathrm{depth}(J(R), M) + \mathrm{depth}(J(S), N/J(R)N),$$
$$\mathrm{cd}(J(S), M \otimes_R N) = \mathrm{cd}(J(R), M) + \mathrm{cd}(J(S), N/J(R)N).$$

Thus $M \otimes_R N$ is a $J(S)$-RCM S-module if and only if M is a $J(R)$-RCM R-module and $N/J(R)N$ is a $J(S)$-RCM S-module. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.6. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a faithfully flat ring homomorphism. Then S is Cohen-Macaulay over S if and only if R is Cohen-Macaulay over R and $R/\mathfrak{m}R$ is Cohen-Macaulay over S.*

The next corollary shows that the relative Cohen-Macaulayness is stable under $J(R)$-adic completion of R.

Corollary 3.7. *Let $J(R) = J$ be the Jacosbson radical of R, M a finitely generated R-module and $\widehat{M}^J$ its J-adic completion. If R/J is semisimple and $\mathrm{cd}(J, R) = \mathrm{ara}(J, R)$, then*

(1) $\mathrm{depth}(J, M) = \mathrm{depth}_{\widehat{R}}(\widehat{J}, \widehat{M}^J)$.

(2) *M is J-RCM if and only if $\widehat{M}^J$ is $\widehat{J}$-RCM.*

Proof. This follows from $\mathrm{depth}(\widehat{J}, \widehat{R}^J/J\widehat{R}^J) = 0$ and the ring homomorphism $R \to \widehat{R}^J$ is faithfully flat. □

The relative Cohen-Macaulayness is stable under polynomial rings and formal power series.

Corollary 3.8. *Let $J(R) = J$ be the Jacosbson radical of R. If R/J is semisimple and $\mathrm{cd}(J, R) = \mathrm{ara}(J, R)$, then*

(1) *R is J-RCM if and only if $R[x_1, \dots, x_n]$ is $J[x_1, \dots, x_n]$-RCM.*

(2) *R is J-RCM if and only if $R[[x_1, \dots, x_n]]$ is $J[[x_1, \dots, x_n]]$-RCM.*

Proof. This follows from

$$\operatorname{depth}_{R[x_1,\dots,x_n]}(J[x_1,\dots,x_n], R[x_1,\dots,x_n]/J[x_1,\dots,x_n]) = 0,$$
$$\operatorname{depth}_{R[[x_1,\dots,x_n]]}(J[[x_1,\dots,x_n]], R[[x_1,\dots,x_n]]/J[[x_1,\dots,x_n]]) = 0$$

and the ring homomorphisms $R \to R[x_1,\dots,x_n]$ and $R \to R[[x_1,\dots,x_n]]$ are faithfully flat. □

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, second edition, Graduate Texts in Mathematics, 13, Springer, New York, 1992. https://doi.org/10.1007/978-1-4612-4418-9

[2] K. Divaani-Aazar, A. Ghanbari Doust, M. Tousi, and H. Zakeri, *Cohomological dimension and relative Cohen-Maculayness*, Comm. Algebra **47** (2019), no. 12, 5417–5427. https://doi.org/10.1080/00927872.2019.1623242

[3] K. Divaani-Aazar, A. Ghanbari Doust, M. Tousi, and H. Zakeri, *Modules whose finiteness dimensions coincide with their cohomological dimensions*, J. Pure Appl. Algebra **226** (2022), no. 4, Paper No. 106900, 19 pp. https://doi.org/10.1016/j.jpaa.2021.106900

[4] K. Divaani-Aazar, R. Naghipour, and M. Tousi, *Cohomological dimension of certain algebraic varieties*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3537–3544. https://doi.org/10.1090/S0002-9939-02-06500-0

[5] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, 60, Cambridge Univ. Press, Cambridge, 1998. https://doi.org/10.1017/CBO9780511629204

[6] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge Univ. Press, Cambridge, 1993.

[7] O. Celikbas and H. Holm, *Equivalences from tilting theory and commutative algebra from the adjoint functor point of view*, New York J. Math. **23** (2017), 1697–1721.

[8] M. Hochster, *Some applications of the Frobenius in characteristic* 0, Bull. Amer. Math. Soc. **84** (1978), no. 5, 886–912. https://doi.org/10.1090/S0002-9904-1978-14531-5

[9] M. R. Zargar, *Relative canonical modules and some duality results*, Algebra Colloq. **26** (2019), no. 2, 351–360. https://doi.org/10.1142/S1005386719000269

[10] M. R. Zargar and H. Zakeri, *On injective and Gorenstein injective dimensions of local cohomology modules*, Algebra Colloq. **22** (2015), Special Issue no. 1, 935–946. https://doi.org/10.1142/S1005386715000784

[11] M. R. Zargar and H. Zakeri, *On flat and Gorenstein flat dimensions of local cohomology modules*, Canad. Math. Bull. **59** (2016), no. 2, 403–416. https://doi.org/10.4153/CMB-2015-080-x

Zhongkui Liu
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: Liuzk@nwnu.edu.cn

Pengju Ma
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: 2642293920@qq.com

Xiaoyan Yang
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: yangxy@nwnu.edu.cn