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TEMPORAL DECAY OF SOLUTIONS FOR A CHEMOTAXIS MODEL OF ANGIOGENESIS TYPE

Jaewook Ahn and Myeonghyeon Kim

Abstract. This paper considers a parabolic-hyperbolic-hyperbolic type chemotaxis system in $\mathbb{R}^d$, $d \geq 3$, describing tumor-induced angiogenesis. The global existence result and temporal decay estimate for a unique mild solution are established under the assumption that some Sobolev norms of initial data are sufficiently small.

1. Introduction

To facilitate angiogenesis, the tumor secrets TAF (tumor angiogenic factor), which induces EC (endothelial-cells) to move towards the tumor (see [6, 11]). As EC migrate, an extracellular protein, so-called fibronectin, is produced and its main function is the adhesion of the cells to the matrix (see [9, 10]). It is known that the movement of EC is regulated by the haptotactic effect of fibronectin as well as the chemotactic effect of TAF (see [2, 13]).

In this paper, we investigates a parabolic-hyperbolic-hyperbolic type chemotaxis system so-called the Anderson-Chaplain model [1] describing the tumor-induced angiogenesis:

$$\begin{aligned} n_t &= \Delta n - \nabla \cdot \left(\frac{\chi}{1+c} n \nabla c \right) - \nabla \cdot (\kappa n \nabla f), \\ f_t &= \beta n - \gamma n f, \\ c_t &= -nc \end{aligned} \tag{1.1}$$

in $\mathbb{R}^d \times (0,T)$, $d \geq 3$, subject to the initial conditions

$$n(x,0) = n_0(x), \quad f(x,0) = f_0(x), \quad c(x,0) = c_0(x). \tag{1.2}$$

Here, $\chi \geq 0$, $\kappa \geq 0$, $\beta \geq 0$ and $\gamma > 0$ are given constants and the unknowns n represents the EC density per unit area, f represents the fibronectin concentration, and c represents the TAF concentration. A closely related variant of

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(1.1) is the following angiogenesis model:

$$n_t = \Delta n - \nabla\cdot(nS(c)\nabla c), \quad c_t = -nc, \quad \mathbb{R}^d\times(0,T), \tag{1.3}$$

where the fibronectin factor is ignored, i.e., $f\equiv 0$ and $\beta = 0$, and S is smooth.

In the one-dimensional setting, smooth solutions to (1.1) or (1.3) are known to exist globally in time for large data [7,8,15]. In two-dimensional and higher-dimensional settings, smooth solutions have been found to exist globally in time only under a smallness assumption on the initial data, and large data global existence results are available for weak solution concepts. Indeed, in the case of $d\geq 2$, it was proved in [4] that if

$$\alpha := \frac{1}{2}\inf_{c\geq 0}\left(\frac{cS'}{S}+1\right) > 0, \tag{1.4}$$

then weak solutions (n,c) to (1.3) exist globally in time and satisfy $n\in L^\infty(0,\infty;\,L^1(\mathbb{R}^d))$, $c\in L^\infty(0,\infty;L^\infty(\mathbb{R}^d))$, and

$$\int_{\mathbb{R}^d}\left[\frac{1}{2}|\nabla\Upsilon(c)|^2 + n\ln n\right] + \int_0^t\int_{\mathbb{R}^d} n\left[|\nabla\ln n|^2 + \alpha|\nabla\Upsilon(c)|^2\right] < \infty$$

with $\Upsilon'(c) = \sqrt{S(c)/c}$. Later on, for $d\geq 2$ and finite $p\geq\max\{1,\frac{d}{2}-1\}$, Corrias-Perthame-Zaag [5] constructed global-in-time weak solutions to (1.3) satisfying $n\in L^\infty(0,\infty;(L^1\cap L^p)(\mathbb{R}^d))$ and $c\in L^\infty(0,\infty;L^\infty(\mathbb{R}^d))$ under the assumption, as a replacement of the condition (1.4), that $L^{\frac{d}{2}}(\mathbb{R}^d)$-norm of n_0 is sufficiently small. Perthame-Vasseur [12] also proved that the global solutions of (1.3) constructed in [5] for $d\geq 2$ satisfy the temporal decay property,

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-1}, \quad t>0. \tag{1.5}$$

For the temporal decay property (1.5) of smooth solutions to (1.3) coupled with fluid equations under the smallness condition on $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$, we refer to Chae-Kang-Lee [3].

Since the linear counterpart of $(1.3)_1$, $n_t=\Delta n$, with $L^1(\mathbb{R}^d)$ initial data has

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}}, \quad t>0, \tag{1.6}$$

one can expect that the system (1.3) also has (1.6) instead of (1.5) if the initial data are sufficiently small in a certain norm to weaken the nonlinear effect.

Motivated by the above observation, the main objective of this paper is to establish the existence of global solutions to either (1.1) or (1.3) in $\mathbb{R}^d$, $d\geq 3$, with the decay property (1.6). The previous decay result (1.5) was proved in [12] based on De Giorgi's technique and the scaling invariant property (see also [3]), but it is unclear whether or not such a method can be applied to derive (1.6). Instead, we use the method of successive approximation to construct a mild solution of (1.1)–(1.2) satisfying (1.6). Here and throughout this paper, we call $n\in C([0,\tau);L^\sigma(\mathbb{R}^d))$ with $\sigma\in[1,\infty)$ and $\tau\in(0,\infty]$ is a mild solution to (1.1)–(1.2) if

$$n(t) = e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(\frac{\chi}{1+c}n\nabla c + \kappa n\nabla f)(s)ds, \tag{1.7}$$

where

$$f(t) - \frac{\beta}{\gamma} = (f_0 - \frac{\beta}{\gamma})e^{-\gamma\int_0^t n(s)ds}, \quad c(t) = c_0 e^{-\int_0^t n(s)ds}. \tag{1.8}$$

Our main result reads as follows.

Theorem 1.1. *Let $d \geq 3$ and $\frac{d}{d-1} < q < d < r \leq p < \infty$. Assume that the nonnegative functions n_0, f_0, and c_0 satisfy*

$$n_0 \in (L^1 \cap L^p)(\mathbb{R}^d) \quad \text{and} \quad f_0 - \frac{\beta}{\gamma}, c_0 \in (W^{1,q} \cap W^{1,r})(\mathbb{R}^d).$$

Then, there exists a constant $\varepsilon_0 = \varepsilon_0(d, p, q, r, \chi, \kappa, \beta, \gamma) > 0$ such that if

$$\|n_0\|_{L^1\cap L^p(\mathbb{R}^d)} + \|\gamma f_0 - \beta\|_{W^{1,q}\cap W^{1,r}(\mathbb{R}^d)} + \|c_0\|_{W^{1,q}\cap W^{1,r}(\mathbb{R}^d)} \leq \varepsilon_0,$$

then a unique global mild solution $n \in C([0,\infty); (L^1 \cap L^p)(\mathbb{R}^d))$ to (1.1)–(1.2) exists and with some $M > 0$,

$$\int_0^\infty \|n(t)\|_{L^\infty(\mathbb{R}^d)}dt + \sup_{t>0}\left\|\int_0^t \nabla n(s)ds\right\|_{L^q\cap L^r(\mathbb{R}^d)} + \sup_{t>0}\|n(t)\|_{L^1(\mathbb{R}^d)} \leq M,$$

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)} \leq Mt^{-\frac{d}{2}} \quad \textit{for all } t > 0.$$

Remark 1.2. In Theorem 1.1, $f_0 - \frac{\beta}{\gamma}, c_0 \in L^\infty(\mathbb{R}^d)$ since $W^{1,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Moreover, $f - \frac{\beta}{\gamma}$ and c in (1.8) are bounded in $(L^\infty \cap W^{1,q} \cap W^{1,r})(\mathbb{R}^d)$ uniformly in time since $\int_0^\infty \|n(t)\|_{L^\infty(\mathbb{R}^d)}dt$ and $\sup_{t>0}\|\int_0^t \nabla n(s)ds\|_{L^q\cap L^r(\mathbb{R}^d)}$ are finite.

Remark 1.3. As in Anderson-Chaplain [1], we adopted $S(c) = \frac{\chi}{1+c}$ as a chemotactic function but it is not difficult to extend Theorem 1.1 to more general $S \in C^1([0,\infty))$. Moreover, Theorem 1.1 is also applicable to the system (1.3) if $f_0 \equiv 0$ and $\beta = 0$. Thus, compared to (1.5), the decay rate of n is improved to $\frac{d}{2}$ for $d \geq 3$.

Remark 1.4. We expect that by using more technical estimates, the initial smallness condition required for the decay (1.6) can be relaxed to $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} + \|(c_0, \gamma f_0 - \beta)\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon$. Since it is out of our scope, we left it as a future work.

The outline of this paper is as follows. Section 2 is devoted to introducing notations and useful lemmas. In Section 3, we present the proofs of Theorem 1.1.

2. Preliminaries

In this section, we collect some notations, definitions, and lemmas. Let us first introduce basic notations.

(i) $\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y$.
(ii) $g \approx h$ means that $M_a g \le h \le M_b g$ for some positive constants M_a and M_b. We denote by $g \lesssim h$ if $g \le Mh$ for some positive constant M.

Next, we introduce the definition of the homogeneous Besov space $\dot{B}^s_{p,q}$. Let φ be a compactly supported function belongs to Schwartz class in $\mathbb{R}^d$ such that

$$\operatorname{supp}\varphi \subset \{1/2 \le |\xi| \le 2\} \quad \text{and} \quad \sum_{\mathrm{j}\in\mathcal{S}} \varphi(2^{-\mathrm{j}}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^{\mathrm{d}} \setminus \{0\}.$$

We define ϕ_j for $j = 0, \pm1, \dots$ by $\mathcal{F}\phi_j(\xi) = \varphi(2^{-j}\xi)$, where $\mathcal{F}$ denotes the Fourier transform. The homogeneous Besov space $\dot{B}^s_{p,q}$ is the collection of $u \in \mathcal{S}'$ such that $\|u\|_{\dot{B}^s_{p,q}} < \infty$, where

$$\|u\|_{\dot{B}^s_{p,q}} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\phi_j * u\|^q_{L^p}\Big)^{\frac{1}{q}}, \quad q < \infty,$$
$$\|u\|_{\dot{B}^s_{p,\infty}} = \sup_{j\in\mathbb{Z}} 2^{js} \|\phi_j * u\|_{L^p}, \quad q = \infty.$$

Next, we collect some lemmas which are crucial for proving Theorem 1.1.

Lemma 2.1. *Assume that $d \ge 1$. Let $u \in \dot{B}^s_{p,q}$, $U \in L^p \cap L^q(\mathbb{R}^d)$.*

(i) *Suppose $0 < p, q < \infty$, $s \in \mathbb{R}$ with nonnegative integer $z > \frac{s}{2}$. Then,*

$$\left(\int_0^\infty (t^{z-\frac{s}{2}} \left\|(-\Delta)^z e^{t\Delta} u\right\|_{L^p(\mathbb{R}^d)})^q \frac{dt}{t}\right)^{\frac{1}{q}} \approx \|u\|_{\dot{B}^s_{p,q}}.$$

(ii) *For $1 \le p < d < q \le \infty$,*

$$\int_0^\infty \Big\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot U(\tau) d\tau \Big\|_{L^\infty(\mathbb{R}^d)} dt \lesssim \int_0^\infty \|U(\tau)\|_{L^p\cap L^q(\mathbb{R}^d)} d\tau.$$

(iii) *For all $1 < p < \infty$,*

$$\Big\| \int_0^T \nabla \int_0^t e^{(t-\tau)\Delta} \nabla \cdot U(\tau) d\tau dt \Big\|_{L^p(\mathbb{R}^d)} \lesssim \int_0^T \|U(\tau)\|_{L^p(\mathbb{R}^d)} d\tau.$$

Proof. We skip the proof of Lemma 2.1 and refer [14, Lem. 2.1] for details. □

Lemma 2.2. *Let $d \ge 3$, $\frac{d}{d-1} < q < d < r \le p \le \infty$ and $u \in (L^1 \cap L^p)(\mathbb{R}^d)$. Then*

$$\int_0^\infty \|e^{t\Delta} u\|_{L^\infty(\mathbb{R}^d)} dt + \sup_{t>0} \Big\| \int_0^t \nabla e^{\tau\Delta} u d\tau \Big\|_{(L^q\cap L^r(\mathbb{R}^d))} \lesssim \|u\|_{(L^1\cap L^p(\mathbb{R}^d))}.$$

Proof. From Lemma 2.1, we obtain

$$\int_0^\infty \|e^{t\Delta}u\|_{L^\infty(\mathbb{R}^d)}dt + \sup_{t>0}\left\| \int_0^t \nabla \qquad e^{\tau\Delta}ud\tau\right\|_{(L^q\cap L^r(\mathbb{R}^d))}$$
$$\lesssim \|u\|_{\dot{B}^{-2}_{\infty,1}} + \|u\|_{\dot{B}^{-1}_{r,1}} + \|u\|_{\dot{B}^{-1}_{q,1}}.$$

Then,

$$\|u\|_{\dot{B}^{-2}_{\infty,1}} \lesssim \sum_{j\le 0} 2^{j(d-2)}\|u\|_{L^1(\mathbb{R}^d)} + \sum_{j\ge 1} 2^{j(\frac{d}{p}-2)}\|u\|_{L^p(\mathbb{R}^d)}, \quad \text{and}$$
$$\|u\|_{\dot{B}^{-1}_{l,1}} \lesssim \sum_{j\le 0} 2^{j(\frac{d}{l'}-1)}\|u\|_{L^1(\mathbb{R}^d)} + \sum_{j\ge 1} 2^{-j}\|u\|_{L^l(\mathbb{R}^d)} \quad \text{for} \quad l=r,\ q,$$

where $\frac{1}{l}+\frac{1}{l'}=1$. Since q and r are less than or equal to p, we deduce the desired bound from the interpolation inequality. □

3. Proof of Theorem 1.1

In this section, we assume $\chi=\kappa=1$ for simplicity unless any confusion is to be expected. A generic constant $M>0$ may change from one to the other. We use function spaces $X_t = X_t^1 \cap X_t^2$, $Y_t = Y_t^1 \cap Y_t^\infty$ and Z_t, where

$$\|J\|_{X_t^1} = \int_0^t \|J(s)\|_{L^\infty(\mathbb{R}^d)}ds,$$
$$\|J\|_{X_t^2} = \sup_{0<s<t}\left\|\int_0^s \nabla J(\tau)d\tau\right\|_{L^q\cap L^r(\mathbb{R}^d)},$$
$$\|J\|_{Y_t^1} = \sup_{0<s<t}\|J(s)\|_{L^1(\mathbb{R}^d)},$$
$$\|J\|_{Y_t^\infty} = \sup_{0<s<t} s^{\frac{d}{2}}\|J(s)\|_{L^\infty(\mathbb{R}^d)},$$

and

$$\|J\|_{Z_t} = \sup_{0<s<t}\|J(s)\|_{W^{1,q}\cap W^{1,r}(\mathbb{R}^d)}.$$

Note that $X_\infty \cap Y_\infty$ is a Banach space endowed with norm $\|\cdot\|_{X_\infty} + \|\cdot\|_{Y_\infty}$ and Z_∞ is a Banach space endowed with norm $\|\cdot\|_{Z_\infty}$.

Proof of Theorem 1.1. We use the method of successive approximation. Let

$$\varepsilon \in (0,1)$$

which will be specified later and we assume that

$$\|n_0\|_{L^1\cap L^p(\mathbb{R}^d)} + \|\gamma f_0 - \beta\|_{W^{1,q}\cap W^{1,r}(\mathbb{R}^d)} + \|c_0\|_{W^{1,q}\cap W^{1,r}(\mathbb{R}^d)} \le \varepsilon. \tag{3.1}$$

Define

$$n_1(t) := e^{t\Delta}n_0. \tag{3.2}$$

From Lemma 2.2 and the heat kernel estimates, we observe that there exist positive constants M_X and M_Y such that

$$\begin{aligned} \|n_1\|_{X_\infty} &\le M_X \|n_0\|_{L^1\cap L^p(\mathbb{R}^d)}, \\ \|n_1\|_{Y_\infty} &\le M_Y \|n_0\|_{L^1(\mathbb{R}^d)}. \end{aligned} \tag{3.3}$$

Thus, with some $M_* > 0$,

$$\|n_1\|_{X_\infty \cap Y_\infty} \le M_* \|n_0\|_{L^1\cap L^p(\mathbb{R}^d)}. \tag{3.4}$$

Let us define a closed convex subset $\mathcal{S}$ of the Banach space $X_\infty \cap Y_\infty$ by

$$\mathcal{S} := \{J \in (X_\infty \cap Y_\infty) : \|J\|_{\mathcal{S}} := \|J\|_{X_\infty} + \|J\|_{Y_\infty} \le 2M_*\varepsilon\}.$$

We note that if $n_k \in \mathcal{S}$ for some $k \ge 1$, then

$$f_k(t) - \frac{\beta}{\gamma} := (f_0 - \frac{\beta}{\gamma})e^{-\gamma\int_0^t n_k(s)ds}, \quad c_k(t) := c_0 e^{-\int_0^t n_k(s)ds} \tag{3.5}$$

belong to Z_∞. Indeed, we have

$$\begin{aligned} \nabla f_k(t) &= e^{-\gamma\int_0^t n_k(s)ds}\left(\nabla f_0 - (\gamma f_0 - \beta)\int_0^t \nabla n_k(s)ds\right), \\ \nabla c_k(t) &= e^{-\int_0^t n_k(s)ds}\left(\nabla c_0 - c_0\int_0^t \nabla n_k(s)ds\right), \end{aligned}$$

and direct computations yield for $l = q, r$ that

$$\begin{aligned} \left\|f_k(t) - \frac{\beta}{\gamma}\right\|_{L^l(\mathbb{R}^d)} &\le \left\|f_0 - \frac{\beta}{\gamma}\right\|_{L^l(\mathbb{R}^d)} e^{\gamma\int_0^t \|n_k(s)\|_{L^\infty(\mathbb{R}^d)}ds} \\ &\le \left\|f_0 - \frac{\beta}{\gamma}\right\|_{L^l(\mathbb{R}^d)} e^{2\gamma M_*\varepsilon} \\ &\le M_0\varepsilon, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\|\nabla f_k(t)\|_{L^l(\mathbb{R}^d)} \\ &\le e^{\gamma\int_0^t \|n_k(s)\|_{L^\infty(\mathbb{R}^d)}ds} \\ &\quad \cdot \left(\|\nabla f_0\|_{L^l(\mathbb{R}^d)} + \|\gamma f_0 - \beta\|_{L^\infty(\mathbb{R}^d)} \left\|\int_0^t \nabla n_k(s)ds\right\|_{L^l(\mathbb{R}^d)}\right) \\ &\le e^{2\gamma M_*\varepsilon}\left(\|\nabla f_0\|_{L^l(\mathbb{R}^d)} + \|\gamma f_0 - \beta\|_{L^\infty(\mathbb{R}^d)} 2M_*\varepsilon\right) \\ &\le M_1\varepsilon, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|c_k(t)\|_{L^l(\mathbb{R}^d)} &\le \|c_0\|_{L^l(\mathbb{R}^d)} e^{\int_0^t \|n_k(s)\|_{L^\infty(\mathbb{R}^d)}ds} \\ &\le \|c_0\|_{L^l(\mathbb{R}^d)} e^{2M_*\varepsilon} \\ &\le M_2\varepsilon, \end{aligned} \tag{3.8}$$

and

$$\|\nabla c_k(t)\|_{L^l(\mathbb{R}^d)} \tag{3.9}$$

$$\le e^{\int_0^t \|n_k(s)\|_{L^\infty(\mathbb{R}^d)}ds}\left(\|\nabla c_0\|_{L^l(\mathbb{R}^d)} + \|c_0\|_{L^\infty(\mathbb{R}^d)}\left\|\int_0^t \nabla n_k(s)ds\right\|_{L^l(\mathbb{R}^d)}\right)$$
$$\le e^{2M_*\varepsilon}\left(\|\nabla c_0\|_{L^l(\mathbb{R}^d)} + \|c_0\|_{L^\infty(\mathbb{R}^d)} 2M_*\varepsilon\right)$$
$$\le M_3\varepsilon,$$

where M_i, $i = 0,1,2,3$, are positive constants independent of ε and t. With such M_i, we define a closed convex subset $\mathcal{T}$ of the Banach space Z_∞ as

$$\mathcal{T} := \{J \in Z_\infty : \|J\|_{Z_\infty} \le 2(M_0 + M_1 + M_2 + M_3)\varepsilon\}.$$

In view of (3.6)–(3.9), it is obvious that $f_k - \frac{\beta}{\gamma}$ and c_k belong to $\mathcal{T}$ provided that $n_k \in \mathcal{S}$. Now, we divide the remaining part of the proof into three steps.

Step 1) First, we prove that $(n_k, f_k - \frac{\beta}{\gamma}, c_k)$ given by (3.2), (3.5), and

$$(3.10)\qquad n_{k+1}(t) := e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(n_k\frac{1}{1+c_k}\nabla c_k + n_k\nabla f_k)(s)ds, \quad k \ge 1,$$

belongs to $\mathcal{S}\times\mathcal{T}\times\mathcal{T}$ for any $k \ge 1$ if ε is sufficiently small. We use mathematical induction. Note from (3.4) that $n_1 \in \mathcal{S}$. We now assume that

$$(3.11)\qquad n_m \in \mathcal{S} \quad \text{for some } m \ge 1$$

and show $n_{m+1} \in \mathcal{S}$. By (3.6)–(3.9) and (3.11), we have

$$(3.12)\qquad f_m - \frac{\beta}{\gamma}, c_m \in \mathcal{T}, \quad \text{and} \quad c_m \ge 0.$$

We abbreviate

$$B_{1,m}(t) = -\int_0^t e^{(t-s)\Delta}\nabla\cdot(n_m\frac{1}{1+c_m}\nabla c_m)(s)ds,$$
$$B_{2,m}(t) = -\int_0^t e^{(t-s)\Delta}\nabla\cdot(n_m\nabla f_m)(s)ds,$$

so that

$$(3.13)\qquad n_{m+1}(t) = e^{t\Delta}n_0 + B_{1,m}(t) + B_{2,m}(t).$$

By Lemma 2.2, we see that

$$(3.14)\qquad \|n_{m+1}\|_{X_\infty} \le M_X\|n_0\|_{L^1\cap L^p(\mathbb{R}^d)} + \|B_{1,m}\|_{X_\infty} + \|B_{2,m}\|_{X_\infty},$$

where M_X is a constant given in (3.3). Using Lemma 2.1, $\frac{1}{1+c_m} \le 1$ and the Hölder inequality, we estimate X_∞-norm of $B_{1,m}$ as

$$(3.15)\qquad \|B_{1,m}\|_{X_\infty} \le M\int_0^\infty \|n_m\nabla c_m(s)\|_{L^q\cap L^r(\mathbb{R}^d)}ds$$
$$\le M\big\|\|n_m\|_{L^\infty(\mathbb{R}^d)}\big\|_{L^1(0,\infty)}\big\|\|\nabla c_m\|_{L^q\cap L^r(\mathbb{R}^d)}\big\|_{L^\infty(0,\infty)}$$

$$\le M\|n_m\|_{X^1_\infty} e^{\|n_m\|_{X^1_\infty}} \Big(\|\nabla c_0\|_{L^q\cap L^r(\mathbb{R}^d)} + \|c_0\|_{L^\infty(\mathbb{R}^d)} \|n_m\|_{X^2_\infty} \Big).$$

Similarly, we can estimate X_∞-norm of $B_{2,m}$ as

$$\begin{aligned}(3.16)\quad &\|B_{2,m}\|_{X_\infty}\\ &\le M\|n_m\|_{X^1_\infty} e^{\gamma\|n_m\|_{X^1_\infty}} \Big(\|\nabla f_0\|_{L^q\cap L^r(\mathbb{R}^d)} + \|\gamma f_0 - \beta\|_{L^\infty(\mathbb{R}^d)} \|n_m\|_{X^2_\infty} \Big).\end{aligned}$$

Combining (3.14)–(3.16), due to (3.1) and (3.11), we have

$$\begin{aligned}(3.17)\quad &\|n_{m+1}\|_{X_\infty}\\ &\le M_X\|n_0\|_{L^1\cap L^p(\mathbb{R}^d)}\\ &\quad + M\|n_m\|_{X_\infty} e^{\|n_m\|_{X_\infty}} \|\nabla c_0\|_{L^q\cap L^r(\mathbb{R}^d)}\\ &\quad + M\|n_m\|^2_{X_\infty} e^{\|n_m\|_{X_\infty}} \|c_0\|_{L^\infty(\mathbb{R}^d)}\\ &\quad + M\|n_m\|_{X_\infty} e^{\gamma\|n_m\|_{X_\infty}} \|\nabla f_0\|_{L^q\cap L^r(\mathbb{R}^d)}\\ &\quad + M\|n_m\|^2_{X_\infty} e^{\gamma\|n_m\|_{X_\infty}} \|\beta - \gamma f_0\|_{L^\infty(\mathbb{R}^d)}\\ &\le M_X\|n_0\|_{L^1\cap L^p(\mathbb{R}^d)} + M^*_X\varepsilon^2,\end{aligned}$$

where M^*_X is a positive constant independent of m and ε.

Next, we consider Y_∞-norm of n_{m+1}. Using (3.10)–(3.12) and Hölder's inequality, we compute $L^1(\mathbb{R}^d)$-norm of n_{m+1} as

$$\begin{aligned}(3.18)\quad &\|n_{m+1}(t)\|_{L^1(\mathbb{R}^d)}\\ &\le \|e^{t\Delta}n_0\|_{L^1(\mathbb{R}^d)}\\ &\quad + M\int_0^t (t-s)^{-\frac12} \Big(\|n_m\nabla c_m(s)\|_{L^1(\mathbb{R}^d)} + \|n_m\nabla f_m(s)\|_{L^1(\mathbb{R}^d)} \Big)\, ds\\ &\le \|e^{t\Delta}n_0\|_{L^1(\mathbb{R}^d)}\\ &\quad + M\int_0^t (t-s)^{-\frac12} \|n_m(s)\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} (\|\nabla c_m(s)\|_{L^d(\mathbb{R}^d)} + \|\nabla f_m(s)\|_{L^d(\mathbb{R}^d)}) ds\\ &\le \|e^{t\Delta}n_0\|_{L^1(\mathbb{R}^d)} + M\varepsilon\int_0^t (t-s)^{-\frac12} \|n_m(s)\|^{1-\frac1d}_{L^1(\mathbb{R}^d)} \|n_m(s)\|^{\frac1d}_{L^\infty(\mathbb{R}^d)} s^{\frac12} s^{-\frac12} ds\\ &\le \|e^{t\Delta}n_0\|_{L^1(\mathbb{R}^d)} + M\varepsilon \|n_m\|^{1-\frac1d}_{Y^1_\infty} \|n_m\|^{\frac1d}_{Y^\infty_\infty} \int_0^t (t-s)^{-\frac12} s^{-\frac12} ds\\ &\le \|e^{t\Delta}n_0\|_{L^1(\mathbb{R}^d)} + M\varepsilon^2 \quad \text{for all} \quad t > 0,\end{aligned}$$

where we used

$$\int_0^t (t-s)^{-\frac12} s^{-\frac12} ds = \int_0^1 (1-\tau)^{-\frac12} \tau^{-\frac12} d\tau < \infty.$$

Next, we compute $L^\infty(\mathbb{R}^d)$-norm of n_{m+1}. Using (3.10) and Hölder's inequality, we compute

$$(3.19)\qquad \|n_{m+1}(t)\|_{L^\infty(\mathbb{R}^d)} \le \|e^{t\Delta}n_0\|_{L^\infty(\mathbb{R}^d)} + I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^{\frac{t}{2}} \|e^{(t-s)\Delta}\nabla\cdot(n_k\frac{1}{1+c_k}\nabla c_k + n_k\nabla f_k)(s)\|_{L^\infty(\mathbb{R}^d)}ds,$$
$$I_2(t) = \int_{\frac{t}{2}}^{t} \|e^{(t-s)\Delta}\nabla\cdot(n_k\frac{1}{1+c_k}\nabla c_k + n_k\nabla f_k)(s)\|_{L^\infty(\mathbb{R}^d)}ds.$$

Using the heat kernel estimates, Hölder's inequality and (3.11)–(3.12), we compute I_1 as

$$\begin{aligned}
(3.20)\quad & I_1(t)\\
&\le \int_0^{\frac{t}{2}} (t-s)^{-\frac12-\frac d2}\left(\|n_m\nabla c_m(s)\|_{L^1(\mathbb{R}^d)} + \|n_m\nabla f_m(s)\|_{L^1(\mathbb{R}^d)}\right)ds\\
&\le \int_0^{\frac{t}{2}} (t-s)^{-\frac12-\frac d2}\|n_m(s)\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}(\|\nabla c_m(s)\|_{L^d(\mathbb{R}^d)} + \|\nabla f_m(s)\|_{L^d(\mathbb{R}^d)})ds\\
&\le M\varepsilon\int_0^{\frac{t}{2}} (t-s)^{-\frac12-\frac d2}\|n_m(s)\|_{L^1(\mathbb{R}^d)}^{1-\frac1d}\|n_m(s)\|_{L^\infty(\mathbb{R}^d)}^{\frac1d}s^{\frac12}s^{-\frac12}ds\\
&\le M\varepsilon\|n_m\|_{Y^1_\infty}^{1-\frac1d}\|n_m\|_{Y^\infty_\infty}^{\frac1d}\int_0^{\frac{t}{2}}(t-s)^{-\frac12-\frac d2}s^{-\frac12}ds\\
&\le M\varepsilon^2 t^{-\frac d2}.
\end{aligned}$$

We treat I_2 as follows. If $t \le 2$, then using the heat kernel estimates, Hölder's inequality and (3.11)–(3.12), we compute I_2 as

$$\begin{aligned}
(3.21)\quad & I_2(t)\\
&\le \int_{\frac{t}{2}}^{t} (t-s)^{-\frac12-\frac d2\frac1r}\|n_m(s)\|_{L^\infty(\mathbb{R}^d)}(\|\nabla c_m(s)\|_{L^r(\mathbb{R}^d)} + \|\nabla f_m(s)\|_{L^r(\mathbb{R}^d)})ds\\
&\le M\varepsilon\int_{\frac{t}{2}}^{t} (t-s)^{-\frac12-\frac d2\frac1r}\|n_m(s)\|_{L^\infty(\mathbb{R}^d)}s^{\frac d2}s^{-\frac d2}ds\\
&\le M\varepsilon\|n_m\|_{Y^\infty_\infty}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12-\frac d2\frac1r}s^{-\frac d2}ds\\
&\le M\varepsilon^2 t^{-\frac d2}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12-\frac d2\frac1r}ds\\
&\le M\varepsilon^2 t^{-\frac d2}.
\end{aligned}$$

Otherwise, if $t > 2$, then using the heat kernel estimates and $\frac{1}{1+c_m} \le 1$, we compute I_2 as

$$I_2(t) \le I_{21}(t) + I_{22}(t), \tag{3.22}$$

where

$$I_{21}(t) = \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac12-\frac d2\frac1q}(\|n_m\nabla c_m(s)\|_{L^q(\mathbb{R}^d)} + \|n_m\nabla f_m(s)\|_{L^q(\mathbb{R}^d)})ds,$$

$$I_{22}(t) = \int_{t-1}^{t} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{r}} (\|n_m \nabla c_m(s)\|_{L^r(\mathbb{R}^d)} + \|n_m \nabla f_m(s)\|_{L^r(\mathbb{R}^d)}) ds.$$

By Hölder's inequality and (3.11)–(3.12), we have that

$$\begin{aligned}
(3.23)\quad I_{21}(t) &\\
&\le \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{q}} \|n_m(s)\|_{L^\infty(\mathbb{R}^d)} (\|\nabla c_m(s)\|_{L^q(\mathbb{R}^d)} + \|\nabla f_m(s)\|_{L^q(\mathbb{R}^d)}) ds \\
&\le M\varepsilon \|n_m\|_{Y_\infty^\infty} \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{q}} s^{-\frac{d}{2}} ds \\
&\le M\varepsilon^2 t^{-\frac{d}{2}} \left[\sup_{t>0} \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{q}} ds \right] \\
&\le M\varepsilon^2 t^{-\frac{d}{2}},
\end{aligned}$$

and similarly, we also compute that

$$\begin{aligned}
(3.24)\quad I_{22}(t) &\\
&\le \int_{t-1}^{t} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{r}} \|n_m(s)\|_{L^\infty(\mathbb{R}^d)} (\|\nabla c_m(s)\|_{L^r(\mathbb{R}^d)} + \|\nabla f_m(s)\|_{L^r(\mathbb{R}^d)}) ds \\
&\le M\varepsilon \|n_m\|_{Y_\infty^\infty} \int_{t-1}^{t} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{r}} s^{-\frac{d}{2}} ds \\
&\le M\varepsilon^2 (t-1)^{-\frac{d}{2}} \left[\sup_{t>0} \int_{t-1}^{t} (t-s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{r}} ds \right] \\
&\le M\varepsilon^2 t^{-\frac{d}{2}}.
\end{aligned}$$

Combining (3.19)–(3.24), we obtain

$$(3.25)\qquad \|n_{m+1}(t)\|_{L^\infty(\mathbb{R}^d)} \le \|e^{t\Delta} n_0\|_{L^\infty(\mathbb{R}^d)} + M\varepsilon^2 t^{-\frac{d}{2}} \quad \text{for all } t > 0.$$

Thus, with (3.18), it follows that

$$(3.26)\qquad \|n_{m+1}\|_{Y_\infty} \le M_Y \|n_0\|_{L^1(\mathbb{R}^d)} + M_Y^* \varepsilon^2,$$

where M_Y is a constant given in (3.3) and $M_Y^* > 0$ is a constant independent of m and ε. Now, taking ε sufficiently small so that

$$(3.27)\qquad \varepsilon \le \frac{M_*}{2\max\{M_X^*, M_Y^*\}},$$

by (3.17) and (3.26), we have $n_{m+1} \in \mathcal{S}$. Then inductively we can deduce that if (3.27) holds, then

$$(3.28)\qquad (n_k, f_k - \frac{\beta}{\gamma}, c_k) \in \mathcal{S} \times \mathcal{T} \times \mathcal{T} \quad \text{for all } k \in \mathbb{N}.$$

Step 2) Next, we show that $\{(n_k, f_k - \frac{\beta}{\gamma}, c_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S} \times \mathcal{T} \times \mathcal{T}$ for sufficiently small ε. Let (3.27) hold and $k \ge 2$. We note from

(3.13) that

$$(n_{k+1}-n_k)(t)=(B_{1,k}-B_{1,k-1})(t)+(B_{2,k}-B_{2,k-1})(t), \tag{3.29}$$

where

$$\begin{aligned}
&(B_{1,k}-B_{1,k-1})(t)\\
&= -\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(n_k\frac{1}{1+c_k}\nabla c_k-n_{k-1}\frac{1}{1+c_{k-1}}\nabla c_{k-1}\Big)ds\\
&= -\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big((n_k-n_{k-1})\frac{1}{1+c_k}\nabla c_k-n_{k-1}\nabla c_k\frac{(c_k-c_{k-1})}{(1+c_k)(1+c_{k-1})}\\
&\qquad\qquad +n_{k-1}\frac{1}{1+c_{k-1}}\nabla(c_k-c_{k-1})\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
&(B_{2,k}-B_{2,k-1})(t)\\
&= -\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big((n_k-n_{k-1})\nabla f_k+n_{k-1}\nabla(f_k-f_{k-1})\Big)ds.
\end{aligned}$$

We note also that

$$(c_k-c_{k-1})(t)=c_0(e^{-\int_0^t n_k(s)ds}-e^{-\int_0^t n_{k-1}(s)ds}),$$

$$(f_k-f_{k-1})(t)=(f_0-\frac{\beta}{\gamma})(e^{-\gamma\int_0^t n_k(s)ds}-e^{-\gamma\int_0^t n_{k-1}(s)ds}),$$

$$\begin{aligned}
\nabla(c_k-c_{k-1})(t)&=\nabla c_0(e^{-\int_0^t n_k(s)ds}-e^{-\int_0^t n_{k-1}(s)ds})\\
&\quad -c_0\left((e^{-\int_0^t n_k(s)ds}-e^{-\int_0^t n_{k-1}(s)ds})\int_0^t\nabla n_k(s)ds\right)\\
&\quad -c_0e^{-\int_0^t n_{k-1}(s)ds}\left(\int_0^t\nabla(n_k-n_{k-1})(s)ds\right),
\end{aligned}$$

and

$$\begin{aligned}
&\nabla(f_k-f_{k-1})(t)\\
&=\nabla f_0(e^{-\gamma\int_0^t n_k(s)ds}-e^{-\gamma\int_0^t n_{k-1}(s)ds})\\
&\quad -(\gamma f_0-\beta)\left((e^{-\gamma\int_0^t n_k(s)ds}-e^{-\gamma\int_0^t n_{k-1}(s)ds})\int_0^t\nabla n_k(s)ds\right)\\
&\quad -(\gamma f_0-\beta)e^{-\gamma\int_0^t n_{k-1}(s)ds}\left(\int_0^t\nabla(n_k-n_{k-1})(s)ds\right).
\end{aligned}$$

Since the mean value theorem yields

$$|e^a-e^b|\le e^{\max\{|a|,|b|\}}|a-b|\quad\text{for all } a,b\in\mathbb{R}, \tag{3.30}$$

using (3.30), Hölder's inequality and (3.28), we compute

$$\|c_k-c_{k-1}\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^d))} \tag{3.31}$$

$$\leq \|c_0\|_{L^\infty(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds$$
$$\leq \|c_0\|_{L^\infty(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \|n_k - n_{k-1}\|_{X^1_\infty}.$$

Similarly, again by (3.30), Hölder's inequality and (3.28), we have for $l = q, r$ that

$$\begin{aligned}
(3.32)\quad & \|\nabla(c_k - c_{k-1})\|_{L^\infty(0,\infty;L^l(\mathbb{R}^d))} \\
&\leq \|\nabla c_0\|_{L^l(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \\
&\quad + \|c_0\|_{L^\infty(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \|n_k\|_{X^2_\infty} \\
&\quad + \|c_0\|_{L^\infty(\mathbb{R}^d)} e^{\|n_{k-1}\|_{X^1_\infty}} \|n_k - n_{k-1}\|_{X^2_\infty} \\
&\leq M \|n_k - n_{k-1}\|_{X_\infty}
\end{aligned}$$

and

$$\begin{aligned}
(3.33)\quad & \|\nabla(f_k - f_{k-1})\|_{L^\infty(0,\infty;L^l(\mathbb{R}^d))} \\
&\leq \|\nabla f_0\|_{L^l(\mathbb{R}^d)} e^{\gamma \max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \gamma \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \\
&\quad + \|\gamma f_0 - \beta\|_{L^\infty(\mathbb{R}^d)} e^{\gamma \max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \gamma \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \|n_k\|_{X^2_\infty} \\
&\quad + \|\gamma f_0 - \beta\|_{L^\infty(\mathbb{R}^d)} e^{\gamma \|n_{k-1}\|_{X^1_\infty}} \|n_k - n_{k-1}\|_{X^2_\infty} \\
&\leq M \|n_k - n_{k-1}\|_{X_\infty},
\end{aligned}$$

where M is a positive constant independent of ε and k. Thus, using (3.29), (3.31)–(3.33) and utilizing computations similar to (3.14)–(3.17), we can obtain

$$(3.34)\qquad \|n_{k+1} - n_k\|_{X_\infty} \leq M^{**}_X \varepsilon \|n_k - n_{k-1}\|_{X_\infty},$$

where M^{**}_X is a positive constant independent of ε and k. Moreover, treating (3.29) similarly as in (3.18)–(3.26), again by (3.31)–(3.33), we have

$$(3.35)\qquad \|n_{k+1} - n_k\|_{Y_\infty} \leq M^{**}_Y \varepsilon \|n_k - n_{k-1}\|_{\mathcal{S}},$$

where M^{**}_Y is a positive constant independent of ε and k. Now, choosing ε satisfying (3.27) and

$$(3.36)\qquad \varepsilon \leq \frac{1}{2(M^{**}_X + M^{**}_Y)},$$

we can see from (3.34)–(3.35) that $\{n_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}$.

Moreover, analogues to (3.31), we can compute for $l = q, r$ that

$$\begin{aligned}
(3.37)\qquad & \|c_k - c_{k-1}\|_{L^\infty(0,\infty;L^l(\mathbb{R}^d))} \\
&\leq \|c_0\|_{L^l(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \\
&\leq \|c_0\|_{L^l(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \|n_k - n_{k-1}\|_{X^1_\infty},
\end{aligned}$$

and

$$\begin{aligned} (3.38) \quad & \|f_k - f_{k-1}\|_{L^\infty(0,\infty;L^l(\mathbb{R}^d))} \\ & \le \|\gamma f_0 - \beta\|_{L^l(\mathbb{R}^d)} e^{\gamma \max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \int_0^\infty \|n_k - n_{k-1}\|_{L^\infty(\mathbb{R}^d)} ds \\ & \le \|\gamma f_0 - \beta\|_{L^l(\mathbb{R}^d)} e^{\max\{\|n_k\|_{X^1_\infty}, \|n_{k-1}\|_{X^1_\infty}\}} \|n_k - n_{k-1}\|_{X^1_\infty}. \end{aligned}$$

With (3.34)–(3.35) and ε satisfying (3.27) and (3.36), we can deduce from (3.32)–(3.33) and (3.37)–(3.38) that $\{f_k - \frac{\beta}{\gamma}\}_{k\in\mathbb{N}}$ and $\{c_k\}_{k\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{T}$.

Step 3) Due to *Step 1–Step 2*, for (n_0, f_0, c_0) and ε satisfying (3.1), (3.27) and (3.36), (1.7)–(1.8) has a global solution

$$(3.39) \qquad (n, f - \frac{\beta}{\gamma}, c) \in \mathcal{S} \times \mathcal{T} \times \mathcal{T}.$$

To show its uniqueness, we suppose that $(n^1, f^1 - \frac{\beta}{\gamma}, c^1)$ and $(n^2, f^2 - \frac{\beta}{\gamma}, c^2)$ are two solutions of (1.7)–(1.8). By repeating computations similar to (3.34)–(3.35), we can obtain

$$\|n^2 - n^1\|_{\mathcal{S}} \le (M^{**}_X + M^{**}_Y)\varepsilon \|n^2 - n^1\|_{\mathcal{S}}$$

and thus, by (3.36), $\|n^2 - n^1\|_{\mathcal{S}} = 0$. Namely, $n^1 = n^2$. Since $n^1 = n^2$, it is direct to see from (1.8) that $(f^1, c^1) = (f^2, c^2)$, which yields the uniqueness of the solution.

Finally, we show $n \in C([0,\infty); (L^1 \cap L^p)(\mathbb{R}^d))$. Note that, by a similar argument as in (3.18), we have $n \in C([0,\infty); L^1(\mathbb{R}^d))$. To show $n \in C([0,\infty); L^p(\mathbb{R}^d))$, we let

$$\frac{1}{p_k} = 1 - \frac{k+1}{2}\left(\frac{1}{d} - \frac{1}{r}\right), \quad k = 1, 2, \ldots$$

and show inductively that $n \in C([0,\infty); L^{p_k}(\mathbb{R}^d))$ for all $k = 1, \ldots, m$. Here, m is uniquely chosen so that

$$\frac{1}{p_m} \in \left[\frac{1}{p}, \ \frac{1}{p} + \frac{1}{2}\left(\frac{1}{d} - \frac{1}{r}\right)\right).$$

Note that $\frac{1}{p_1} > \frac{1}{p} + \frac{1}{2}(\frac{1}{d} - \frac{1}{r})$ and $n_0 \in L^{p_1}(\mathbb{R}^d)$. Using the heat kernel estimates and $\frac{1}{1+c} \le 1$, we compute

$$\begin{aligned} & \|n(t)\|_{L^{p_1}(\mathbb{R}^d)} \\ & \lesssim \|n_0\|_{L^{p_1}(\mathbb{R}^d)} + \int_0^t (t-s)^{-\frac{1}{2} - \frac{d}{2}(1 - \frac{1}{p_1})} (\|n\nabla c(s)\|_{L^1(\mathbb{R}^d)} + \|n\nabla f(s)\|_{L^1(\mathbb{R}^d)}) ds. \end{aligned}$$

Applying Hölder's inequality to the rightmost term, after using (3.39), we observe that

$$\int_0^t (t-s)^{-\frac{1}{2} - \frac{d}{2}(1 - \frac{1}{p_1})} (\|n\nabla c(s)\|_{L^1(\mathbb{R}^d)} + \|n\nabla f(s)\|_{L^1(\mathbb{R}^d)}) ds$$

$$\lesssim \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(1-\frac{1}{p_1})} \|n(s)\|_{L^1(\mathbb{R}^d)}^{1-\frac{1}{r}} \|n(s)\|_{L^\infty(\mathbb{R}^d)}^{\frac{1}{r}} \cdot (\|\nabla c(s)\|_{L^r(\mathbb{R}^d)} + \|\nabla f(s)\|_{L^r(\mathbb{R}^d)}) ds$$

$$\lesssim \|n\|_{Y^1_\infty}^{1-\frac{1}{r}} \|n\|_{Y^\infty_\infty}^{\frac{1}{r}} (\|c\|_{Z_\infty} + \|\gamma f - \beta\|_{Z_\infty}) \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(1-\frac{1}{p_1})} s^{-\frac{1}{r}\frac{d}{2}} ds$$

$$\lesssim \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(1-\frac{1}{p_1})} s^{-\frac{1}{r}\frac{d}{2}} ds.$$

Since two exponents in the last integral representation satisfy

$$-\frac{1}{2}-\frac{d}{2}\left(1-\frac{1}{p_1}\right) = -\frac{1}{2}-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{r}\right) > -1, \quad -\frac{1}{r}\frac{d}{2} > -1,$$

and

$$\left[-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{r}\right)\right] + \left[-\frac{1}{r}\frac{d}{2}\right] = -1,$$

we can deduce that $n \in C([0,\infty); L^{p_1}(\mathbb{R}^d))$. Using this $L^{p_1}(\mathbb{R}^d)$ bound of n with $\frac{1}{1+c} \le 1$, (3.39), and the heat kernel estimates, we next compute

$$\begin{aligned}
&\|n(t)\|_{L^{p_2}(\mathbb{R}^d)} \\
&\lesssim \|n_0\|_{L^{p_2}(\mathbb{R}^d)} \\
&\quad + \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}+\frac{1}{p_1}-\frac{1}{p_2})} (\|n\nabla c(s)\|_{L^{\frac{p_1 r}{p_1+r}}(\mathbb{R}^d)} + \|n\nabla f(s)\|_{L^{\frac{p_1 r}{p_1+r}}(\mathbb{R}^d)}) ds \\
&\lesssim \|n_0\|_{L^{p_2}(\mathbb{R}^d)} \\
&\quad + \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}+\frac{1}{p_1}-\frac{1}{p_2})} \|n(s)\|_{L^{p_1}(\mathbb{R}^d)} (\|\nabla c(s)\|_{L^r(\mathbb{R}^d)} + \|\nabla f(s)\|_{L^r(\mathbb{R}^d)}) ds \\
&\lesssim \|n_0\|_{L^{p_2}(\mathbb{R}^d)} \\
&\quad + \|n\|_{L^\infty(0,t;L^{p_1}(\mathbb{R}^d))} (\|c\|_{Z_\infty} + \|\gamma f - \beta\|_{Z_\infty}) \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}+\frac{1}{p_1}-\frac{1}{p_2})} ds.
\end{aligned}$$

Since

$$\begin{aligned}
-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{r}+\frac{1}{p_k}-\frac{1}{p_{k+1}}\right) &= -\frac{1}{2}-\frac{d}{2}\left(\frac{1}{r}+\frac{1}{2}\Big(\frac{1}{d}-\frac{1}{r}\Big)\right) \\
&> -1 \quad \text{for all } k = 1, 2, \ldots,
\end{aligned}$$

it follows that $n \in C([0,\infty); L^{p_2}(\mathbb{R}^d))$. Moreover, by repeating the similar procedures, we have for $k = 1, \ldots, m-1$ that

$$\begin{aligned}
&\|n(t)\|_{L^{p_{k+1}}(\mathbb{R}^d)} \\
&\lesssim \|n_0\|_{L^{p_{k+1}}(\mathbb{R}^d)} \\
&\quad + \|n\|_{L^\infty(0,t;L^{p_k}(\mathbb{R}^d))} (\|c\|_{Z_\infty} + \|\gamma f - \beta\|_{Z_\infty}) \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}+\frac{1}{p_k}-\frac{1}{p_{k+1}})} ds
\end{aligned}$$

and thus, by inductive reasoning, $n \in C([0,\infty); L^{p_m}(\mathbb{R}^d))$ follows. Now, we can deduce $n \in C([0,\infty); L^p(\mathbb{R}^d))$ from the estimates

$$\begin{aligned}
&\|n(t)\|_{L^p(\mathbb{R}^d)} \\
&\lesssim \|n_0\|_{L^p(\mathbb{R}^d)} \\
&\quad + \|n\|_{L^\infty(0,t;L^{p_m}(\mathbb{R}^d))}(\|c\|_{Z_\infty} + \|\gamma f - \beta\|_{Z_\infty}) \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{r}+\frac{1}{p_m}-\frac{1}{p})} ds,
\end{aligned}$$

where

$$\begin{aligned}
-\frac{1}{2} - \frac{d}{2}\left(\frac{1}{r} + \frac{1}{p_m} - \frac{1}{p}\right) &> -\frac{1}{2} - \frac{d}{2}\left(\frac{1}{r} + \frac{1}{2}\left(\frac{1}{d} - \frac{1}{r}\right)\right) \\
&= -\frac{3}{4} - \frac{d}{4r} \\
&> -1.
\end{aligned}$$

In summary, for ε satisfying (3.1), (3.27) and (3.36), there exists a unique global mild solution to (1.1)–(1.2) satisfying $n \in C([0,\infty); (L^1 \cap L^p)(\mathbb{R}^d))$ and (3.39). □

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Jaewook Ahn
Department of Mathematics
Dongguk University
Seoul 04620, Korea
Email address: jaewookahn@dgu.ac.kr

Myeonghyeon Kim
Department of Mathematics
Kyungsung University
Busan 48434, Korea
Email address: mkim@ks.ac.kr