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TIME PERIODIC SOLUTION FOR THE COMPRESSIBLE MAGNETO-MICROPOLAR FLUIDS WITH EXTERNAL FORCES IN $\mathbb{R}^3$

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Abstract. In this paper, we consider the existence of time periodic solutions for the compressible magneto-micropolar fluids in the whole space $\mathbb{R}^3$. In particular, we first solve the problem in a sequence of bounded domains by the topological degree theory. Then we obtain the existence of time periodic solutions in $\mathbb{R}^3$ by a limiting process.

1. Introduction

This paper is concerned with the existence of time periodic solutions to the following $3D$ magneto-micropolar for compressible fluids:

$$
(1)\quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = (\mu+\nu)\Delta u + (\mu+\lambda-\nu)\nabla \operatorname{div} u \\ \qquad\qquad +2\nu \nabla \times \omega + (\nabla \times H) \times H + \rho f, \\ (\rho\omega)_t + \operatorname{div}(\rho u \otimes \omega) + 4\nu\omega = \mu'\Delta\omega + (\mu'+\lambda')\nabla \operatorname{div}\omega \\ \qquad\qquad +2\nu\nabla \times u + \rho g, \\ H_t - \nabla \times (u \times H) = -\nabla \times (\sigma \nabla \times H), \\ \operatorname{div} H = 0. \end{cases}
$$

Here $\rho(x,t)$ denotes the fluid density, $u(x,t) = (u_1,u_2,u_3)(x,t)$ denotes the fluid velocity field, $\omega(x,t) = (\omega_1,\omega_2,\omega_3)(x,t)$ denotes the micro-rotational velocity, and $H(x,t) = (H_1,H_2,H_3)(x,t)$ denotes the magnetic field. The pressure $P(\rho) = \rho^\gamma$ is a smooth function with the specific heat ratio $\gamma > 1$. The parameters μ, ν, λ, μ', λ' and σ are constants denoting the viscosity coefficients of the flows satisfying

$$\mu,\nu,\mu',\sigma > 0, \quad 2\mu + 3\lambda - 4\nu \geq 0, \quad 2\mu' + 3\lambda' \geq 0.$$

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$f = f(x,t)$ and $g = g(x,t)$ are given external forces periodic in time. In what follows, we always assume that $f = f(x,t)$ and $g = g(x,t)$ satisfy the conditions

$$\begin{aligned} f(x,t+T) &= f(x,t), \qquad g(x,t+T) = g(x,t),\\ f(x,t) &= -f(-x,t), \qquad g(x,t) = -g(-x,t), \end{aligned} \tag{2}$$

for some constant $T > 0$.

Let us first review some previous works about the model (1) and the related models. In particular, when the magnetic field is absent ($H = 0$), then the system (1) reduces to the compressible micropolar fluids system, introduced by Eringen [7] in 1966. The existence of time periodic solutions in the whole space $\mathbb{R}^3$ was obtained by Tan-Xu [18] for some suitable smallness and structure conditions on time periodic forces. On the other hand, when the micro-rotation effects are neglected ($\omega = 0$), the magneto-micropolar system (1) reduces to the compressible magnetohydrodynamic equations (MHD) (see for instance the books [6,15]). For the time periodic solutions, Tan and Wang [19] studied the existence, uniqueness and time-asymptotic stability of time periodic solutions only when the space dimension $n \geq 5$, by using the time decay of the solution operator generated by the linearized system. Later, Cai and Tan [4] established the existence and uniqueness of periodic solutions for MHD system in an n-dimensional periodic domain ($n \geq 1$). For the $3D$ case, based on the topological degree theorem, Cai and Tan [5] showed the existence of time periodic solutions to MHD system in the whole space.

When the magnetic field is absent ($H = 0$) and there is no micro-rotational effects ($\omega = 0$), the compressible magneto-micropolar fluids (1) becomes the classical compressible Navier-Stokes equations, which has been extensively investigated on the existence of time periodic solutions, we refer to [3,8,9,11–14, 20,21] and the references therein. More precisely, based on the energy method and the spectral analysis for the optimal decay estimates on the linearized solution operator, Ma-Ukai-Yang [14] showed that a time periodic solution exists when the space dimension is greater than or equal to 5. Jin and Yang [12] studied the existence of periodic solutions in a periodic domain in $\mathbb{R}^3$. Later, Jin and Yang [11] considered the existence of time periodic solutions to the whole space $\mathbb{R}^3$ through the topological degree theory. By the spectral properties, the authors in [13] obtained a time periodic solution for sufficiently small and symmetry condition on the time periodic external force when the space dimension is greater than or equal to 3. Later, without the symmetry condition on the external force, Tsuda [20] also showed the existence of a time periodic solution of the compressible Navier-Stokes equations on the whole space $\mathbb{R}^n$ ($n \geq 3$).

The full system (1) can be used to describe the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids under the action of magnetic fields, such as water, hydrocarbon, ester, fluorocarbon, etc, which is of great importance in practical and mathematical applications [2,10,16,17]. For multi-dimensional compressible magneto-micropolar equations, Amirat and Hamdache [1] studied the global existence of weak solutions with finite energy.

In [24], a blow-up criterion of strong solutions to $3D$ compressible viscous magneto-micropolar fluids with initial vacuum has been established by Zhang. Later, Wei-Guo-Li [22] studied the global existence and decay rates of smooth solutions under the condition that the initial data are small perturbation of some given constant state. Also see other research results for (1) in [23,25] and the references therein. Recently, the authors in [26] concerned with the time periodic solutions for compressible magnetic-micropolar fluids in a periodic domain. In this paper, we will generalized the result in [26] to the three-dimensional whole space.

Before stating our main results, we need to introduce some function spaces that will be used later. Let L^p, $1 \le p \le \infty$ denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$, and H^s to denote the usual L^2-Sobolev spaces with normal $\|\cdot\|_{H^s}$. Further we put the t–anisotropic Sobolev spaces as

$W_p^{m,k}((0,T)\times\mathbb{R}^3)=\{u : D^\alpha u, D_t^\beta u \in L^p((0,T)\times\mathbb{R}^3) \text{ for any } |\alpha| \le m, |\beta| \le k\}$, endowed with the norm

$$\|u\|_{W_p^{m,k}} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p} + \sum_{|\beta|\le k} \|D_t^\beta u\|_{L^p}.$$

For $0 < \alpha < 1$, denote $C^{\alpha,\frac{\alpha}{2}}((0,T)\times\mathbb{R}^3)$ be the set of all functions u such that $|u|_{\alpha,\frac{\alpha}{2}} < \infty$, where

$$|u|_{\alpha,\frac{\alpha}{2}} = [u]_{\alpha,\frac{\alpha}{2}} + \|u\|_{L^\infty},$$

where $[\cdot]_{\alpha,\frac{\alpha}{2}}$ is the semi–norm defined by

$$[u]_{\alpha,\frac{\alpha}{2}} = \sup_{(x,t)\ne(y,s)} \frac{|u(x,t)-u(y,s)|}{(|x-y|^2+|t-s|)^{\frac{\alpha}{2}}}.$$

With the above notations in hand, we now define some solution spaces.

Definition. The function spaces of solutions in a bounded domain $\Omega^L = (-L,L)^3 \subset \mathbb{R}^3$ and the whole space $\mathbb{R}^3$ are given by

$$S^L = \left\{ (n,u,\omega,H)(x,t) \;\middle|\; \begin{array}{l} (n,u,\omega,H) \in L^\infty(0,T,L^6(\Omega^L)) \text{ satisfies (a), (b), (c);} \\ (n_t,u_t,\omega_t,H_t) \in L^\infty(0,T;L^2(\Omega^L)) \cap L^2(0,T;H^1(\Omega^L)); \\ \nabla n \in L^\infty(0,T;H^1(\Omega^L)) \cap L^2(0,T;H^1(\Omega^L)); \\ (\nabla u,\nabla\omega,\nabla H) \in L^\infty(0,T;H^1(\Omega^L)) \cap L^2(0,T;H^2(\Omega^L)); \end{array} \right\}$$

and

$$S = \left\{ (n,u,\omega,H)(x,t) \;\middle|\; \begin{array}{l} (n,u,\omega,H) \in L^\infty(0,T,L^6(\mathbb{R}^3)) \text{ satisfies (c);} \\ (n_t,u_t,\omega_t,H_t) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)); \\ \nabla n \in L^\infty(0,T;H^1(\mathbb{R}^3))) \cap L^2(0,T;H^1(\mathbb{R}^3)); \\ (\nabla u,\nabla\omega,\nabla H) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)); \end{array} \right\},$$

with

(a) (n,u,ω,H) are time periodic functions with periodic boundary;
(b) $\int_{\Omega^L} n(x,t)\,dx = 0$;
(c) $n(x,t) = n(-x,t)$, $u(x,t) = -u(-x,t)$, $\omega(x,t) = -\omega(-x,t)$, $H(x,t) = -H(-x,t)$.

The norm in S^L or S is defined as

$$\begin{aligned}&|||(n,u,\omega,H)|||^2\\ =&\sup_{0\le t\le T}\left(\|(n,u,\omega,H)\|_{L^6}^2+\|(\rho,u,\omega,H)_t\|_{L^2}^2+\|\nabla(\rho,u,\omega,H)\|_{H^1}^2\right)\\ &+\int_0^T\left(\|(n,u,\omega,H)_t\|_{H^1}^2+\|\nabla\rho\|_{H^1}^2+\|\nabla(u,\omega,H)\|_{H^2}^2\right)dt.\end{aligned}$$

Moreover, we put

$$S_\delta^L=\left\{(n,u,\omega,H)\in S^L:|||(\rho,u,\omega,H)|||<\delta\right\}.$$

The aim of this paper is to show that the problem (1) admits a time periodic solution around the constant state $(\bar\rho,0,0,0)$ in $\mathbb{R}^3$, which has the same period as the external forces. Let $n=\rho-\overline{\rho}$. Then (1) can be reformulated as

$$\text{(3)}\qquad\begin{cases}n_t+\overline{\rho}\operatorname{div}u=-\operatorname{div}(nu),\\ (\overline{\rho}+n)u_t+(\overline{\rho}+n)(u\cdot\nabla)u+P'(\overline{\rho}+n)\nabla n=(\mu+\nu)\Delta u\\ +(\mu+\lambda-\nu)\nabla\operatorname{div}u+2\nu\nabla\times\omega+(\nabla\times H)\times H+(\overline{\rho}+n)f_L,\\ (\overline{\rho}+n)\omega_t+(\overline{\rho}+n)(u\cdot\nabla)\omega+4\nu\omega=\mu'\Delta\omega+(\mu'+\lambda')\nabla\operatorname{div}\omega\\ \qquad\qquad+2\nu\nabla\times u+(\overline{\rho}+n)g_L,\\ H_t-\sigma\Delta H=\nabla\times(u\times H),\quad\operatorname{div}H=0.\end{cases}$$

Our main result in this paper is stated as follows.

Theorem 1.1. *Suppose that the external forces* $(f,g)(x,t)\in L^2((0,T;L^{\frac{6}{5}}(\mathbb{R}^3))$ $\cap W_2^{1,1}((0,T)\times\mathbb{R}^3))$ *and satisfy the conditions* (2). *If*

$$\int_0^T\left(\|(f,g)\|_{L^{\frac{6}{5}}}^2+\|(f,g)\|_{H^1}^4\right)dt+\|(f,g)\|_{W_2^{1,1}}^2\le\delta^*$$

for some small constant $\delta^*>0$*, then there is a constant* $\delta_0>0$ *such that the system* (3) *admits a time periodic solution* $(n,u,\omega,H)\in\mathcal{S}_{\delta_0}$.

Now we outline the main ideas used in proving our main results. Firstly, in the same spirit as [11], we introduce a regularized system (4) and a completely continuous operator. Then by elaborate calculations, a series of uniform estimates on the regularized problem is obtained. Thus, the existence of a time periodic solution in a sequence of bounded domains follows by the topological degree theory. Compared with the work for the Navier-Stokes system [11], the magneto-micropolar fluids are more difficult to deal with because of the strong nonlinearities and interactions among the physical quantities. Finally, letting the sequence tend to the original unbounded domain $\mathbb{R}^3$, we obtain the desired time periodic solution to the original compressible magneto-micropolar fluids under some smallness and structure conditions on the external forces. Here, some Sobolev imbedding estimates with coefficients independent of the domain play an important role in passing the limit of the approximate solutions in the last section.

The rest of this paper is organized as follows. In Section 2, we prove the existence of time-periodic solutions for the regularized problem (4) in a bounded domain by the topological degree theory. The proof of the main theorem will be studied in Section 3.

2. Existence of periodic solutions in a bounded domain

This section is concerned with the following regularized problem:

$$
(4)\qquad \begin{cases} n_t + \overline{\rho}\,\mathrm{div}\, u - \varepsilon\Delta n = -\,\mathrm{div}(nu), \\ (\overline{\rho}+n)u_t + (\overline{\rho}+n)(u\cdot\nabla)u + P'(\overline{\rho}+n)\nabla n = (\mu+\nu)\Delta u \\ +(\mu+\lambda-\nu)\nabla\,\mathrm{div}\, u + 2\nu\nabla\times\omega + (\nabla\times H)\times H + (\overline{\rho}+n)f_L, \\ (\overline{\rho}+n)\omega_t + (\overline{\rho}+n)(u\cdot\nabla)\omega + 4\nu\omega = \mu'\Delta\omega + (\mu'+\lambda')\nabla\,\mathrm{div}\,\omega \\ \qquad\qquad\qquad +2\nu\nabla\times u + (\overline{\rho}+n)g_L, \\ H_t - \sigma\Delta H = \nabla\times(u\times H), \quad \mathrm{div}\, H = 0, \\ \displaystyle\int_{\Omega^L} n\,dx = 0, \end{cases}
$$

where f_L, g_L are sufficiently smooth time periodic functions and odd functions on the space variable x with periodic boundary, satisfying

$$
(f_L, g_L) \to (f, g) \ \text{ in } \ L^2(0,T; L^{\frac{6}{5}}(\mathbb{R}^3)) \cap W_2^{1,1}((0,T)\times\mathbb{R}^3),
$$

and

$$
\int_0^T \left(\|(f_L,g_L)\|^2_{L^{\frac{6}{5}}(\Omega^L)} + \|(f_L,g_L)\|^4_{H^1(\Omega^L)} \right) dt + \|(f_L,g_L)\|^2_{W_2^{1,1}((0,T)\times\Omega^L)}
$$
$$
\le \int_0^T \left(\|(f,g)\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|(f,g)\|^4_{H^1(\mathbb{R}^3)} \right) dt + \|(f,g)\|^2_{W_2^{1,1}((0,T)\times\mathbb{R}^3)}.
$$

Our goal of this section is to prove the existence of time periodic solutions for the regularized problem (4), that is,

Proposition 2.1. *Suppose that the external forces $(f_L, g_L)(x,t) \in L^2(0,T; L^{\frac{6}{5}}(\Omega^L)) \cap W_2^{1,1}((0,T)\times\Omega^L)$. If*

$$
\int_0^T \left(\|(f^L,g^L)\|^2_{L^{\frac{6}{5}}} + \|(f^L,g^L)\|^4_{H^1} \right) dt + \|(f^L,g^L)\|^2_{W_2^{1,1}} \le \delta^*
$$

for some small constant $\delta^* > 0$, then the problem (4) admits a solution $(n_L, u_L, \omega_L, H_L) \in \mathcal{S}^L_{\delta_0}$, where δ_0 is a small constant independent of L and ϵ.

For later use, we state some basic inequalities.

Lemma 2.2 ([11]). *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $\partial\Omega$ is locally Lipschitz continuous. If $u|_{\partial\Omega} = 0$ (or $\int_\Omega u\,dx = 0$), then for any $1 \le p < N$, $1 \le q \le p^* = \frac{Np}{N-p}$,*

$$
\Big(\int_\Omega |u|^q\,dx\Big)^{1/q} \le C(N,p,q)|mes\Omega|^{\frac{1}{q}-\frac{1}{p^*}}\Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}.
$$

In particular, if $q = p^* = \frac{Np}{N-p}$*, then*

$$\Big(\int_\Omega |u|^{p^*}\,dx\Big)^{1/p^*} \le C(N,p,q)\Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}.$$

Lemma 2.3 ([11])**.** *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain, and* $\partial\Omega$ *is locally Lipschitz continuous. If* $u|_{\partial\Omega} = 0$ *(or* $\int_\Omega u\,dx = 0$*), then*

$$\begin{aligned} \|u\|_{L^3} &\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2},\\ \|u\|_{L^4} &\le C\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4},\\ \|u\|_{L^\infty} &\le C\|\nabla u\|_{H^1}, \end{aligned}$$

where C *is independent of* Ω*. Moreover, the above inequalities also hold in* $\mathbb{R}^3$ *if* $u(x) \to 0$ *as* $|x| \to \infty$*.*

In the following, we are devoted to proving Proposition 2.1 by the topological degree theory. To make the presentation easy to follow, we divide this section into several subsections and the first one is to introduce a completely continuous operator $\mathcal{P}$ to problem (5).

2.1. Construction of an operator $\mathcal{P}$

For any $\tau \in [0,1]$, we define an operator

$$\begin{aligned} \mathcal{P}:\ \ &\mathcal{S}_\delta^L \times [0,1] \to \mathcal{S}^L,\\ &((\rho, v, \overline{\omega}, B), \tau) \to (n, u, \omega, H) \end{aligned}$$

with δ being suitably small. Here (n, u, ω, H) is the solution of the following problem:

$$\begin{cases} n_t + \overline{\rho}\,\mathrm{div}\, u - \varepsilon\Delta n = G_1(\rho, v, \tau),\\ (\overline{\rho} + \tau\rho)u_t - (\mu+\nu)\Delta u - (\mu+\lambda-\nu)\nabla\,\mathrm{div}\, u\\ \quad + \dfrac{P'(\overline{\rho})}{\overline{\rho}}(\overline{\rho}+\tau\rho)\nabla n = G_2(\rho, v, \overline{\omega}, \tau) + \tau(\overline{\rho}+\tau\rho)f_L,\\ (\overline{\rho}+\tau\rho)\omega_t - \mu'\Delta\omega - (\mu'+\lambda')\nabla\,\mathrm{div}\,\omega + 4\nu\omega = G_3(\rho, v, \overline{\omega}, \tau)\\ \qquad\qquad + \tau(\overline{\rho}+\tau\rho)g_L,\\ H_t - \sigma\Delta H = G_4(v, B, \tau), \quad \mathrm{div}\, H = 0,\\ \displaystyle\int_{\Omega^L} n\,dx = 0, \end{cases} \tag{5}$$

with

$$\begin{aligned} G_1(\rho, v, \tau) &= -\tau\,\mathrm{div}(\rho v),\\ G_2(\rho, v, \overline{\omega}, \tau) &= 2\tau\nu\nabla\times\overline{\omega} - \tau(\overline{\rho}+\tau\rho)(v\cdot\nabla)v\\ &\quad + (\overline{\rho}+\tau\rho)\left(\frac{P'(\overline{\rho})}{\overline{\rho}} - \frac{P'(\overline{\rho}+\tau\rho)}{\overline{\rho}+\tau\rho}\right)\nabla\rho + \tau(\nabla\times B)\times B,\\ G_3(\rho, v, \overline{\omega}, \tau) &= 2\tau\nu\nabla\times v - (\overline{\rho}+\tau\rho)(v\cdot\nabla)\overline{\omega}, \quad G_4(v, B, \tau) = \tau\nabla\times(v\times B). \end{aligned}$$

Remark 2.4. The condition $\int_{\Omega^L} ndx = 0$ is given to ensure the uniqueness of solutions. In fact, notice that $\frac{d}{dt}\int_{\Omega^L} ndx = 0$, if (n, u, ω, H) is a solution for the problem (5), then $(n + c, u, \omega, H)$ is also a solution with any constant c.

Now we show that the operator $\mathcal{P}$ is completely continuous. For this purpose, we first give the following lemma which implies that $\mathcal{P}$ is well-defined.

Lemma 2.5. *If δ is sufficiently small, then for any $(\rho, v, \overline{\omega}, B) \in \mathcal{S}^L_\delta$, $\tau \in [0, 1]$, the problem* (5) *admits a unique time periodic solution $(n, u, \omega, H) \in \mathcal{S}^L$.*

Proof. By Lemma 2.2, we have

$$\|\rho\|_{L^\infty} \le C\|\nabla\rho\|_{H^1} \le C\delta. \tag{6}$$

Choosing δ suitable small gives that

$$\frac{1}{2\overline{\rho}} \le \frac{1}{\overline{\rho} + \tau\rho} \le \frac{2}{\overline{\rho}}.$$

Define the operator

$$\mathbb{A} = \begin{pmatrix} \varepsilon\Delta & -\overline{\rho}\operatorname{div} & 0 & 0 \\ -\frac{P'(\overline{\rho})}{\overline{\rho}}\nabla & \frac{\mu+\nu}{\tau\rho+\overline{\rho}}\Delta + \frac{\mu+\lambda-\nu}{\tau\rho+\overline{\rho}}\nabla\operatorname{div} & 0 & 0 \\ 0 & 0 & \frac{\mu'}{\tau\rho+\overline{\rho}}\Delta + \frac{\mu'+\lambda'}{\tau\rho+\overline{\rho}}\nabla\operatorname{div} - \frac{4\nu}{\tau\rho+\overline{\rho}} & 0 \\ 0 & 0 & 0 & \sigma\Delta \end{pmatrix}$$

and set $U = (n, u, \omega, H)$, $W = (\rho, v, \overline{\omega}, B)$, $G(W) = (G_1, \frac{G_2}{\overline{\rho}+\tau\rho}, \frac{G_3}{\overline{\rho}+\tau\rho}, G_4)$, $F = (0, \tau f_L, \tau g_L, 0)$, then the system (5) can be written as

$$U_t = \mathbb{A}U + G(W) + F.$$

Investigate the corresponding homogenous linear system $U_t = \mathbb{A}U$ of (5), that is consider the following initial value problem in Ω^L with periodic boundary

$$\begin{cases} n_t + \overline{\rho}\operatorname{div} u - \varepsilon\Delta n = 0, \\ u_t - \dfrac{\mu+\nu}{\tau\rho+\overline{\rho}}\Delta u - \dfrac{\mu+\lambda-\nu}{\tau\rho+\overline{\rho}}\nabla\operatorname{div} u + \dfrac{P'(\overline{\rho})}{\overline{\rho}}\nabla n = 0, \\ \omega_t - \dfrac{\mu'}{\tau\rho+\overline{\rho}}\Delta\omega - \dfrac{\mu'+\lambda'}{\tau\rho+\overline{\rho}}\nabla\operatorname{div}\omega + \dfrac{4\nu}{\tau\rho+\overline{\rho}}\omega = 0, \\ H_t - \sigma\Delta H = 0, \quad \operatorname{div} H = 0, \\ (n, u, \omega, H)(x, 0) = (n_0, u_0, \omega_0, H_0), \end{cases} \tag{7}$$

where $n_0(x)$ is an even function with $\int_{\Omega^L} n_0 dx = 0$, and $u_0(x)$, $\omega_0(x)$, $H_0(x)$ are odd functions. It is not difficult to prove that this properties remain unchanged for the solution of the problem (7).

Multiplying the second equation in (7) by u, and then integrating it over Ω^L we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^L} |u|^2 dx + \int_{\Omega^L} \left(\frac{\mu+\nu}{\tau\rho+\overline{\rho}}|\nabla u|^2 + \frac{\mu+\lambda-\nu}{\tau\rho+\overline{\rho}}|\operatorname{div} u|^2\right) dx$$

$$
\begin{aligned}
&+\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}\nabla n u dx\\
=&\int_{\Omega^L}\frac{\tau(\mu+\nu)}{(\tau\rho+\overline{\rho})^2}\nabla u\nabla\rho u dx+\int_{\Omega^L}\frac{\tau(\mu+\lambda-\nu)}{(\tau\rho+\overline{\rho})^2}\operatorname{div}u\nabla\rho u dx\\
\le&\frac{\tau(2\mu+\lambda)}{(\overline{\rho}-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6}\\
\le& C\frac{\tau(2\mu+\lambda)}{(\overline{\rho}-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{H^1}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where C is a constant independent of Ω^L. By (6), for sufficiently small δ, we see that

$$
\begin{aligned}
&\int_{\Omega^L}\left(\frac{\mu+\nu}{3\overline{\rho}}|\nabla u|^2+\frac{\mu+\lambda-\nu}{3\overline{\rho}}|\operatorname{div}u|^2dx+\frac{P'(\overline{\rho})}{\overline{\rho}}\nabla n u\right)dx\\
&+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|u|^2dx\le 0.
\end{aligned}
\tag{8}
$$

Similarly, multiplying the third equation in (7) by ω, and integrating it over Ω^L we deduced that

$$
\int_{\Omega^L}\left(\frac{\mu'}{3\overline{\rho}}|\nabla\omega|^2+\frac{\mu'+\lambda'}{3\overline{\rho}}|\operatorname{div}\omega|^2+\frac{2\nu}{\overline{\rho}}|\omega|^2\right)dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\omega|^2dx\le 0.
\tag{9}
$$

Then multiplying the first and the forth equations in (7) by $\frac{P'(\overline{\rho})}{\overline{\rho}^2}n$ and H, respectively, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}^2}|n|^2+|H|^2\right)dx\\
&+\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}}n\operatorname{div}u+\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}^2}|\nabla n|^2+\sigma|\nabla H|^2\right)dx=0.
\end{aligned}
\tag{10}
$$

Thus it follows from (8)–(10) that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}^2}|n|^2+|u|^2+|\omega|^2+|H|^2\right)dx+2\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}^2}\int_{\Omega^L}|\nabla n|^2\,dx\\
&+2\int_{\Omega^L}\sigma|\nabla H|^2\,dx+\frac{2}{3\overline{\rho}}\int_{\Omega^L}\big((\mu+\nu)|\nabla u|^2+(\mu+\lambda-\nu)|\operatorname{div}u|^2\\
&+\mu'|\nabla\omega|^2+(\mu'+\lambda')|\operatorname{div}\omega|^2+4\nu|\omega|^2\big)\,dx\le 0.
\end{aligned}
\tag{11}
$$

Moreover, multiplying the second equation of (7) by $(\mu+\nu)\Delta u+(\mu+\lambda-\nu)\nabla\operatorname{div}u$, and integrating it over Ω^L, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\big((\mu+\nu)|\nabla u|^2+(\mu+\lambda-\nu)|\operatorname{div}u|^2\big)\,dx\\
&+\int_{\Omega^L}\frac{1}{\overline{\rho}+\tau\rho}\left((\mu+\nu)\Delta u+(\mu+\lambda-\nu)\nabla\operatorname{div}u\right)^2dx\\
&-(2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}\nabla n\nabla\operatorname{div}u\,dx=0.
\end{aligned}
\tag{12}
$$

Multiplying the third equations by $\mu'\Delta\omega + (\mu'+\lambda')\nabla \operatorname{div}\omega$, and integrating it to obtain

$$
\begin{aligned}
(13)\qquad & \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\mu'|\nabla\omega|^2 + (\mu'+\lambda')|\operatorname{div}\omega|^2\right)dx \\
& + \int_{\Omega^L}\frac{1}{\overline{\rho}+\tau\rho}\left(\mu'\Delta\omega + (\mu'+\lambda')\nabla\operatorname{div}\omega\right)^2 dx \\
& + \int_{\Omega^L}\frac{4\nu}{\overline{\rho}+\tau\rho}(\mu'|\nabla\omega|^2 + (\mu'+\lambda')|\operatorname{div}\omega|^2)dx \\
= & \int_{\Omega^L}\frac{4\tau\nu}{(\overline{\rho}+\tau\rho)^2}\nabla\rho(\mu'\nabla\omega + (\mu'+\lambda')\operatorname{div}\omega)\omega dx \\
\le & \, C\frac{\tau\nu}{(\overline{\rho}-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{L^3}\|\nabla\omega\|_{L^2}\|\omega\|_{L^6} \\
\le & \, C\|\nabla\rho\|_{H^1}\|\nabla\omega\|_{L^2}^2 \le C\delta\|\nabla\omega\|_{L^2}^2.
\end{aligned}
$$

Applying ∇ to the first equation and the fourth equation of (7), and multiplying the resultant identities by $(2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}^2}\nabla n$ and ∇H, respectively, we obtain

$$
\begin{aligned}
(14)\qquad & \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left((2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}^2}|\nabla n|^2 + |\nabla H|^2\right)dx \\
& + \int_{\Omega^L}(2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}}\nabla n\nabla\operatorname{div} u dx \\
& + \int_{\Omega^L}\left(\varepsilon(2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}^2}|\Delta n|^2 + \sigma|\Delta H|^2\right)dx = 0.
\end{aligned}
$$

Combining (12)–(14), we derive

$$
\begin{aligned}
(15)\qquad & \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left((\mu+\nu)|\nabla u|^2 + (\mu+\lambda-\nu)|\operatorname{div} u|^2 + \mu'|\nabla\omega|^2\right. \\
& \left. +(\mu'+\lambda')|\operatorname{div}\omega|^2 + |\nabla H|^2 + (2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}^2}|\nabla n|^2\right)dx \\
& + \int_{\Omega^L}\frac{1}{\overline{\rho}+\tau\rho}((\mu+\nu)\Delta u + (\mu+\lambda-\nu)\nabla\operatorname{div} u)^2 dx \\
& + \int_{\Omega^L}\frac{1}{\overline{\rho}+\tau\rho}(\mu'\Delta\omega + (\mu'+\lambda')\nabla\operatorname{div}\omega)^2 dx \\
& + \int_{\Omega^L}\frac{4\nu}{\overline{\rho}+\tau\rho}\left(\frac{\mu'}{2}|\nabla\omega|^2 + (\mu'+\lambda')|\operatorname{div}\omega|^2\right)dx \\
& + \sigma\int_{\Omega^L}|\Delta H|^2\,dx + \varepsilon(2\mu+\lambda)\frac{P'(\overline{\rho})}{\overline{\rho}^2}\int_{\Omega^L}|\Delta n|^2\,dx \le 0.
\end{aligned}
$$

By the Poincaré inequality, (11) and (15), we can get

$$
\frac{d}{dt}\|(n,u,\omega,H)(t)\|_{H^1} + C\varepsilon\|(n,u,\omega,H)(t)\|_{H^1} \le 0,
$$

which means that

$$\|(n,u,\omega,H)(x,t)\|_{H^1}\le \|(n_0,u_0,\omega_0,H_0)\|_{H^1}e^{-C\varepsilon t}.$$

Next, by Duhamel's principle, the solution to the system (5) can be written in a mild form as

$$U(t)=\int_{-\infty}^{t} e^{(t-s)\mathbb{A}}\left(G(W)(s)+F(s)\right)ds.$$

Moreover, $U(t)$ satisfies that

$$\begin{aligned}\|U(t)\|_{H^1}&\le \int_{-\infty}^{t}\|e^{(t-s)\mathbb{A}}\left(G(W)(s)+F(s)\right)\|_{H^1}ds\\&\le \int_{-\infty}^{t} e^{-C\varepsilon(t-s)}\|\left(G(W)(s)+F(s)\right)\|_{H^1}ds\\&\le C_\varepsilon\left(\int_0^T\|\left(G(W)(s)+F(s)\right)\|_{H^1}^2ds\right)^{\frac12},\end{aligned}$$

where we have used the time periodic property of W and F, also we have

$$\begin{aligned}U(t+T)&=\int_{-\infty}^{t+T} e^{(t+T-s)\mathbb{A}}\left(G(W)(s)+F(s)\right)ds\\&=\int_{-\infty}^{t+T} e^{(t-(s-T))\mathbb{A}}\left(G(W)(s-T)+F(s-T)\right)ds\\&=\int_{-\infty}^{t} e^{(t-s)\mathbb{A}}\left(G(W)(s)+F(s)\right)ds=U(t),\end{aligned}$$

which means that $(n,u,\omega,H)\in L^\infty(0,T;H^1(\Omega^L))$ is a time periodic solution of (5) with time period T.

Assume that the problem (5) has two solutions $U_1=(n_1,u_1,\omega_1,H_1)$ and $U_2=(n_2,u_2,\omega_2,H_2)$ for some $(\rho,v,\overline{\omega},B)\in\mathcal{S}^L_\delta$. Then we have

$$(U_1-U_2)_t=\mathbb{A}(U_1-U_2).$$

Let $\widetilde{n}=n_1-n_2$, $\widetilde{u}=u_1-u_2$, $\widetilde{\omega}=\omega_1-\omega_2$, $\widetilde{H}=H_1-H_2$. Then we easily check from (8)–(11) that

$$\begin{aligned}&\frac{2}{3\overline{\rho}}\int_0^T\int_{\Omega^L}\big((\mu+\nu)|\nabla\widetilde{u}|^2+(\mu+\lambda-\nu)|\operatorname{div}\widetilde{u}|^2+\mu'|\nabla\widetilde{\omega}|^2+(\mu'+\lambda')|\operatorname{div}\widetilde{\omega}|^2\\&+4\nu|\widetilde{\omega}|^2\big)\,dxdt+2\int_0^T\int_{\Omega^L}\left(\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}^2}|\nabla\widetilde{n}|^2+\sigma|\nabla\widetilde{H}|^2\right)dxdt\le 0.\end{aligned}$$

By the Poincaré inequality, we have $(n_1-n_2,u_1-u_2,\omega_1-\omega_2,H_1-H_2)=(0,0,0,0)$ which implies $U_1=U_2$.

On the other hand, by the classical theory of the parabolic equation, the system (5) admits a unique time periodic solution $(n,u,\omega,H)\in S^L$ if $G_1,G_2+\tau(\overline{\rho}+\tau\rho)f_L,G_3+\tau(\overline{\rho}+\tau\rho)g_L,G_4\in W_2^{1,1}((0,T)\times\Omega^L)$. Then when $(\rho,v,\overline{\omega},B)\in$

S^L_δ, (n, u, ω, H) is in a sub-space of S^L and is the unique solution of the problem (5).

In addition, if $(n(x,t), u(x,t), \omega(x,t), H(x,t))$ is the periodic solution of (5), then $(n(-x,t), -u(-x,t), -\omega(-x,t), -H(-x,t))$ is also the periodic solution of (5). By the uniqueness, we easily obtain that $(n(x,t), u(x,t), \omega(x,t), H(x,t)) = (n(-x,t), -u(-x,t), -\omega(-x,t), -H(-x,t))$. This completes the proof of this lemma. □

Similar to the proof of Lemma 2.4 and Lemma 2.5 in [26], we have that the operator $\mathcal{P}$ is completely continuous.

Lemma 2.6. *If δ is sufficiently small, then the operator $\mathcal{P}$ is compact and continuous.*

2.2. Energy estimates

With the above preparations in hand, we now turn to give a series of uniform estimates on the solutions to the following system:

$$
\begin{cases}
n_t + \overline{\rho}\operatorname{div} u - \varepsilon\Delta n = -\tau\operatorname{div}(nu),\\
(\overline{\rho}+\tau n)u_t - (\mu+\nu)\Delta u - (\mu+\lambda-\nu)\nabla\operatorname{div} u + P'(\overline{\rho}+\tau n)\nabla n\\
= 2\tau\nu\nabla\times\omega - \tau(\tau n+\overline{\rho})(u\cdot\nabla)u + \tau(\nabla\times H)\times H + \tau(\overline{\rho}+\tau n)f_L,\\
(\overline{\rho}+\tau n)\omega_t - \mu'\Delta\omega - (\mu'+\lambda')\nabla\operatorname{div}\omega + 4\nu\omega\\
= 2\tau\nu\nabla\times u - \tau(\overline{\rho}+\tau n)(u\cdot\nabla)\omega + \tau(\overline{\rho}+\tau n)g_L,\\
H_t - \sigma\Delta H = \tau\nabla\times(u\times H), \quad \operatorname{div} H = 0,\\
\displaystyle\int_{\Omega^L} n dx = 0.
\end{cases}
\tag{16}
$$

Similar to the proof of uniqueness in Lemma 2.5, we can show that $n = u = \omega = H = 0$ when $\tau = 0$. Thus, in the following, we only investigate the case when $\tau \in (0,1]$.

Lemma 2.7. *Assume that $\tau \in (0,1]$. If $(n, u, \omega, H) \in S^L$ is a solution to the system (16) and $|n| \le \frac{\overline{\rho}}{2}$, then it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^L}\left((\overline{\rho}+\tau n)(u^2+\omega^2) + \frac{2}{\tau^2(\gamma-1)}P(\overline{\rho}+\tau n) + H^2\right)dx\\
&\quad + \int_{\Omega^L}\left(\mu'|\nabla\omega|^2 + \mu|\nabla u|^2\right)dx + \frac{\varepsilon}{(\gamma-1)}\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2 dx\\
&\quad + 2\int_{\Omega^L}\left((\mu'+\lambda')|\operatorname{div}\omega|^2 + (\mu+\lambda-\nu)|\operatorname{div} u|^2 + \sigma|\nabla H|^2\right)dx\\
&\le C\tau\varepsilon(\|\nabla u\|^3_{L^2}\|\Delta u\|_{L^2} + \|\nabla\omega\|^3_{L^2}\|\Delta\omega\|_{L^2}) + C\tau\|f_L\|^2_{L^{\frac{6}{5}}} + C\tau\|g_L\|^2_{L^{\frac{6}{5}}},
\end{aligned}
\tag{17}
$$

where C is a constant independent of L and ε.

Proof. Multiplying the second equation of (16) by u, then integrating it over Ω^L, and combining the periodic boundary conditions, we have

$$
\frac{1}{2}\int_{\Omega^L}(\overline{\rho}+\tau n)\frac{d}{dt}u^2 dx + \int_{\Omega^L}\left((\mu+\nu)|\nabla u|^2 + (\mu+\lambda-\nu)|\operatorname{div} u|^2\right)dx
\tag{18}
$$

$$
\begin{aligned}
&+\int_{\Omega^L} P'(\overline{\rho}+\tau n)\nabla n\cdot udx+\int_{\Omega^L}\tau(\overline{\rho}+\tau n)u\nabla u\cdot udx\\
&=2\tau\nu\int_{\Omega^L}\operatorname{curl}\omega\cdot udx+\int_{\Omega^L}\tau(\nabla\times H)\times H\cdot udx+\int_{\Omega^L}\tau(\overline{\rho}+\tau n)f_L udx.
\end{aligned}
$$

It follows from the first equation of (16) that

$$
\begin{aligned}
\int_{\Omega^L}\tau(\overline{\rho}+\tau n)u\nabla u\cdot udx&=\int_{\Omega^L}\tau(\overline{\rho}+\tau n)u\nabla\left(\frac{u^2}{2}\right)dx\\
&=-\frac{\tau}{2}\int_{\Omega^L}\left[\tau\nabla nu+(\overline{\rho}+\tau n)\operatorname{div}u\right]\cdot u^2dx\\
&=\frac{\tau}{2}\int_{\Omega^L}n_tu^2dx-\frac{\tau\varepsilon}{2}\int_{\Omega^L}\Delta nu^2dx.
\end{aligned}\tag{19}
$$

From (18) and (19) it follows

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(\overline{\rho}+\tau n)u^2dx+\int_{\Omega^L}\left((\mu+\nu)|\nabla u|^2+(\mu+\lambda-\nu)|\operatorname{div}u|^2\right)dx\\
&\quad+\int_{\Omega^L}P'(\overline{\rho}+\tau n)\nabla n\cdot udx\\
&=\tau\left[\int_{\Omega^L}\frac{\varepsilon u^2}{2}\Delta ndx+2\nu\int_{\Omega^L}\operatorname{curl}\omega\cdot udx+\int_{\Omega^L}(\overline{\rho}+\tau n)f_Ludx\right]\\
&\quad-\tau\int_{\Omega^L}(H^T\nabla uH+\frac{1}{2}\nabla(H^2)\cdot u)dx.
\end{aligned}\tag{20}
$$

Similarly, multiplying the third equation of (16) by ω, then integrating it over Ω^L, and combining the first equation of (16) and the periodic boundary conditions, we can deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(\overline{\rho}+\tau n)\omega^2dx+\int_{\Omega^L}\left(\mu'|\nabla\omega|^2+(\mu'+\lambda')|\operatorname{div}\omega|^2+4\nu\omega^2\right)dx\\
&=\frac{\tau\varepsilon}{2}\int_{\Omega^L}\Delta n\omega^2dx+2\tau\nu\int_{\Omega^L}\operatorname{curl}u\cdot\omega dx+\int_{\Omega^L}\tau(\overline{\rho}+\tau n)g_L\omega dx.
\end{aligned}\tag{21}
$$

Multiplying the first equation of (16) by $P'(\overline{\rho}+\tau n)$, then integrating it over Ω^L, we obtain

$$
\begin{aligned}
&\frac{1}{\tau}\frac{d}{dt}\int_{\Omega^L}P(\overline{\rho}+\tau n)dx+\varepsilon\tau\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2dx\\
&=\tau\overline{\rho}\int_{\Omega^L}P''(\overline{\rho}+\tau n)\nabla n\cdot udx+\tau^2\int_{\Omega^L}P''(\overline{\rho}+\tau n)\nabla n\cdot nudx\\
&=\tau(\gamma-1)\int_{\Omega^L}P'(\overline{\rho}+\tau n)\nabla n\cdot udx.
\end{aligned}\tag{22}
$$

Multiplying the fourth equation of (16) by H, then integrating it over Ω^L, we get

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}H^2dx+\sigma\int_{\Omega^L}|\nabla H|^2dx=\tau\int_{\Omega^L}\nabla\times(u\times H)\cdot Hdx\tag{23}
$$

$$= \tau \int_{\Omega^L} (H^T \nabla u H + \frac{1}{2}\nabla(H^2)\cdot u) dx.$$

Combining (20)–(23), we can arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left((\overline{\rho}+\tau n)(u^2+\omega^2)+\frac{2}{\tau^2(\gamma-1)}P(\overline{\rho}+\tau n)+H^2\right)dx\\
&+\int_{\Omega^L}\left(\mu'|\nabla\omega|^2+(\mu'+\lambda')|\operatorname{div}\omega|^2+4\nu\omega^2\right)dx\\
&+\int_{\Omega^L}\left((\mu+\nu)|\nabla u|^2+(\mu+\lambda-\nu)|\operatorname{div}u|^2+\sigma|\nabla H|^2\right)dx\\
&+\frac{\varepsilon}{\gamma-1}\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2dx\\
= &-\frac{\tau\varepsilon}{2}\int_{\Omega^L}\nabla(u^2+\omega^2)\nabla n dx+4\tau\nu\int_{\Omega^L}\operatorname{curl}u\cdot\omega dx\\
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)(f_L u+g_L\omega)\,dx\\
\le &\frac{\varepsilon}{2(\gamma-1)}\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2dx+C\tau\varepsilon\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2\\
&+C\tau\varepsilon\|\omega\|_{L^6}^2\|\nabla\omega\|_{L^3}^2+\tau\nu\|\nabla u\|_{L^2}^2+4\tau\nu\|\omega\|_{L^2}^2\\
&+C\tau\|f_L\|_{L^{\frac{6}{5}}}\|u\|_{L^6}+C\tau\|g_L\|_{L^{\frac{6}{5}}}\|\omega\|_{L^6}\\
\le &\frac{\varepsilon}{2(\gamma-1)}\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2dx+C\tau\varepsilon\|\nabla u\|_{L^2}^3\|\Delta u\|_{L^2}\\
&+C\tau\varepsilon\|\nabla\omega\|_{L^2}^3\|\Delta\omega\|_{L^2}+\tau\nu\|\nabla u\|_{L^2}^2+4\tau\nu\|\omega\|_{L^2}^2+C\tau\|f_L\|_{L^{\frac{6}{5}}}^2\\
&+C\tau\|g_L\|_{L^{\frac{6}{5}}}^2+\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\frac{\mu'}{2}\|\nabla\omega\|_{L^2}^2.
\end{aligned}$$

This implies the estimate (17) immediately. □

Lemma 2.8. *Under the assumptions in Lemma 2.7, we have*

$$\begin{aligned}
&\int_{\Omega^L}\left[(\overline{\rho}+\tau n)(u_t^2+\omega_t^2)+H_t^2\right]dx+\frac{d}{dt}\int_{\Omega^L}\left((\mu+\nu)|\nabla u|^2\right.\\
&\left.+(\mu+\lambda-\nu)|\operatorname{div}u|^2\right)dx+\frac{d}{dt}\int_{\Omega^L}\left(\sigma|\nabla H|^2+\mu'|\nabla\omega|^2\right.\\
&\left.+(\mu'+\lambda')|\operatorname{div}\omega|^2+4\nu\omega^2-\frac{2}{\tau}P(\overline{\rho}+\tau n)\operatorname{div}u\right)dx\\
\le\ &C\|\nabla u\|_{L^2}^2+C\tau\|\nabla\omega\|_{L^2}^2+\varepsilon^2\|\Delta n\|_{L^2}^2+C\tau\|\nabla n\|_{L^2}^4\|\nabla^2u\|_{L^2}^2\\
&+C\tau\|\nabla^2u\|_{L^2}\|\nabla u\|_{L^2}^3+C\tau\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2+C\tau\|f_L\|_{L^2}^2\\
&+C\tau\|g_L\|_{L^2}^2+C\tau\|\nabla^2H\|_{L^2}\|\nabla H\|_{L^2}^3+C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2,
\end{aligned}\tag{24}$$

where C is a constant independent of L and ε.

Proof. Multiplying the first equation of (16) by $P'(\overline{\rho}+\tau n)\operatorname{div} u$, and integrating it over Ω^L, one has

$$\begin{aligned}\int_{\Omega^L} P'(\overline{\rho}+\tau n)n_t \operatorname{div} u dx &+ \int_{\Omega^L} (\overline{\rho}+\tau n)P'(\overline{\rho}+\tau n)|\operatorname{div} u|^2 dx \\ &- \varepsilon \int_{\Omega^L} P'(\overline{\rho}+\tau n)\Delta n \operatorname{div} u dx \\ = &- \tau \int_{\Omega^L} P'(\overline{\rho}+\tau n)\nabla n \cdot u \operatorname{div} u \, dx.\end{aligned} \tag{25}$$

Moreover, multiplying the last three equation of (16) by u_t, ω_t and H_t, respectively, then integrating them over Ω^L, we can deduce that

$$\begin{aligned}&\int_{\Omega^L} [\overline{\rho}+\tau n)((u_t^2+\omega_t^2)+H_t^2]\, dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega^L} \big((\mu+\nu)|\nabla u|^2 \\ &+(\mu+\lambda-\nu)|\operatorname{div} u|^2\big)\, dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega^L} \big(\mu'|\nabla\omega|^2 + (\mu'+\lambda')|\operatorname{div}\omega|^2 \\ &+4\nu\omega^2 + \sigma|\nabla H|^2\big)\, dx - \int_{\Omega^L} \frac{1}{\tau} P(\overline{\rho}+\tau n)\operatorname{div} u_t dx \\ = &\, 2\tau\nu \int_{\Omega^L} \operatorname{curl}\omega \cdot u_t dx - \int_{\Omega^L} \tau(\overline{\rho}+\tau n)(u\cdot\nabla u)\cdot u_t dx \\ &+ \tau \int_{\Omega^L} (\nabla\times H)\times H \cdot u_t dx + \int_{\Omega^L} \tau(\overline{\rho}+\tau n) f_L u_t dx \\ &+ 2\tau\nu \int_{\Omega^L} \operatorname{curl} u \cdot \omega_t dx - \int_{\Omega^L} \tau(\overline{\rho}+\tau n)(u\cdot\nabla\omega)\cdot\omega_t dx \\ &+ \int_{\Omega^L} \tau(\overline{\rho}+\tau n) g_L \omega_t dx + \tau \int_{\Omega^L} \nabla\times(u\times H)\cdot H_t dx.\end{aligned} \tag{26}$$

In light of (25)–(26), we see that

$$\begin{aligned}&\int_{\Omega^L} (\overline{\rho}+\tau n)[(u_t^2+\omega_t^2)+H_t^2]\, dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega^L} \big((\mu+\nu)|\nabla u|^2 \\ &+(\mu+\lambda-\nu)|\operatorname{div} u|^2\big)\, dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega^L} \big(\mu'|\nabla\omega|^2 + (\mu'+\lambda')|\operatorname{div}\omega|^2 \\ &+ 4\nu\omega^2 + \sigma|\nabla H|^2 - \frac{2}{\tau}P(\overline{\rho}+\tau n)\operatorname{div} u\big)\, dx \\ = &\, \gamma \int_{\Omega^L} P(\overline{\rho}+\tau n)|\operatorname{div} u|^2 dx - \varepsilon \int_{\Omega^L} P'(\overline{\rho}+\tau n)\Delta n \operatorname{div} u dx \\ &+ \tau \int_{\Omega^L} P'(\overline{\rho}+\tau n)\nabla n \cdot u \operatorname{div} u dx + 2\tau\nu \int_{\Omega^L} \operatorname{curl}\omega \cdot u_t dx \\ &- \int_{\Omega^L} \tau(\overline{\rho}+\tau n)(u\cdot\nabla u)\cdot u_t dx + 2\tau\nu \int_{\Omega^L} \operatorname{curl} u \cdot \omega_t dx \\ &- \int_{\Omega^L} \tau(\overline{\rho}+\tau n)(u\cdot\nabla\omega)\cdot\omega_t dx + \int_{\Omega^L} \tau(\overline{\rho}+\tau n) f_L u_t dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega^L}\tau(\overline{\rho}+\tau n)g_L\omega_t dx+\tau\int_{\Omega^L}\nabla\times(u\times H)\cdot H_t dx\\
&+\tau\int_{\Omega^L}(\nabla\times H)\times H\cdot u_t dx\\
\le\ & C\|\operatorname{div}u\|_{L^2}^2+C\varepsilon\|\Delta n\|_{L^2}\|\operatorname{div}u\|_{L^2}+C\tau\|\nabla n\|_{L^2}\|u\|_{L^6}\|\operatorname{div}u\|_{L^3}\\
&+\frac{1}{2}\int_{\Omega^L}(\overline{\rho}+\tau n)(u_t^2+\omega_t^2)dx+C\tau\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2+C\tau\|u\|_{L^6}^2\|\nabla\omega\|_{L^3}^2\\
&+C\tau\|\nabla u\|_{L^2}^2+C\tau\|\nabla\omega\|_{L^2}^2+C\tau\|f_L\|_{L^2}^2+C\tau\|g_L\|_{L^2}^2\\
&+C\tau\|\nabla H\|_{L^3}^2\|H\|_{L^6}^2+C\tau\|\nabla H\|_{L^2}^2\|u\|_{L^\infty}^2\\
&+C\tau\|\nabla u\|_{L^2}^2\|H\|_{L^\infty}^2+\frac{1}{2}\int_{\Omega^L}H_t^2dx\\
\le\ & \frac{1}{2}\int_{\Omega^L}(\overline{\rho}+\tau n)(u_t^2+\omega_t^2)dx+\frac{1}{2}\int_{\Omega^L}H_t^2dx+C\|\nabla u\|_{L^2}^2+\varepsilon^2\|\Delta n\|_{L^2}^2\\
&+C\tau\|\nabla n\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2+C\tau\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^3+C\tau\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2\\
&+C\tau\|\nabla\omega\|_{L^2}^2+C\tau\|f_L\|_{L^2}^2+C\tau\|g_L\|_{L^2}^2\\
&+C\tau\|\nabla^2 H\|_{L^2}\|\nabla H\|_{L^2}^3+C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2.
\end{aligned}
$$

Hence, the estimate (24) follows from the above inequality immediately. □

Lemma 2.9. *Under the assumptions in Lemma 2.7, one has*

$$
\begin{aligned}
(27)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\overline{\rho}(|\nabla u|^2+|\nabla\omega|^2)+|\nabla H|^2+\frac{P'(\overline{\rho})}{\overline{\rho}}|\nabla n|^2\right)dx\\
&+\frac{1}{2}\int_{\Omega^L}(\mu|\Delta u|^2+\sigma|\Delta H|^2)dx+\int_{\Omega^L}\Big((\mu+\lambda-\nu)|\nabla\operatorname{div}u|^2\\
&+\frac{\mu'}{2}|\Delta\omega|^2+(\mu'+\lambda')|\nabla\operatorname{div}\omega|^2\Big)dx+\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}|\Delta n|^2dx\\
\le\ & C\tau\|\nabla n\|_{L^2}^3\|\nabla^2 n\|_{L^2}+C\tau\|\nabla n\|_{H^1}^2\|u_t\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla u\|_{L^2}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|\omega_t\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{L^2}^2+C\tau\|\nabla H\|_{H^1}^2\|\nabla H\|_{L^2}^2\\
&+C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2+C\tau\|f_L\|_{L^2}^2+C\tau\|g_L\|_{L^2}^2,
\end{aligned}
$$

where C is a constant independent of L and ε.

Proof. Applying ∇ to the first equation of (16), and taking the L^2 inner product with ∇n, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla n|^2dx+\int_{\Omega^L}\overline{\rho}\nabla n\nabla\operatorname{div}udx+\varepsilon\int_{\Omega^L}|\Delta n|^2dx\\
=\ &-\tau\int_{\Omega^L}\nabla\operatorname{div}(nu)\nabla ndx\\
=\ &-\tau\int_{\Omega^L}\left(\frac{1}{2}|\nabla n|^2\operatorname{div}u+\nabla n\nabla u\nabla n+n\nabla n\nabla\operatorname{div}u\right)dx.
\end{aligned}
$$

Multiplying the second, the third and the forth equation of (16) by Δu, $\Delta\omega$ and ΔH, respectively, and then integrating them over Ω^L, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\overline{\rho}(|\nabla u|^2+|\nabla\omega|^2)+|\nabla H|^2\right)dx+\int_{\Omega^L}\big((\mu+\nu)|\Delta u|^2\\
&+(\mu+\lambda-\nu)|\nabla\operatorname{div}u|^2\big)\,dx+\int_{\Omega^L}\big(\mu'|\Delta\omega|^2+(\mu'+\lambda')|\nabla\operatorname{div}\omega|^2\\
&+4\nu|\nabla\omega|^2+\sigma|\Delta H|^2\big)\,dx\\
=&\,P'(\overline{\rho})\int_{\Omega^L}\nabla n\Delta udx+\tau\int_{\Omega^L}nu_t\Delta udx+\int_{\Omega^L}P'(\overline{\rho}+\tau n)\nabla n\Delta udx\\
&-\int_{\Omega^L}P'(\overline{\rho})\nabla n\Delta udx+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)(u\cdot\nabla u)\cdot\Delta udx\\
&-2\tau\nu\int_{\Omega^L}(\operatorname{curl}\omega)\Delta udx-\tau\int_{\Omega^L}(\nabla\times H)\times H\cdot\Delta udx\\
&-\tau\int_{\Omega^L}(\overline{\rho}+\tau n)f_L\Delta udx+\tau\int_{\Omega^L}n\omega_t\Delta\omega dx\\
&-2\tau\nu\int_{\Omega^L}(\operatorname{curl}u)\Delta\omega dx-\tau\int_{\Omega^L}(\overline{\rho}+\tau n)g_L\Delta udx\\
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)(u\cdot\nabla\omega)\cdot\Delta\omega dx-\tau\int_{\Omega^L}\nabla\times(u\times H)\cdot\Delta Hdx.
\end{aligned}$$

Combing the above two estimates gives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\overline{\rho}(|\nabla u|^2+|\nabla\omega|^2)+|\nabla H|^2+\frac{P'(\overline{\rho})}{\overline{\rho}}|\nabla n|^2\right)dx\\
&+\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}|\Delta n|^2dx\\
&+\int_{\Omega^L}\left((\mu+\nu)|\Delta u|^2+(\mu+\lambda-\nu)|\nabla\operatorname{div}u|^2+\sigma|\Delta H|^2\right)dx\\
&+\int_{\Omega^L}\left(\mu'|\Delta\omega|^2+(\mu'+\lambda')|\nabla\operatorname{div}\omega|^2+4\nu|\nabla\omega|^2\right)dx\\
=&\,-\frac{P'(\overline{\rho})}{\overline{\rho}}\tau\int_{\Omega^L}\left(\frac{1}{2}|\nabla n|^2\operatorname{div}u+\nabla n\nabla u\nabla n+n\nabla n\nabla\operatorname{div}u\right)dx\\
&+\tau\int_{\Omega^L}nu_t\Delta udx+\int_{\Omega^L}(P'(\overline{\rho}+\tau n)-P'(\overline{\rho}))\nabla n\Delta udx\\
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)(u\cdot\nabla u)\cdot\Delta udx-2\tau\nu\int_{\Omega^L}(\operatorname{curl}\omega)\Delta udx\\
&-\tau\int_{\Omega^L}(\nabla\times H)\times H\cdot\Delta udx-\tau\int_{\Omega^L}(\overline{\rho}+\tau n)f_L\Delta udx\\
&+\tau\int_{\Omega^L}n\omega_t\Delta\omega dx-2\tau\nu\int_{\Omega^L}(\operatorname{curl}u)\Delta\omega dx-\tau\int_{\Omega^L}(\overline{\rho}+\tau n)g_L\Delta udx
\end{aligned}$$

$$
\begin{aligned}
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)(u\cdot\nabla\omega)\cdot\Delta\omega dx-\tau\int_{\Omega^L}\nabla\times(u\times H)\cdot\Delta H dx\\
&\le C\tau\|\nabla n\|_{L^2}\|\nabla n\|_{L^3}\|\Delta u\|_{L^2}+C\tau\|\nabla n\|_{H^1}\|u_t\|_{L^2}\|\Delta u\|_{L^2}\\
&\quad+C\tau\|\nabla u\|_{H^1}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}+4\tau\nu\|\nabla\omega\|_{L^2}^2+\tau\nu\|\Delta u\|_{L^2}^2\\
&\quad+C\tau\|\nabla H\|_{H^1}\|\nabla H\|_{L^2}\|\Delta u\|_{L^2}+C\tau\|f_L\|_{L^2}\|\Delta u\|_{L^2}\\
&\quad+C\tau\|\nabla n\|_{H^1}\|\omega_t\|_{L^2}\|\Delta\omega\|_{L^2}+C\tau\|g_L\|_{L^2}\|\Delta u\|_{L^2}\\
&\quad+C\tau\|\nabla u\|_{H^1}\|\nabla\omega\|_{L^2}\|\Delta\omega\|_{L^2}\\
&\quad+C\tau\|\nabla u\|_{H^1}\|\nabla H\|_{L^2}\|\Delta H\|_{L^2}+C\tau\|\nabla H\|_{H^1}\|\nabla u\|_{L^2}\|\Delta H\|_{L^2}\\
&\le C\tau\|\nabla n\|_{L^2}^3\|\nabla^2 n\|_{L^2}+C\tau\|\nabla n\|_{H^1}^2\|u_t\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla u\|_{L^2}^2\\
&\quad+C\tau\|\nabla n\|_{H^1}^2\|\omega_t\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{L^2}^2+C\tau\|\nabla H\|_{H^1}^2\|\nabla H\|_{L^2}^2\\
&\quad+C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2+C\tau\|f_L\|_{L^2}^2+C\tau\|g_L\|_{L^2}^2+4\tau\nu\|\nabla\omega\|_{L^2}^2\\
&\quad+\tau\nu\|\Delta u\|_{L^2}^2+\frac{\mu}{2}\|\Delta u\|_{L^2}^2+\frac{\mu'}{2}\|\Delta\omega\|_{L^2}^2+\frac{\sigma}{2}\|\Delta H\|_{L^2}^2.
\end{aligned}
$$

Thus we have (27). □

Lemma 2.10. *Under the assumptions in Lemma 2.7, one has*

$$
\begin{aligned}
(28)\quad &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\Big(\overline{\rho}(|\nabla\operatorname{div}u|^2)+|\nabla\operatorname{div}\omega|^2)+|\Delta H|^2+\frac{P'(\overline{\rho})}{\overline{\rho}}|\Delta n|^2\\
&+D_1(\overline{\rho}+\tau n)(u_t^2+\omega_t^2)\Big)\,dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{D_1P'(\overline{\rho})}{\overline{\rho}}n_t^2+D_1H_t^2\right)dx\\
&+D_1\int_{\Omega^L}\left(\frac{\mu}{4}|\nabla u_t|^2+\frac{\mu+\lambda-\nu}{2}|\operatorname{div}u_t|^2\right)dx\\
&+D_1\int_{\Omega^L}\left(\frac{\mu'}{2}|\nabla\omega_t|^2+(\mu'+\lambda')|\operatorname{div}\omega_t|^2+\frac{\sigma}{2}|\nabla H_t|^2\right)dx\\
&+\frac{\sigma}{2}\int_{\Omega^L}|\nabla\Delta H|^2dx+4\nu\int_{\Omega^L}\left(|\nabla\operatorname{div}\omega|^2+|\nabla\operatorname{curl}\omega|^2\right)dx\\
&+\frac{1}{2}\int_{\Omega^L}\left((\mu+\lambda)|\Delta\operatorname{div}u|^2+(\mu+\nu)|\operatorname{curl}\Delta u|^2\right)dx\\
&+\int_{\Omega^L}\left((\mu'+\lambda')|\Delta\operatorname{div}\omega|^2+\mu'|\operatorname{curl}\Delta\omega|^2)\right)dx\\
&+\varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}(|\nabla\Delta n|^2+D_1|\nabla n_t|^2)dx\\
&\le C\tau\|n_t\|_{L^2}^2\|\nabla n\|_{H^1}^2+C\tau\|n_t\|_{L^2}^4\|u_t\|_{L^2}^2+C\tau\|n_t\|_{L^2}^4\|\omega_t\|_{L^2}^2\\
&\quad+C\tau\|u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2+C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\\
&\quad+C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^2+C\tau\|u_t\|_{L^2}^2\|\nabla\omega\|_{H^1}^2\\
&\quad+C\tau\|\omega_t\|_{L^2}^2\|\nabla u\|_{H^1}^2+C\tau\|H_t\|_{L^2}^2\|\nabla H\|_{H^1}^2\\
&\quad+C\tau\|H_t\|_{L^2}^2\|\nabla u\|_{H^1}^2+C\tau\|u_t\|_{L^2}^2\|\nabla H\|_{H^1}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 \\
&+ C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla \omega\|_{H^1}^2 + C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2 \\
&+ C\tau\|\nabla u\|_{H^1}^2\|\nabla \omega\|_{H^1}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla \omega_t\|_{L^2}^2 \\
&+ C\tau\|\nabla n\|_{H^1}^4\|u_t\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^4\|\omega_t\|_{L^2}^2 \\
&+ C\tau\|\nabla H\|_{H^1}^2\|\Delta H\|_{L^2}^2 + C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2 + C\tau\|\nabla^2 u\|_{L^2}^2 \\
&+ C\tau\|\nabla^2 \omega\|_{L^2}^2 + C_{\eta_1}\tau\|n_t\|_{L^2}^4 + C_{\eta_1}\tau\|\nabla n\|_{H^1}^4 + \eta_1\|\nabla u\|_{H^1}^2 + \eta_2\|u_t\|_{L^2}^2 \\
&+ \eta_2\|\omega_t\|_{L^2}^2 + C_{\eta_2}\tau\|f_{Lt}\|_{L^2}^2 + C_{\eta_2}\tau\|g_{Lt}\|_{L^2}^2 \\
&+ C\tau\|f_L\|_{L^3}^4 + C\tau\|\nabla f_L\|_{L^2}^2 + C\tau\|g_L\|_{L^3}^4 + C\tau\|\nabla g_L\|_{L^2}^2,
\end{aligned}
$$

where C, D_1, η_1, η_2, C_{η_1}, C_{η_2} are constants independent of L and ε. Moreover, η_i, $i = 1, 2$ can be chosen to be arbitrarily small, and C_{η_i}, $i = 1, 2$ is constant depending on η_i, $i = 1, 2$.

Proof. Applying ∂_t to (16), and then taking the L^2 inner product with $\frac{P'(\overline{\rho})}{\overline{\rho}} n_t$, u_t, ω_t, H_t, respectively, we have from integrations by parts that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}}n_t^2 + (\overline{\rho}+\tau n)(u_t^2+\omega_t^2) + H_t^2\right)dx + \varepsilon\int_{\Omega^L}\frac{P'(\overline{\rho})}{\overline{\rho}}|\nabla n_t|^2 dx \\
&+ \int_{\Omega^L}\left((\mu+\nu)|\nabla u_t|^2 + (\mu+\lambda-\nu)|\operatorname{div} u_t|^2\right)dx + \int_{\Omega^L}\sigma|\nabla H_t|^2 dx \\
&+ \int_{\Omega^L}\left(\mu'|\nabla\omega_t|^2 + (\mu'+\lambda')|\operatorname{div}\omega_t|^2 + 4\nu|\omega_t|^2\right)dx \\
= &-\tau\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}\left(\frac{1}{2}|n_t|^2\operatorname{div} u + \nabla n\cdot n_t u_t + n n_t \operatorname{div} u_t\right)dx \\
&- \frac{\tau}{2}\int_{\Omega^L} n_t(u_t^2+\omega_t^2)dx + \int_{\Omega^L}(P'(\overline{\rho}+\tau n) - P'(\overline{\rho}))n_t\operatorname{div} u_t dx \\
&+ 4\tau\nu\int_{\Omega^L}\omega_t\operatorname{curl} u_t dx - \tau^2\int_{\Omega^L} n_t u\nabla u\cdot u_t dx - \tau\int_{\Omega^L}(\overline{\rho}+\tau n)u_t\nabla u\cdot u_t dx \\
&- \tau\int_{\Omega^L}(\overline{\rho}+\tau n)u\nabla u_t\cdot u_t dx - \tau^2\int_{\Omega^L} n_t u\nabla\omega\cdot\omega_t dx \\
&- \tau\int_{\Omega^L}(\overline{\rho}+\tau n)u_t\nabla\omega\cdot\omega_t dx - \tau\int_{\Omega^L}(\overline{\rho}+\tau n)u\nabla\omega_t\cdot\omega_t dx \\
&+ \tau\int_{\Omega^L}(\nabla\times H_t)\times H\cdot u_t dx + \tau\int_{\Omega^L}(\nabla\times H)\times H_t\cdot u_t dx \\
&+ \tau\int_{\Omega^L}\nabla\times(u_t\times H)\cdot H_t dx + \tau\int_{\Omega^L}\nabla\times(u\times H_t)\cdot H_t dx \\
&+ \tau\int_{\Omega^L}(\tau n_t f_L u_t + (\overline{\rho}+\tau n)f_{Lt}u_t)dx + \tau\int_{\Omega^L}(\tau n_t g_L\omega_t + (\overline{\rho}+\tau n)g_{Lt}\omega_t)dx \\
\le &\, C\tau\|n_t\|_{L^2}^2\|\operatorname{div} u\|_{H^2} + C\tau\|n_t\|_{L^2}\|\nabla n\|_{H^1}\|\nabla u_t\|_{L^2} \\
&+ C\tau\|n_t\|_{L^2}\|\nabla n\|_{H^1}\|\operatorname{div} u_t\|_{L^2} + C\tau\|n_t\|_{L^2}\|u_t\|_{L^3}\|\nabla u_t\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\tau\|n_t\|_{L^2}\|\omega_t\|_{L^3}\|\nabla\omega_t\|_{L^2} + \tau\nu(\|\nabla u_t\|_{L^2}^2 + 4\|\omega_t\|_{L^2}^2)\\
&+ C\tau\|n_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2} + C\tau\|u_t\|_{L^2}\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}\\
&+ C\tau\|n_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2\omega\|_{L^2}\|\nabla\omega_t\|_{L^2} + C\tau\|u_t\|_{L^2}\|\nabla\omega\|_{H^1}\|\nabla\omega_t\|_{L^2}\\
&+ C\tau\|\nabla u\|_{H^1}\|\nabla\omega_t\|_{L^2}\|\omega_t\|_{L^2} + C\tau\|\nabla H_t\|_{L^2}\|\nabla H\|_{H^1}\|u_t\|_{L^2}\\
&+ C\tau\|f_{Lt}\|_{L^2}\|u_t\|_{L^2} + C\tau\|\nabla u_t\|_{L^2}\|\nabla H\|_{H^1}\|H_t\|_{L^2}\\
&+ C\tau\|\nabla H_t\|_{L^2}\|\nabla u\|_{H^1}\|H_t\|_{L^2} + C\tau\|g_{Lt}\|_{L^2}\|\omega_t\|_{L^2}\\
&+ C\tau\|n_t\|_{L^2}\|\nabla u_t\|_{L^2}\|f_L\|_{L^3} + C\tau\|n_t\|_{L^2}\|\nabla\omega_t\|_{L^2}\|g_L\|_{L^3}\\
\le\ & C\tau\|n_t\|_{L^2}^2\|\operatorname{div} u\|_{H^2} + C\tau\|n_t\|_{L^2}^2\|\nabla n\|_{H^1}^2 + C\tau\|n_t\|_{L^2}^4\|u_t\|_{L^2}^2\\
&+ C\tau\|n_t\|_{L^2}^4\|\omega_t\|_{L^2}^2 + C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 + C\tau\|u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2\\
&+ C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^2 + C\tau\|u_t\|_{L^2}^2\|\nabla\omega\|_{H^1}^2 + C\tau\|\nabla u\|_{H^1}^2\|\omega_t\|_{L^2}^2\\
&+ C\tau\|\nabla H\|_{L^2}\|^2\|u_t\|_{L^2} + C\tau\|\nabla H\|_{H^1}^2\|H_t\|_{L^2}^2 + C\tau\|\nabla u\|_{H^1}^2\|H_t\|_{L^2}^2\\
&+ C\tau\|f_{Lt}\|_{L^2}\|u_t\|_{L^2} + C\tau\|n_t\|_{L^2}^2\|f_L\|_{L^3}^2 + C\tau\|g_{Lt}\|_{L^2}\|\omega_t\|_{L^2}\\
&+ C\tau\|n_t\|_{L^2}^2\|g_L\|_{L^3}^2 + \frac{\mu}{2}\|\nabla u_t\|_{L^2}^2 + \frac{\mu+\lambda-\nu}{2}\|\operatorname{div} u_t\|_{L^2}^2 + \frac{\mu'}{2}\|\nabla\omega_t\|_{L^2}^2\\
&+ \frac{\sigma}{2}\|\nabla H_t\|_{L^2}^2 + 4\tau\nu\|\omega_t\|_{L^2}^2 + \tau\nu\|\nabla u_t\|_{L^2}^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(29)\quad &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}}n_t^2 + (\overline{\rho}+\tau n)(u_t^2+\omega_t^2) + H_t^2\right)dx\\
&+ \varepsilon\int_{\Omega^L}\frac{P'(\overline{\rho})}{\overline{\rho}}|\nabla n_t|^2dx + \frac{1}{2}\int_{\Omega^L}\big(\mu|\nabla u_t|^2 + (\mu+\lambda-\nu)|\operatorname{div} u_t|^2\\
&+\mu'|\nabla\omega_t|^2 + 2(\mu'+\lambda')|\operatorname{div}\omega_t|^2 + \sigma|\nabla H_t|^2\big)\,dx\\
\le\ & C\tau\|n_t\|_{L^2}^2\|\operatorname{div} u\|_{H^2} + C\tau\|n_t\|_{L^2}^2\|\nabla n\|_{H^1}^2 + C\tau\|n_t\|_{L^2}^4\|u_t\|_{L^2}^2\\
&+ C\tau\|n_t\|_{L^2}^4\|\omega_t\|_{L^2}^2 + C\tau\|u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\\
&+ C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^2 + C\tau\|u_t\|_{L^2}^2\|\nabla\omega\|_{H^1}^2\\
&+ C\tau\|\nabla u\|_{H^1}^2\|\omega_t\|_{L^2}^2 + C\tau\|\nabla H\|_{L^2}\|u_t\|_{L^2}^2 + C\tau\|\nabla H\|_{H^1}^2\|H_t\|_{L^2}^2\\
&+ C\tau\|\nabla u\|_{H^1}^2\|H_t\|_{L^2}^2 + C\tau\|f_{Lt}\|_{L^2}\|u_t\|_{L^2} + C\tau\|n_t\|_{L^2}^2\|f_L\|_{L^3}^2\\
&+ C\tau\|g_{Lt}\|_{L^2}\|\omega_t\|_{L^2} + C\tau\|n_t\|_{L^2}^2\|g_L\|_{L^3}^2.
\end{aligned}
$$

On the other hand, applying the operator Δ to the first equation of (16), and taking the L^2 inner product with Δn, it is easy to see that

$$
\begin{aligned}
(30)\quad &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\Delta n|^2dx + \overline{\rho}\int_{\Omega^L}\Delta\operatorname{div} u\Delta n dx + \varepsilon\int_{\Omega^L}|\nabla\Delta n|^2dx\\
&= -\tau\int_{\Omega^L}\Big(\frac{1}{2}|\Delta n|^2\operatorname{div} u + \nabla n\Delta u\Delta n + 2\nabla^2 n\nabla u\Delta n
\end{aligned}
$$

$$+ n\Delta \operatorname{div} u\Delta n + 2\nabla n\nabla \operatorname{div} u\Delta n\Big)dx.$$

Multiply the second equation of (16) by $\nabla\Delta \operatorname{div} u$, and integrate it over Ω^L, it holds that

$$\begin{aligned}
(31)\quad & \frac{\overline{\rho}}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla \operatorname{div} u|^2dx + (2\mu+\lambda)\int_{\Omega^L}|\Delta \operatorname{div} u|^2dx \\
&= P'(\overline{\rho})\int_{\Omega^L}\Delta n\Delta \operatorname{div} u dx + \tau\int_{\Omega^L} n \operatorname{div} u_t\Delta \operatorname{div} u dx \\
&\quad + \tau\int_{\Omega^L}\nabla n u_t\Delta \operatorname{div} u dx + \tau\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2\Delta \operatorname{div} u dx \\
&\quad + \int_{\Omega^L}(P'(\overline{\rho}+\tau n)-P'(\overline{\rho}))\Delta n\Delta \operatorname{div} u dx - 2\tau\nu\int_{\Omega^L}\operatorname{div}(\operatorname{curl}\omega)\Delta \operatorname{div} u dx \\
&\quad + \tau\int_{\Omega^L}\operatorname{div}\big((\overline{\rho}+\tau n)(u\cdot\nabla)u\big)\Delta \operatorname{div} u dx - \int_{\Omega^L}\operatorname{div}((\nabla\times H)\times H)\Delta \operatorname{div} u dx \\
&\quad - \tau\int_{\Omega^L}\operatorname{div}((\overline{\rho}+\tau n)f_L)\Delta \operatorname{div} u\, dx.
\end{aligned}$$

Similarly, multiply the third equation of (16) by $\nabla\Delta \operatorname{div}\omega$, and integrate it over Ω^L, we can also obtain

$$\begin{aligned}
(32)\quad & \frac{\overline{\rho}}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla \operatorname{div}\omega|^2dx + (2\mu'+\lambda')\int_{\Omega^L}|\Delta \operatorname{div}\omega|^2dx \\
&\quad + 4\nu\int_{\Omega^L}|\nabla \operatorname{div}\omega|^2dx \\
&= \tau\int_{\Omega^L} n \operatorname{div}\omega_t\Delta \operatorname{div}\omega dx + \tau\int_{\Omega^L}\nabla n\omega_t\Delta \operatorname{div}\omega dx \\
&\quad - 2\nu\tau\int_{\Omega^L}\operatorname{div}(\operatorname{curl} u)\Delta \operatorname{div}\omega dx + \tau\int_{\Omega^L}\operatorname{div}\big((\overline{\rho}+\tau n)(u\cdot\nabla)\omega\big)\Delta \operatorname{div}\omega dx \\
&\quad - \tau\int_{\Omega^L}\operatorname{div}((\overline{\rho}+\tau n)g_L)\Delta \operatorname{div}\omega\, dx.
\end{aligned}$$

Applying Δ to the forth equation of (16), and taking the L^2 inner product with ΔH, we have

$$(33)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\Delta H|^2dx + \int_{\Omega^L}\sigma|\nabla\Delta H|^2dx = -\tau\int_{\Omega^L}\nabla\nabla\times(u\times H)\nabla\Delta H dx.$$

Putting (30)–(33) together, we deduce

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}}|\Delta n|^2 + \overline{\rho}(|\nabla \operatorname{div} u|^2 + |\nabla \operatorname{div}\omega|^2) + |\Delta H|^2\right)dx \\
&+ (2\mu+\lambda)\int_{\Omega^L}|\Delta \operatorname{div} u|^2dx + \varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}|\nabla\Delta n|^2dx \\
&+ (2\mu'+\lambda')\int_{\Omega^L}|\Delta \operatorname{div}\omega|^2dx + 4\nu\int_{\Omega^L}|\nabla \operatorname{div}\omega|^2 + \int_{\Omega^L}\sigma|\nabla\Delta H|^2dx
\end{aligned}$$

$$
\begin{aligned}
= & -\frac{\tau P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}\big(\frac{1}{2}|\Delta n|^2\operatorname{div}u+\nabla n\Delta u\Delta n+2\nabla^2 n\nabla u\Delta n+n\Delta\operatorname{div}u\Delta n\\
&+2\nabla n\nabla\operatorname{div}u\Delta n)\,dx+\tau\int_{\Omega^L}n\operatorname{div}u_t\Delta\operatorname{div}udx+\tau\int_{\Omega^L}\nabla nu_t\Delta\operatorname{div}udx\\
&+\tau\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2\Delta\operatorname{div}udx+\int_{\Omega^L}(P'(\overline{\rho}+\tau n)-P'(\overline{\rho}))\Delta n\Delta\operatorname{div}udx\\
&+\tau^2\int_{\Omega^L}\nabla n\cdot u\nabla u\Delta\operatorname{div}udx+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)|\nabla u|^2\Delta\operatorname{div}udx\\
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)u\Delta u\Delta\operatorname{div}udx+\tau^2\int_{\Omega^L}\nabla n\cdot u\nabla\omega\Delta\operatorname{div}\omega dx\\
&+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)\nabla u\nabla\omega\Delta\operatorname{div}udx+\tau\int_{\Omega^L}(\overline{\rho}+\tau n)u\Delta\omega\Delta\operatorname{div}udx\\
&+\tau\int_{\Omega^L}n\operatorname{div}\omega_t\Delta\operatorname{div}\omega dx+\tau\int_{\Omega^L}\nabla n\omega_t\Delta\operatorname{div}\omega dx\\
&-\int_{\Omega^L}\operatorname{div}((\nabla\times H)\times H)\Delta\operatorname{div}udx-\tau\int_{\Omega^L}\nabla\nabla\times(u\times H)\nabla\Delta Hdx\\
&-\tau\int_{\Omega^L}\operatorname{div}((\overline{\rho}+\tau n)f_L)\Delta\operatorname{div}udx-\tau\int_{\Omega^L}\operatorname{div}((\overline{\rho}+\tau n)g_L)\Delta\operatorname{div}\omega dx\\
\le & \, C\tau\|\Delta n\|_{L^2}^2\|\nabla u\|_{H^2}+C\tau\|\nabla n\|_{H^1}\|\Delta n\|_{L^2}\|\nabla u\|_{H^1}\\
&+C\tau\|\nabla n\|_{H^1}\|\Delta n\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\nabla n\|_{H^1}\|\operatorname{div}u_t\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}\\
&+C\tau\|\nabla n\|_{H^1}\|\nabla u_t\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\nabla n\|_{L^4}^2\|\Delta\operatorname{div}u\|_{L^2}\\
&+C\tau\|\nabla u\|_{L^4}^2\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\nabla n\|_{H^1}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}\|\Delta\operatorname{div}u\|_{L^2}\\
&+C\tau\|\nabla u\|_{H^1}\|\Delta u\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\nabla n\|_{H^1}\|\nabla u\|_{L^2}\|\nabla\omega\|_{H^1}\|\Delta\operatorname{div}\omega\|_{L^2}\\
&+C\tau\|\nabla u\|_{H^1}\|\Delta\omega\|_{L^2}\|\Delta\operatorname{div}\omega\|_{L^2}+C\tau\|\nabla u\|_{H^1}\|\nabla\omega\|_{H^1}\|\Delta\operatorname{div}\omega\|_{L^2}\\
&+C\tau\|\nabla n\|_{H^1}\|\nabla\omega_t\|_{L^2}\|\Delta\operatorname{div}\omega\|_{L^2}+C\tau\|\nabla H\|_{L^4}^2\|\Delta\operatorname{div}u\|_{L^2}\\
&+C\tau\|\nabla H\|_{L^2}\|\Delta H\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\nabla H\|_{H^1}\|\Delta u\|_{L^2}\|\nabla\Delta H\|_{L^2}\\
&+C\tau\|\nabla u\|_{H^1}\|\nabla H\|_{H^1}\|\nabla\Delta H\|_{L^2}++C\tau\|\Delta H\|_{L^2}\|\nabla u\|_{H^1}\|\nabla\Delta H\|_{L^2}\\
&+C\tau\|\nabla n\|_{H^1}\|f_L\|_{L^3}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\operatorname{div}f_L\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}\\
&+C\tau\|\nabla n\|_{H^1}\|g_L\|_{L^3}\|\Delta\operatorname{div}u\|_{L^2}+C\tau\|\operatorname{div}g_L\|_{L^2}\|\Delta\operatorname{div}u\|_{L^2}\\
\le & \, C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^2}+C\tau\|\nabla n\|_{H^1}^2\|\Delta n\|_{L^2}^2+C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2\\
&+C\tau\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2}^3+C\tau\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^3+C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{H^1}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2+C\tau\|\nabla n\|_{H^1}^2\|\nabla\omega_t\|_{L^2}^2\\
&+C\tau\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2}^3+C\tau\|\Delta H\|_{L^2}^2\|\nabla H\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|f_L\|_{L^3}^2+C\tau\|\operatorname{div}f_L\|_{L^2}^2+C\tau\|\nabla n\|_{H^1}^2\|g_L\|_{L^3}^2
\end{aligned}
$$

$$+ C\tau\|\operatorname{div} g_L\|_{L^2}^2 + \mu\|\Delta \operatorname{div} u\|_{L^2}^2 + \mu'\|\Delta \operatorname{div}\omega\|_{L^2} + \frac{\sigma}{2}\|\nabla\Delta H\|_{L^2}^2.$$

Hence, the above estimate implies that

$$\begin{aligned}
(34)\quad & \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\overline{\rho})}{\overline{\rho}}|\Delta n|^2 + \overline{\rho}(|\nabla \operatorname{div} u|^2 + |\nabla \operatorname{div}\omega|^2) + |\Delta H|^2\right)dx \\
& + (\mu+\lambda)\int_{\Omega^L}|\Delta \operatorname{div} u|^2 dx + \varepsilon\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}|\nabla\Delta n|^2 dx \\
& + (\mu'+\lambda')\int_{\Omega^L}|\Delta \operatorname{div}\omega|^2 dx + 4\nu\int_{\Omega^L}|\nabla \operatorname{div}\omega|^2 + \int_{\Omega^L}\frac{\sigma}{2}|\nabla\Delta H|^2 dx \\
\le\ & C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^2} + C\tau\|\nabla n\|_{H^1}^2\|\Delta n\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2 \\
& + C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 \\
& + C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{H^1}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2 \\
& + C\tau\|\nabla n\|_{H^1}^2\|\nabla\omega_t\|_{L^2}^2 + C\tau\|\Delta H\|_{L^2}^2\|\nabla H\|_{H^1}^2 \\
& + C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2 + C\tau\|\nabla n\|_{H^1}^2\|f_L\|_{L^3}^2 + C\tau\|\operatorname{div} f_L\|_{L^2}^2 \\
& + C\tau\|\nabla n\|_{H^1}^2\|g_L\|_{L^3}^2 + C\tau\|\operatorname{div} g_L\|_{L^2}^2.
\end{aligned}$$

Applying the operator curl to the second and the third of (16), we arrive at

$$\begin{aligned}
(35)\quad & \operatorname{curl}((\overline{\rho}+\tau n)u_t) - (\mu+\nu)\operatorname{curl}\Delta u \\
& + \tau\operatorname{curl}\big((\overline{\rho}+\tau n)(u\cdot\nabla)u\big) - \tau\operatorname{curl}((\nabla\times H)\times H) \\
& = 2\tau\nu\operatorname{curl}(\operatorname{curl}\omega) + \tau\operatorname{curl}((\overline{\rho}+\tau n)f_L)
\end{aligned}$$

and

$$\begin{aligned}
(36)\quad & \operatorname{curl}((\overline{\rho}+\tau n)\omega_t) - \mu'\operatorname{curl}\Delta\omega + 4\nu\operatorname{curl}\omega + \tau\operatorname{curl}\big((\overline{\rho}+\tau n)(u\cdot\nabla)\omega\big) \\
& = 2\tau\nu\operatorname{curl}(\operatorname{curl} u) + \tau\operatorname{curl}((\overline{\rho}+\tau n)g_L).
\end{aligned}$$

Multiplying (35) and (36) by $\operatorname{curl}\Delta u$ and $\operatorname{curl}\Delta\omega$, respectively, and integrating them over Ω^L, we get

$$\begin{aligned}
(37)\quad & (\mu+\nu)\int_{\Omega^L}|\operatorname{curl}\Delta u|^2 + \mu'\int_{\Omega^L}|\operatorname{curl}\Delta\omega|^2 + 4\nu\int_{\Omega^L}|\nabla\operatorname{curl}\omega|^2 dx \\
\le\ & C\|\nabla u_t\|_{L^2}^2 + C\|\nabla\omega_t\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^4\|u_t\|_{L^2}^2 + C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2 \\
& + C\tau\|\nabla^2\omega\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\tau\|\nabla H\|_{H^1}^2\|\nabla^2 H\|_{L^2}^2 \\
& + C\tau\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2}^3 + C\tau\|\nabla n\|_{H^1}^4\|\omega_t\|_{L^2}^2 + C\tau\|\nabla^2 u\|_{L^2}^2 \\
& + C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{H^1}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2 \\
& + C\tau\|\nabla n\|_{H^1}^2\|f_L\|_{L^3}^2 + C\tau\|\nabla f_L\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|g_L\|_{L^3}^2 \\
& + C\tau\|\nabla g_L\|_{L^2}^2.
\end{aligned}$$

Finally, multiplying (29) by a suitably large constant D_1, and combining it with (34), (37), we obtain (28) immediately. □

Now, we give the uniform estimates for $n_t, \nabla n_t, \nabla n$ and Δn. The proof is similar to [11], we omit it here for brevity.

Lemma 2.11. *Under the assumptions in Lemma 2.7, one has*

$$\begin{aligned}(38)\qquad &\int_{\Omega^L} n_t^2 dx + \varepsilon \frac{d}{dt}\int_{\Omega^L} |\nabla n|^2 dx\\ &\le C\|\operatorname{div} u\|_{L^2}^2 + C\tau\|\nabla n\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}(39)\qquad &\int_{\Omega^L} |\nabla n_t|^2 dx + \varepsilon \frac{d}{dt}\int_{\Omega^L} |\Delta n|^2 dx\\ &\le C\|\nabla \operatorname{div} u\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^1}^2,\end{aligned}$$

$$\begin{aligned}(40)\qquad &\int_{\Omega^L} |\nabla n|^2 dx\\ &\le C(\|u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla \operatorname{div} u\|_{L^2}^2)\\ &\quad + C\tau\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\tau\|\nabla H\|_{L^2}^3\|\Delta H\|_{L^2} + C\tau\|f_L\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}(41)\qquad &\int_{\Omega^L} |\Delta n|^2 dx\\ &\le C(\|\nabla u_t\|_{L^2}^2 + \|\Delta \operatorname{div} u\|_{L^2}^2) + C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2\\ &\quad + C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2\\ &\quad + C\tau\|\nabla H\|_{H^1}^2\|\Delta H\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^4 + C\tau\|f_L\|_{L^3}^4 + C\tau\|\nabla f_L\|_{L^2}^2,\end{aligned}$$

where C is a constant independent of L and ε.

Lemma 2.12. *Under the assumptions in Lemma 2.7, one has*

$$\begin{aligned}(42)\qquad &\frac{D_2}{2}\frac{d}{dt}\int_{\Omega^L}\left(\overline{\rho}(|\nabla \operatorname{div} u|^2) + |\nabla \operatorname{div} \omega|^2) + |\Delta H|^2 + \frac{P'(\overline{\rho})}{\overline{\rho}}|\Delta n|^2\right) dx\\ &\quad + D_2\int_{\Omega^L} n_t^2 dx \varepsilon D_2 \frac{d}{dt}\int_{\Omega^L} |\nabla n|^2 dx + \frac{\overline{\rho}}{2}\int_{\Omega^L}(|\nabla u_t|^2 + |\nabla \omega_t|^2)dx\\ &\quad + P'(\overline{\rho})\frac{d}{dt}\int_{\Omega^L} \Delta n \operatorname{div} u dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\big((\mu+\nu)|\Delta u|^2\\ &\quad +(\mu+\lambda-\nu)|\nabla \operatorname{div} u|^2 + \mu'|\Delta\omega|^2 + (\mu'+\lambda')|\nabla \operatorname{div}\omega|^2 + 4\nu|\nabla\omega|^2\big)\, dx\\ &\quad + D_2\int_{\Omega^L}\big((\mu+\nu)|\Delta \operatorname{div} u|^2 + (\mu'+\lambda')|\Delta \operatorname{div}\omega|^2 + 4\nu|\nabla \operatorname{div}\omega|^2\\ &\quad + \frac{\sigma}{2}|\nabla \Delta H|^2\big)dx + \varepsilon D_2\frac{P'(\overline{\rho})}{\overline{\rho}}\int_{\Omega^L}|\nabla\Delta n|^2 dx\\ &\le C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^2} + C\tau\|\nabla n\|_{H^1}^2\|\Delta n\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2\\ &\quad + C\tau\|\nabla n\|_{H^1}^2\|\nabla \omega_t\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2\\ &\quad + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2\\ &\quad + C\tau\|\nabla u\|_{H^1}^2\|\nabla\omega\|_{H^1}^2 + C\tau\|\Delta n\|_{L^2}^4\|u_t\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+ C\tau\|\Delta n\|_{L^2}^4\|\omega_t\|_{L^2}^2 + C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^1}^2 + C\|\operatorname{div} u\|_{L^2}^2\\
&+ C\tau\|\nabla^2 u\|_{L^2}^2 + C\tau\|\nabla^2\omega\|_{L^2}^2 + C\tau\|\nabla H\|_{H^1}^2\|\Delta H\|_{L^2}^2\\
&+ C\tau\|\nabla u\|_{H^1}^2\|\nabla H\|_{H^1}^2 + C\tau\|f_L\|_{L^3}^2\|\nabla n\|_{H^1}^2 + C\tau\|\nabla f_L\|_{L^2}^2\\
&+ C\tau\|g_L\|_{L^3}^2\|\nabla n\|_{H^1}^2 + C\tau\|\nabla g_L\|_{L^2}^2,
\end{aligned}
$$

where C, D_2 are constants independent of L and ε and D_2 can be chosen to be suitably large.

Proof. Multiplying the second and third equation of (16) by Δu_t and $\Delta\omega_t$, respectively, we have from integrations by parts that

$$
\begin{aligned}
&\int_{\Omega^L}(\overline{\rho}+\tau n)(|\nabla u_t|^2+|\nabla\omega_t|^2)\,dx\\
&+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\big((\mu+\nu)|\Delta u|^2+(\mu+\lambda-\nu)|\nabla\operatorname{div} u|^2\big)\,dx\\
&+\frac{d}{dt}\int_{\Omega^L}P'(\overline{\rho})\Delta n\operatorname{div} u\,dx\\
&+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\big(\mu'|\Delta\omega|^2+(\mu'+\lambda')|\nabla\operatorname{div}\omega|^2+4\nu|\nabla\omega|^2\big)\,dx\\
=&\ \frac{\tau}{2}\int_{\Omega^L}(u_t^2+\omega_t^2)\Delta n\,dx-\tau\int_{\Omega^L}P''(\overline{\rho}+\tau n)|\nabla n|^2\operatorname{div} u_t dx\\
&+P'(\overline{\rho})\int_{\Omega^L}n_t\Delta\operatorname{div} u dx-\int_{\Omega^L}(P'(\overline{\rho}+\tau n)-P'(\overline{\rho}))\Delta n\operatorname{div} u_t dx\\
&-\tau\int_{\Omega^L}\nabla\big((\overline{\rho}+\tau n)(u\cdot\nabla)u\big)\nabla u_t dx+2\tau\nu\int_{\Omega^L}\nabla\operatorname{curl}\omega\nabla u_t dx\\
&+\tau\int_{\Omega^L}\nabla((\nabla\times H)\times H)\nabla u_t dx+\tau\int_{\Omega^L}\nabla((\overline{\rho}+\tau n)f_L)\nabla u_t dx\\
&+2\tau\nu\int_{\Omega^L}\nabla\operatorname{curl} u\nabla\omega_t dx-\tau\int_{\Omega^L}\nabla\big((\overline{\rho}+\tau n)(u\cdot\nabla)\omega\big)\nabla\omega_t dx\\
&+\tau\int_{\Omega^L}\nabla((\overline{\rho}+\tau n)g_L)\nabla\omega_t dx\\
\le&\ \frac{\overline{\rho}}{4}(\|\nabla u_t\|_{L^2}^2+\|\nabla\omega_t\|_{L^2}^2)+C\tau\|\Delta n\|_{L^2}^4\|u_t\|_{L^2}^2+C\tau\|\Delta n\|_{L^2}^4\|\omega_t\|_{L^2}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|\Delta n\|_{L^2}^2+C\|n_t\|_{L^2}\|\Delta\operatorname{div} u\|_{L^2}+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2\\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\Delta\omega\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\Delta u\|_{L^2}^2+C\tau\|\nabla u\|_{H^1}^2\|\Delta\omega\|_{L^2}^2\\
&+C\tau\|\nabla H\|_{H^1}^2\|\Delta H\|_{L^2}^2+C\tau\|\nabla^2\omega\|_{L^2}^2+C\tau\|\nabla n\|_{H^1}^2\|f_L\|_{L^3}^2\\
&+C\tau\|\nabla^2 u\|_{L^2}^2+C\tau\|\nabla f_L\|_{L^2}^2+C\tau\|\nabla n\|_{H^1}^2\|g_L\|_{L^3}^2+C\tau\|\nabla g_L\|_{L^2}^2.
\end{aligned}
$$

This together with (34) and (38), by choosing D_2 appropriately large, we obtain (42) immediately. □

2.3. Proof of Proposition 2.1

Now, we are ready to prove the existence of time periodic solutions of Proposition 2.1. The proof is a combination of the estimates obtained above and the topological degree theory.

Proof of Proposition 2.1. In order to solve the problem (4), we first solve the equation

$$U-\mathcal{P}(U,1)=0,\quad U=(n,u,\omega,H)\in S^L_\delta. \tag{43}$$

By the topological degree theory, we have to choose $\delta_0>0$ such that

$$(I-\mathcal{P}(\cdot,\tau))(\partial B_{\delta_0}(0))\neq 0,\quad \forall\tau\in[0,1], \tag{44}$$

where $B_{\delta_0}(0)$ is the ball of radius δ_0 centered at the origin in S^L.

When δ_0 is sufficiently small, then we have $\|n\|_{L^\infty}\le C\|\nabla n\|_{H^1}\le\frac{\bar\rho}{2}$. For suitably large D_3 and D_4, considering the linear combination $D_3D_4\times(17)+D_4\times(24)+D_3D_4\times(27)+D_4\times(28)+D_4\times(38)+(39)+(40)+(41)$, and integrating from 0 to T, we can deduce that

$$\begin{aligned}
&\frac{D_3D_4}{4}\int_0^T\!\!\int_{\Omega^L}\big(\mu|\nabla u|^2+(\mu+\lambda-\nu)|\operatorname{div}u|^2+2\sigma|\nabla H|^2\\
&+\mu'|\nabla\omega|^2+(\mu'+\lambda')|\operatorname{div}\omega|^2\big)\,dxdt+\frac{D_3D_4}{4}\int_0^T\!\!\int_{\Omega^L}\big(\mu|\Delta u|^2\\
&+(\mu+\lambda-\nu)|\nabla\operatorname{div}u|^2+\mu'|\Delta\omega|^2+(\mu'+\lambda')|\nabla\operatorname{div}\omega|^2\big)\,dxdt\\
&+\frac{D_3D_4}{4}\int_0^T\!\!\int_{\Omega^L}2\sigma|\Delta H|^2\,dxdt+\frac{D_1D_4}{4}\int_0^T\!\!\int_{\Omega^L}\big(\mu|\nabla u_t|^2\\
&+2(\mu+\lambda-\nu)|\operatorname{div}u_t|^2\big)\,dxdt+\frac{D_1D_4}{4}\int_0^T\!\!\int_{\Omega^L}\big(\mu'|\nabla\omega_t|^2\\
&+2(\mu'+\lambda')|\operatorname{div}\omega_t|^2+2\sigma|\nabla H_t|^2\big)\,dxdt+\frac{D_4}{4}\int_0^T\!\!\int_{\Omega^L}\big((\mu+\lambda)|\Delta\operatorname{div}u|^2\\
&+2(\mu+\nu)|\operatorname{curl}\Delta u|^2+2\sigma|\nabla\Delta H|^2+4\mu'|\operatorname{curl}\Delta\omega|^2\big)\,dxdt\\
&+D_4\int_0^T\!\!\int_{\Omega^L}\big((\mu'+\lambda')|\Delta\operatorname{div}\omega|^2+4\nu\big(|\nabla\operatorname{div}\omega|^2+|\nabla\operatorname{curl}\omega|^2\big)\big)\,dxdt\\
&+\frac{D_4}{4}\int_0^T\!\!\int_{\Omega^L}(\bar\rho(u_t^2+\omega_t^2)+H_t^2)dxdt+\int_0^T\!\!\int_{\Omega^L}\big(D_4n_t^2+|\nabla n_t|^2+|\nabla n|^2\\
&+|\Delta n|^2\big)\,dxdt+\varepsilon D_4\frac{P'(\bar\rho)}{\bar\rho}\int_0^T\!\!\int_{\Omega^L}(|\nabla\Delta n|^2+D_1|\nabla n_t|^2)dxdt\\
\le{}& C\tau\sup_{0<t<T}\|\nabla n\|^2_{H^1}\int_0^T(\|\nabla n\|^2_{H^1}+\|\nabla u\|^2_{H^1}+\|\omega_t\|^2_{H^1}\\
&+\|n_t\|^2_{L^2}+\|u_t\|^2_{H^1})dt+C\tau\sup_{0<t<T}\{\|\nabla u\|^2_{H^1}+\|\nabla\omega\|^2_{H^1}+\|u_t\|^2_{L^2}\}
\end{aligned} \tag{45}$$

$$\cdot\int_0^T(\|\nabla u\|_{H^1}^2+\|\nabla\omega\|_{H^1}^2+\|\omega_t\|_{L^2}^2)dt+C\tau\sup_{0<t<T}\|n_t\|_{L^2}^2\int_0^T\|n_t\|_{L^2}^2dt$$
$$+C\tau\sup_{0<t<T}\{\|\nabla u\|_{H^1}^4+\|\nabla\omega\|_{H^1}^4\}\int_0^T(\|n_t\|_{L^2}^2+\|\nabla n\|_{H^1}^2)dt$$
$$+C\tau\sup_{0<t<T}\|\nabla n\|_{L^2}^4\int_0^T\|\nabla^2u\|_{L^2}^2dt+C\tau\sup_{0<t<T}\|\nabla H\|_{H^1}^2\int_0^T\|u_t\|_{L^2}^2dt$$
$$+C\tau\sup_{0<t<T}\{\|\nabla n\|_{H^1}^4+\|n_t\|_{L^2}^4\}\int_0^T(\|u_t\|_{L^2}^2+\|\omega_t\|_{L^2}^2)dt$$
$$+C\tau\sup_{0<t<T}\{\|\nabla H\|_{H^1}^2+\|\nabla u\|_{H^1}^2\}\int_0^T(\|\nabla H\|_{H^1}^2+\|H_t\|_{H^1}^2)dt$$
$$+C\tau\int_0^T\left(\|(f_{Lt},g_{Lt})\|_{L^2}^2+\|(f_L,g_L)\|_{L^3}^4+\|(f_L,g_L)\|_{H^1}^2+\|(f_L,g_L)\|_{L^{\frac{6}{5}}}^2\right)dt$$
$$\le C_1\tau\left(\int_0^T\left(\|(f_L,g_L)\|_{L^{\frac{6}{5}}}^2+\|(f_L,g_L)\|_{H^1}^4\right)dt+\|(f_L,g_L)\|_{W_2^{1,1}}^2\right)$$
$$+C_2\tau\delta_0^4+C_3\tau\delta_0^6.$$

Thus there exists a time $t^*\in(0,T)$ such that

$$\begin{aligned}(46)\qquad &\int_{\Omega^L}\big(u_t^2+|\nabla u_t|^2+|\operatorname{div}u_t|^2+\omega_t^2+|\nabla\omega_t|^2+|\operatorname{div}\omega_t|^2\\
&+n_t^2+|\nabla n_t|^2+H_t^2\big)(x,t^*)dx+\int_{\Omega^L}\big(|\nabla H_t|^2+|\nabla u|^2\\
&+|\operatorname{div}u|^2+|\Delta u|^2+|\nabla\operatorname{div}u|^2+|\Delta\operatorname{div}u|^2\big)(x,t^*)dx\\
&+\int_{\Omega^L}\big(|\operatorname{curl}\Delta u|^2+|\nabla\omega|^2+|\operatorname{div}\omega|^2+|\Delta\omega|^2+|\nabla\operatorname{curl}\omega|^2\\
&+|\nabla\operatorname{div}\omega|^2\big)(x,t^*)dx+\int_{\Omega^L}\big(|\Delta\operatorname{div}\omega|^2+|\operatorname{curl}\Delta\omega|^2\\
&+|\nabla H|^2+|\Delta H|^2+|\nabla n|^2+|\Delta n|^2\big)(x,t^*)dx\\
&+\varepsilon\int_{\Omega^L}\big(|\nabla n_t|^2+|\nabla\Delta n|^2\big)(x,t^*)dx\\
\le{}&\widehat{C}_1\tau\int_0^T\left(\|(f_L,g_L)\|_{L^{\frac{6}{5}}}^2+\|(f_L,g_L)\|_{H^1}^4\right)dt\\
&+\widehat{C}_1\|(f_L,g_L)\|_{W_2^{1,1}}^2+\widehat{C}_2\tau\delta_0^4+\widehat{C}_3\tau\delta_0^6.\end{aligned}$$

Combining (27), (29) and (42) yields

$$\begin{aligned}(47)\qquad &\frac{d}{dt}\int_{\Omega^L}\Big(\overline{\rho}(|\nabla u|^2+|\nabla\omega|^2)+|\nabla H|^2+\frac{P'(\overline{\rho})}{\overline{\rho}}(|\nabla n|^2+|n_t|^2)\\
&+(\overline{\rho}+\tau n)(|u_t|^2+|\omega_t|^2)\Big)dx+\frac{d}{dt}\int_{\Omega^L}\big(|H_t|^2+(\mu+\nu)|\Delta u|^2\end{aligned}$$

$$
\begin{aligned}
&+(\mu+\lambda-\nu)|\nabla \operatorname{div} u|^2+\mu'|\Delta\omega|^2+(\mu'+\lambda')|\nabla \operatorname{div}\omega|^2\big)\,dx \\
&+4\nu\frac{d}{dt}\int_{\Omega^L}|\nabla\omega|^2\,dx \\
&+D_2\frac{d}{dt}\int_{\Omega^L}\left(\overline{\rho}(|\nabla \operatorname{div} u|^2+|\nabla \operatorname{div}\omega|^2)+|\Delta H|^2+\frac{P'(\overline{\rho})}{\overline{\rho}}|\Delta n|^2\right)dx \\
&+\frac{d}{dt}\int_{\Omega^L}2P'(\overline{\rho})\Delta n \operatorname{div} u dx+\varepsilon D_2\frac{d}{dt}\int_{\Omega^L}|\nabla n|^2 dx \\
\le\;& C(\|\nabla u\|_{H^1}^2+\|u_t\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2)+C\tau\|n_t\|_{L^2}^2\|\operatorname{div} u\|_{H^2} \\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^2}+C\tau\|n_t\|_{L^2}^4+C\tau\|u_t\|_{L^2}^4+C\tau\|\omega_t\|_{L^2}^4 \\
&+C\tau\|H_t\|_{L^2}^4+C\tau\|\nabla n\|_{H^1}^4+C\tau\|\nabla u\|_{H^1}^4+C\tau\|\nabla\omega\|_{H^1}^4 \\
&+C\tau\|\nabla H\|_{H^1}^4+C\tau\|n_t\|_{L^2}^4\|u_t\|_{L^2}^2+C\tau\|n_t\|_{L^2}^4\|\omega_t\|_{L^2}^2 \\
&+C\tau\|\Delta n\|_{L^2}^4\|u_t\|_{L^2}^2+C\tau\|\Delta n\|_{L^2}^4\|\omega_t\|_{L^2}^2 \\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u_t\|_{L^2}^2+C\tau\|\nabla n\|_{H^1}^2\|\nabla\omega_t\|_{L^2}^2 \\
&+C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{H^1}^4+C\tau\|n_t\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2\omega\|_{L^2}^2 \\
&+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{H^1}^2+C\tau\|\nabla n\|_{H^1}^2\|\nabla u\|_{H^1}^4 \\
&+C\tau\|f_L\|_{H^1}^2+C\tau\|f_{Lt}\|_{L^2}^2+C\tau\|f_L\|_{L^3}^4 \\
&+C\tau\|g_L\|_{H^1}^2+C\tau\|g_{Lt}\|_{L^2}^2+C\tau\|g_L\|_{L^3}^4.
\end{aligned}
$$

Notice that

$$2P'(\overline{\rho})\int_{\Omega^L}\Delta n \operatorname{div} u dx\le\frac{\overline{\rho}}{2}\int_{\Omega^L}|\nabla u|^2dx+\frac{D_2P'(\overline{\rho})}{2\overline{\rho}}\int_{\Omega^L}|\Delta n|^2dx$$

with D_2 is suitably large. By (45) and (46), we integrate (47) from t^* to t for any $t^*\le t\le t^*+T$ to get

$$
\begin{aligned}
&\sup_{0\le t\le T}\int_{\Omega^L}\big(|\nabla u|^2+|\nabla\omega|^2+|\nabla H|^2+|\nabla n|^2+|n_t|^2 \\
&+|u_t|^2+|\omega_t|^2+|H_t|^2\big)(x,t)dx+\int_{\Omega^L}\big(|\nabla \operatorname{div} u|^2+|\nabla \operatorname{div}\omega|^2 \\
&+|\Delta u|^2+|\Delta\omega|^2+|\Delta H|^2+|\Delta n|^2\big)(x,t)dx \\
\le\;&\int_{\Omega^L}\big(|\nabla u|^2+|\nabla\omega|^2+|\nabla H|^2+|\nabla n|^2+|n_t|^2 \\
&+|u_t|^2+|\omega_t|^2+|H_t|^2\big)(x,t^*)dx+\int_{\Omega^L}\big(|\nabla \operatorname{div} u|^2+|\nabla \operatorname{div}\omega|^2 \\
&+|\Delta u|^2+|\Delta\omega|^2+|\Delta H|^2+|\Delta n|^2\big)(x,t^*)dx \\
&+C\int_0^T(\|\nabla u\|_{H^1}^2+\|u_t\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2)dt
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&+ C\tau \sup_{0<t<T} \|\nabla u\|_{H^1}^2 \int_0^T \|\nabla u\|_{H^1}^2\, dt \\
&+ C\tau \sup_{0<t<T} \|\nabla \omega\|_{H^1}^2 \int_0^T \|\nabla \omega\|_{H^1}^2\, dt \\
&+ C\tau \sup_{0\le t\le T} \|\nabla H\|_{H^1}^2 \int_{\Omega^L} \|\nabla H\|_{H^1}^2 dt \\
&+ C\tau \sup_{0<t<T} \|u_t\|_{L^2}^2 \int_{\Omega^L} \|u_t\|_{L^2}^2 dt + C\tau \sup_{0<t<T} \|\omega_t\|_{L^2}^2 \int_{\Omega^L} \|\omega_t\|_{L^2}^2 dt \\
&+ C\tau \sup_{0<t<T} \|H_t\|_{L^2}^2 \int_{\Omega^L} \|H_t\|_{L^2}^2 dt + C\tau \sup_{0<t<T} \|n_t\|_{L^2}^2 \int_0^T \|n_t\|_{L^2}^2\, dt \\
&+ C\tau \sup_{0<t<T} \{\|n_t\|_{L^2}^2 + \|\nabla n\|_{H^1}^2\} \left(\int_0^T \|\nabla u\|_{H^2}^2 dt\right)^{\frac{1}{2}} \\
&+ C\tau \sup_{0<t<T} \{\|n_t\|_{L^2}^4 + \|\Delta n\|_{L^2}^4\} \int_0^T \left(\|u_t\|_{L^2}^2 + \|\omega_t\|_{L^2}^2\right) dt \\
&+ C\tau \sup_{0<t<T} \{\|\nabla u\|_{H^1}^4 + \|\nabla \omega\|_{H^1}^4\} \int_0^T \left(\|n_t\|_{L^2}^2 + \|\nabla n\|_{H^1}^2\right) dt \\
&+ C\tau \sup_{0<t<T} \|\nabla n\|_{H^1}^2 \int_0^T (\|\nabla n\|_{H^1}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla \omega_t\|_{L^2}^2) dt \\
&+ C\tau \|f_L\|_{W_2^{1,1}}^2 + C\tau \|g_L\|_{W_2^{1,1}}^2 + C\tau \int_0^T \left(\|f_L\|_{H^1}^4 + \|g_L\|_{H^1}^4\right) dt \\
\le{}& C_4\tau \int_0^T \left(\|(f_L, g_L)\|_{L^{\frac{6}{5}}}^2 + \|(f_L, g_L)\|_{H^1}^4\right) dt \\
&+ C_4 \|(f_L, g_L)\|_{W_2^{1,1}}^2 + C_5\tau\delta_0^3 + C_6\tau\delta_0^4 + C_7\tau\delta_0^6.
\end{aligned}
$$

By (45) and (48), it holds that

$$
\begin{aligned}
&\sup_{0<t<T} \int_{\Omega^L} \left(|\nabla u|^2 + |\nabla \omega|^2 + |\nabla H|^2 + |\nabla n|^2 + n_t^2\right. \\
&\left. + u_t^2 + \omega_t^2 + H_t^2\right)(x,t) dx + \sup_{0<t<T} \int_{\Omega^L} \left(|\nabla \operatorname{div} u|^2 + |\nabla \operatorname{div} \omega|^2\right. \\
&\left. + |\Delta u|^2 + |\Delta \omega|^2 + |\Delta H|^2 + |\Delta n|^2\right)(x,t) dx + \int_0^T \int_{\Omega^L} \left(|\nabla u|^2\right. \\
&\left. + |\operatorname{div} u|^2 + |\nabla H|^2 + |\nabla \omega|^2 + |\operatorname{div} \omega|^2 + |\Delta u|^2\right)(x,t) dx dt \\
&+ \int_0^T \int_{\Omega^L} \left(|\nabla \operatorname{div} u|^2 + |\Delta H|^2 + |\Delta \omega|^2 + |\nabla \operatorname{div} \omega|^2\right. \\
&\left. + |\Delta \operatorname{div} u|^2\right)(x,t) dx dt + \int_0^T \int_{\Omega^L} \left(|\operatorname{curl} \Delta u|^2 + |\Delta \operatorname{div} \omega|^2 + |\operatorname{curl} \Delta \omega|^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+|\nabla\Delta H|^2+u_t^2+\omega_t^2\big)(x,t)dxdt+\int_0^T\int_{\Omega^L}(H_t^2+|\nabla u_t|^2+|\operatorname{div}u_t|^2\\
&+|\nabla\omega_t|^2+|\nabla H_t|^2+|\operatorname{div}\omega_t|^2+n_t^2)(x,t)dxdt\\
&+\int_0^T\int_{\Omega^L}\left(|\nabla n_t|^2+|\nabla n|^2+|\Delta n|^2+\varepsilon|\nabla\Delta n|^2+\varepsilon|\nabla n_t|^2\right)(x,t)dxdt\\
\le{}& \widehat{C}_4\tau\left(\int_0^T\left(\|(f_L,g_L)\|^2_{L^{\frac{6}{5}}}+\|(f_L,g_L)\|^4_{H^1}\right)dt+\|(f_L,g_L)\|^2_{W_2^{1,1}}\right)\\
&+\widehat{C}_5\tau\delta_0^3+\widehat{C}_6\tau\delta_0^4+\widehat{C}_7\tau\delta_0^6.
\end{aligned}
$$

Thus, when δ_0 and $\int_0^T\left(\|(f_L,g_L)\|^2_{L^{\frac{6}{5}}}+\|(f_L,g_L)\|^4_{H^1}\right)dt+\|(f_L,g_L)\|^2_{W_2^{1,1}}$ are suitably small, we obtain

$$
\begin{aligned}
(49)\qquad &|||(n,u,\omega,H)|||^2+\varepsilon\int_0^T\int_{\Omega^L}(|\nabla n_t|^2+|\nabla\Delta n|^2)dxdt\\
&\le \widetilde{C}_1\tau\left(\int_0^T\left(\|(f_L,g_L)\|^2_{L^{\frac{6}{5}}}+\|(f_L,g_L)\|^4_{H^1}\right)dt+\|(f_L,g_L)\|^2_{W_2^{1,1}}\right)\\
&\quad+\widetilde{C}_2\tau\delta_0^3+\widetilde{C}_3\tau\delta_0^4+\widetilde{C}_4\tau\delta_0^6\\
&\le\frac{1}{2}\delta_0^2,
\end{aligned}
$$

where we have used the fact that $\|(n,u,\omega,H)\|_{L^6}\le\|\nabla(n,u,\omega,H)\|_{L^2}$. Therefore (44) holds. By $\mathcal{P}(\cdot,0)=0$, we have

$$\deg(I-\mathcal{P}(\cdot,1),B_{\delta_0}(0),0)=\deg(I-\mathcal{P}(\cdot,0),B_{\delta_0}(0),0)=\deg(I,B_{\delta_0}(0),0)=1,$$

which implies that problem (4) has a solution (n,u,ω,H) with $|||(n,u,\omega,H)|||\le\delta_0$. This completes the proof of Proposition 2.1. □

3. Existence of time periodic solutions in $\mathbb{R}^3$

This section is concerned with the existence of a strong small periodic solution in the whole space $\mathbb{R}^3$.

Proof of Theorem 1.1. Let (n_L,u_L,ω_L,H_L) be the solution of the regularized problem (16). By Sobolev imbedding theorem, we have $(n_L,u_L,\omega_L,H_L)\in C^{\alpha,\frac{\alpha}{2}}((0,T)\times\Omega^L)$ and

$$[n_L,u_L,\omega_L,H_L]_{\alpha,\frac{\alpha}{2}}\le C\delta_0.$$

Let $\varepsilon\to0$, and then $L\to\infty$, for any fixed $l>0$, there exists a subsequence $\{(n_m,u_m,\omega_m,H_m)\}_{m=1}^\infty$ and $\{(n,u,\omega,H)\}\in S^l_{\delta_0}$, such that

$$
\begin{aligned}
(n_m,u_m,\omega_m,H_m)&\to(n,u,\omega,H)\ \text{ uniformly in }\Omega^l,\\
(n_m,u_m,\omega_m,H_m)&\to(n,u,\omega,H)\ \text{ strongly in }L^2((0,T),L^6(\Omega^l)),\\
(\nabla n_m,\nabla u_m,\nabla\omega_m,\nabla H_m)&\to(\nabla n,\nabla u,\nabla\omega,\nabla H)\ \text{ weakly-* in }L^\infty((0,T),H^1(\Omega^l)),
\end{aligned}
$$

$$
\begin{gathered}
(n_{mt}, u_{mt}, \omega_{mt}, H_{mt}) \to (n_t, u_t, \omega_t, H_t) \text{ weakly-* in } L^\infty((0,T), L^2(\Omega^l)),\\
\nabla n_m \to \nabla n \quad \text{weakly in } L^2((0,T), H^1(\Omega^l)),\\
(\nabla u_m, \nabla \omega_m, \nabla H_m) \to (\nabla u, \nabla \omega, \nabla H) \text{ weakly in } L^2((0,T), H^2(\Omega^l)),\\
(n_{mt}, u_{mt}, \omega_{mt}, H_{mt}) \to (n_t, u_t, \omega_t, H_t) \text{ weakly in } L^2((0,T), H^1(\Omega^l)).
\end{gathered}
$$

Furthermore, by (47), for any small constant ξ, we get

$$
\begin{aligned}
&\int_0^T \bigg| \int_{\Omega^L} (|\nabla n|^2 + |\nabla u|^2 + |\nabla \omega|^2 + |\nabla H|^2 + n_t^2 + u_t^2 + \omega_t^2 + H_t^2)(x, t+\xi)\, dx \\
&+ \int_{\Omega^L} (|\Delta n|^2 + |\Delta u|^2 + |\Delta \omega|^2 + |\Delta H|^2 + |\nabla \mathrm{div} u|^2 + |\nabla \mathrm{div} \omega|^2)(x, t+\xi) dx \\
&- \int_{\Omega^L} (|\nabla n|^2 + |\nabla u|^2 + |\nabla \omega|^2 + |\nabla H|^2 + n_t^2 + u_t^2 + \omega_t^2 + H_t^2)(x, t) dx \\
&- \int_{\Omega^L} (|\Delta n|^2 + |\Delta u|^2 + |\Delta \omega|^2 + |\Delta H|^2 + |\nabla \mathrm{div} u|^2 + |\nabla \mathrm{div} \omega|^2)(x, t) dx \bigg|\, dt \\
&\le C|\xi|,
\end{aligned}
$$

where C is independent of L. By the Arzela-Ascoli theorem, one can check easily that

$$
\begin{gathered}
(n_{mt}, u_{mt}, \omega_{mt}, H_{mt}) \to (n_t, u_t, \omega_t, H_t) \text{ strongly in } L^2((0,T), L^2(\Omega^l));\\
\nabla n_m \to \nabla n \text{ strongly in } L^2((0,T), L^2(\Omega^l));\\
(\nabla u_m, \nabla \omega_m, \nabla H_m) \to (\nabla u, \nabla \omega, \nabla H) \text{ strongly in } L^2((0,T), H^1(\Omega^l)).
\end{gathered}
$$

Hence we take a sequence L_m with $L_m \to +\infty$ as $m \to +\infty$, and let

$$\{(n_m^k, u_m^k, \omega_m^k, H_m^k)\}$$

be the convergent function sequence in Ω^{L_k} , and $\{(n_m^{k+1}, u_m^{k+1}, \omega_m^{k+1}, H_m^{k+1})\}$ be a subsequence of $\{(n_m^k, u_m^k, \omega_m^k, H_m^k)\}$, which converges in $\Omega^{L_{k+1}}(k = 1, 2, \ldots, m, \ldots)$. Repeating the argument as

$$(n_1^1, u_1^1, \omega_1^1, H_1^1)\ (n_2^1, u_2^1, \omega_2^1, H_2^1)\ \cdots\ (n_m^1, u_m^1, \omega_m^1, H_m^1) \text{ converges in } \Omega^{L_1}$$

$$(n_1^2, u_1^2, \omega_1^2, H_1^2)\ (n_2^2, u_2^2, \omega_2^2, H_2^2)\ \cdots\ (n_m^2, u_m^2, \omega_m^2, H_m^2) \text{ converges in } \Omega^{L_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(n_1^m, u_1^m, \omega_1^m, H_1^m)\ (n_2^m, u_2^m, \omega_2^m, H_2^m) \cdots (n_m^m, u_m^m, \omega_m^m, H_m^m) \text{ converges in } \Omega^{L_m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We get a Cantor diagonal subsequence $\{(n_m^m, u_m^m, \omega_m^m, H_m^m)\}$ which goes to $\{(n, u, \omega, H)\}$ in Ω^L with any $L > 0$. Then we obtain the $\{(n, u, \omega, H)\} \in S_{\delta_0}$ is the solution of (1) in $\mathbb{R}^3$ by the arbitrariness of $L > 0$. □

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