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ON SOLVABILITY OF A CLASS OF DEGENERATE KIRCHHOFF EQUATIONS WITH LOGARITHMIC NONLINEARITY

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Abstract. We study the Dirichlet problem for the degenerate nonlocal parabolic equation

$$u_t - a\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\Delta u = C_b\,\|u\|^{\beta}_{L^2(\Omega)}\,|u|^{q(x,t)-2}\,u\log|u| + f \quad \text{in } Q_T,$$

where $Q_T := \Omega \times (0,T)$, $T > 0$, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a sufficiently smooth boundary, $q(x,t)$ is a measurable function in Q_T with values in an interval $[q^-, q^+] \subset (1,\infty)$ and the diffusion coefficient $a(\cdot)$ is a continuous function defined on $\mathbb{R}_+$. It is assumed that $a(s) \to 0$ or $a(s) \to \infty$ as $s \to 0^+$, therefore the equation degenerates or becomes singular as $\|\nabla u(t)\|_2 \to 0$. For both cases, we show that under appropriate conditions on a, β, q, f the problem has a global in time strong solution which possesses the following global regularity property: $\Delta u \in L^2(Q_T)$ and $a(\|\nabla u\|^2_{L^2(\Omega)})\Delta u \in L^2(Q_T)$.

1. Introduction

We study the Dirichlet problem for the nonlinear parabolic equation with the nonlocal terms

$$\text{(1.1)}\quad \begin{cases} u_t - a\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\Delta u = C_b\,\|u\|^{\beta}_{L^2(\Omega)}\,|u|^{q(x,t)-2}\,u\log|u| + f(z) \text{ in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times (0,T), \end{cases}$$

where $Q_T = \Omega \times (0,T)$ is a cylinder of height $T > 0$, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$, $C_b \in \mathbb{R}$ is a constant, $a(\cdot)$ is a real-valued function defined on $\mathbb{R}_+$ and $q(z)$ is a real-valued measurable function of the argument $z = (x,t) \in Q_T$ with values in an interval $[q^-, q^+] \subset (1,\infty)$. It is assumed that $u_0 \in H^1_0(\Omega)$, $f \in L^2(0,T; H^1_0(\Omega))$.

One of the main features of problem (1.1) is the presence of the term

$$a(\|\nabla u\|^2_{L^2(\Omega)}),$$

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which is said to be nonlocal since it depends not only on the point in Q_T, where the equation is evaluated, but on the norm of the whole solution. Such problems are usually called of Kirchhoff-type, as they are generalizations of the Kirchhoff equation, originally proposed in [21]. More specifically, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, L, E are constants. This nonlocal model extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. After the pioneer work of Lions [23] on the abstract framework to Kirchhoff-type equations, the solvability of these nonlocal problems has been studied in the general dimension by various authors.

There are numerous nonlocal mathematical models of Kirchhoff type studied by many authors to express the processes in physics and engineering see, e.g., [3, 10–14, 17, 25, 28] and references therein. For example, nonlocal PDEs arise in mathematical modelling of migration of a population to describe the density of some biological species are worked in [3, 13, 17], nonlocal models obtained from combustion theory is considered in [25] and in medicine [7].

We note that problem (1.1) might be viewed as a model of spreading of bacteria with u standing for the population density at the point x at the moment t. The term $\|u\|_2^{\beta} |u|^{q(x,t)-2} u \log |u|$ is the external source which represents the death and birth processes whose rates depend on the total population at the instant t, $\|u\|_2$, the pointwise density, and the point (x, t) in the problem domain. As for the diffusion coefficient, we refer to [32], where it is shown that the term $a(\|\nabla u\|_2^2)\Delta u$ appears as the limit of a fractional diffusion operator that represents the balance between the rates at which the individuals arrive at the location x or leave it.

The questions of existence, uniqueness and asymptotic behavior of solutions of the initial and boundary-value problems for the equations

$$u_t - a(l(u))\Delta u = f, \quad u_t - a(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = f,$$

were studied in the series of works [10–12, 14] with a continuous function a whose argument $l(u)$ was a linear continuous functional on $L^2(\Omega)$, or a continuously differentiable function a of the argument $\|\nabla u\|^2_{L^2(\Omega)}$. In these works, the equation is nondegenerate: a is assumed to be bounded away from zero so there exist positive constants $0 < m \le M < \infty$ such that

$$m \le a(s) \le M, \quad \forall s \in \mathbb{R}. \tag{1.2}$$

The nonlocal problems without condition (1.2) were studied in [1, 2]. Paper [2] deals with the homogeneous Dirichlet problem for the degenerate nonlocal equation

$$u_t - \|u(t)\|^{2\gamma}_{L^2(\Omega)} \Delta u = f, \quad \gamma \in \mathbb{R}.$$

It is proven that for $\gamma \geq 0$ the solution exists globally in time, in the case $\gamma < 0$ local in time existence is established. It is shown that in the case $\gamma < 0$ and $f \equiv 0$ every solution extincts in finite time.

In recent years, the logarithmic nonlinearity appears frequently in partial differential equations which describes important physical phenomena (see [5, 6, 8, 9, 20, 24, 26]) and the references therein). This type of nonlinearity was introduced in the nonrelativistic wave equations describing spinning particles moving in an external electromagnetic field and also in the relativistic wave equation for spinless particles [19]. Moreover, the logarithmic nonlinearity appears in several branches of physics such as nuclear physics [34], optics and Q-ball dynamics in theoretical physics [16, 18].

It was Chen et al. [8, 9] who first carried out the research on logarithmic source. They studied the following semilinear heat equation with logarithmic nonlinearity in [8]:

$$u_t - \Delta u = u \log |u|$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with zero Dirichlet boundary condition. By using the logarithmic Sobolev inequality, they proved the existence of global weak solution and showed that the power nonlinearity is a critical condition of blow-up in finite time for the solutions of the considered problem.

There are a few papers devoted to study on Kirchhoff-type equations with logarithmic nonlinearity [15, 29, 33, 35]. The first result due to Ding and Zhou [15]. They considered the following fractional Kirchhoff-type parabolic problem with logarithmic nonlinearity:

$$\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u \log |u| & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $0 < s < 1$, $\mathcal{L}_K$ is a nonlocal integro-differential operator which generalizes the fractional Laplace operator $(-\Delta)^s$. The diffusion coefficient $M : [0,\infty) \to [0,\infty)$ is a continuous function depending on the Gagliardo seminorm $[u]_s$. They combined the Galerkin approximation method and the potential well to prove the existence of a global weak solution with subcritical and critical states. According to the differential inequality, the blow-up solution of the equation is given.

The homogeneous Dirichlet problem for the degenerate Kirchhoff-type fractional diffusion equation

$$u_t + [u]_s^2(-\Delta)^s u = |u|^{q-2}u \ln |u|, \quad (x,t) \in \Omega \times \mathbb{R}^+$$

is studied in [33]. By using potential well method ideas, the authors find a sufficient condition for the existence of global solutions that vanish at infinity or solutions that blow up in finite time under some appropriate assumptions.

Equations with variable nonlinearity and nonlocal equations of Kirchhoff type with logarithmic source appear in numerous applications and are actively studied as mentioned above. Inspired by the above works, in the present article we concern a class of the evolution equations which combine both features. Besides that the reaction part of the equation has a nonlocal source as well.

Our purpose in this paper is to obtain sufficient conditions on the existence of global strong solutions of problem (1.1) by avoiding the nondegeneracy condition (1.2) which allows us to treat a wider class of nonlocal terms $a(\cdot)$. We study problem (1.1) without this condition and assume nonlocal dependence of the reaction term on the unknown function with logarithmic nonlinearity. Instead of (1.2) we impose a restriction on the rate of vanishing and growth of $a(s)$ when $s \to 0^+$ and $s \to \infty$. To be precise, we assume that

$$C_a s^{\frac{\tau}{2}} \leq a(s)s \leq C_\mu \left(s^{\frac{\mu}{2}} + s\right), \quad s \geq 0,$$

with some constants $\tau > 1$, $\mu > 1$ and $C_a > 0$, $C_\mu > 0$, and claim monotonicity of $a(s^2)s$. All these conditions are fulfilled, for example, if $a(s^2) = Cs^{\tau-2}$ with constants $C > 0$ and $\tau > 1$.

We find the conditions on the given parameters and the exponent $q(x,t)$ which ensure global in time existence of so-called a strong solution to problem (1.1) in the function space:

$$\mathbf{V} \equiv \{u(z) : \; u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^0([0,T]; H_0^1(\Omega)), \, u_t \in L^2(Q_T)\}.$$

To the best of our knowledge, there are no papers to deal with the global existence results for problems like (1.1) and this is the first result about global higher regularity of solutions of a Kirchhoff-type degenerate parabolic problem with logarithmic nonlinearity.

In the following section, we state the definition of strong solution and main theorem of this paper besides some notations, related lemmas and some preliminary results are presented. The solution of problem (1.1) is obtained as the limit of the family of solutions of the regularized nondegenerate problems (3.1) with $a(\|\nabla u\|^2_{L^2(\Omega)})$ substituted by $a(\epsilon + \|\nabla u\|^2_{L^2(\Omega)})$, $\epsilon > 0$. The solution of the nondegenerate problem (3.1) is constructed in Section 3 by means of Galerkin's method. The proof is based on the series of a priori estimate for the Galerkin approximations. These estimates do not depend on ϵ and are used in Section 4 to justify passing to the limit in ϵ. At this step we use the monotonicity of $a(s^2)s$.

2. Assumptions and the main result

Although the source term in equation (1.1) involves the variable power of the unknown function, the theory of the variable Lebesgue and Sobolev spaces is not used in this work, except for several basic facts which can be found, e.g., in [4, Ch. 1].

2.1. Function spaces and notations

Let $q(x,t)$ be a measurable function on Q_T, $q(x,t) \in [q^-, q^+] \subset (1,\infty)$ a.e. in Q_T. The set of functions

$$L^{q(\cdot,\cdot)}(Q_T) = \{v : |v(x,t)|^{q(x,t)} \in L^1(Q_T)\}$$

endowed with the norm

$$\|u\|_{q(\cdot,\cdot),Q_T} = \inf\left\{\lambda > 0 : \int_{Q_T} \left|\frac{u}{\lambda}\right|^q dz \le 1\right\}$$

is a Banach space. The relation between the norm $\|u\|_{q(\cdot,\cdot),Q_T}$ and the modular $\int_{Q_T} |u|^q\, dz$ is given by the inequalities

$$\text{(2.1)}\quad \begin{cases} \min\left\{\|u\|^{q^-}_{q(\cdot,\cdot),Q_T}, \|u\|^{q^+}_{q(\cdot,\cdot),Q_T}\right\} \le \displaystyle\int_{Q_T} |u|^{q(x,t)}\, dz \\ \qquad\qquad \le \max\left\{\|u\|^{q^-}_{q(\cdot,\cdot),Q_T}, \|u\|^{q^+}_{q(\cdot,\cdot),Q_T}\right\}, \\ \min\left\{\|u(t)\|^{q^-}_{q(\cdot,t),\Omega}, \|u(t)\|^{q^+}_{q(\cdot,t),\Omega}\right\} \le \displaystyle\int_{\Omega} |u(t)|^{q(x,t)}\, dx \\ \qquad\qquad \le \max\left\{\|u(t)\|^{q^-}_{q(\cdot,t),\Omega}, \|u(t)\|^{q^+}_{q(\cdot,t),\Omega}\right\} \end{cases}$$

for a.e. $t \in (0,T)$. The generalized Hölder inequality holds: for all $u \in L^{q(\cdot,t)}(\Omega)$, $v \in L^{q'(\cdot,t)}(\Omega)$ and a.e. $t \in (0,T)$

$$\text{(2.2)}\qquad \int_\Omega |u(t)||v(t)|\, dx \le 2\|u(t)\|_{q(\cdot,t)}\|v(t)\|_{q'(\cdot,t)}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The analogue of Sobolev's embedding inequality is true: if $q(\cdot,t) \in C^0(\bar\Omega)$ and $q^+ < \frac{2N}{N-2}$, then for a.e. $t \in (0,T)$

$$\text{(2.3)}\qquad \|u\|_{q(\cdot,t)} \le C\|\nabla u(t)\|_2 \quad \forall u \in L^{q(\cdot,t)}(\Omega) \cap H^1_0(\Omega).$$

Throughout the text, the symbol C or C_k ($k \in \mathbb{N}$) represents the constants which can be explicitly calculated or estimated using the known quantities, but whose exact value is not crucial for the argument and may change from line to line even inside the same formula. We recall that the notation z is often used for the points of the cylinder Q_T: $z = (x,t) \in \Omega \times (0,T) = Q_T$.

For the functions $u(t), v(t) : (0,T) \mapsto H^1_0(\Omega)$ we denote

$$(u(t), v(t))_{2,\Omega} = \int_\Omega u(x,t)v(x,t)\, dx, \quad \|u(t)\|_2^2 = \int_\Omega u^2(x,t)\, dx$$
$$\|\nabla u(t)\|_2^2 = \int_\Omega |\nabla u(x,t)|^2\, dx.$$

Similar problem to (1.1) is studied in the article [27]. Differently from this article we consider logarithmic nonlinearity in the reaction part of the equation (1.1). The presence of the logarithmic nonlinearity caused some difficulties to obtain energy inequalities for the function sequences of Galerkin approximations. In order to handle this situation the following two lemmas will be used to get the required a priori estimates. The proof of Lemma 2.1 is straightforward, and will be omitted. For the proof of Lemma 2.2, see [31].

By the help of these lemmas, differently from the paper [27], under certain changes on the parameters of the problem we manage to prove the global existence of the strong solution of the problem (1.1).

Lemma 2.1. *Let ϱ be a positive number. Then the following inequality fulfills*

$$\log s \leq \frac{e^{-1}}{\varrho} s^{\varrho} \quad \text{for all} \quad s \in [1, \infty).$$

Lemma 2.2. *Assume that $\zeta : \Omega \to [1, \infty)$ is a measurable function satisfying $1 \leq \zeta^{-} \leq \zeta(x) \leq \zeta^{+} < \infty$ and $\beta \geq 1$, $\sigma > 0$. Then for every $u \in L^{\zeta(\cdot)+\sigma}(\Omega)$*

$$\int_{\Omega} |u|^{\zeta(x)} \left|\log |u|\right|^{\beta} dx \leq M_1 \int_{\Omega} |u|^{\zeta(x)+\sigma} dx + M_2$$

is fulfilled. Here $M_1 \equiv M_1(\sigma, \beta) > 0$ and $M_2 \equiv M_2(\sigma, \beta, |\Omega|) > 0$ are constants.

2.2. Statement of the main result

Assume that the following conditions are fulfilled:

(A.1) (i) $a : (0, \infty) \to (0, \infty)$, $a(s) \in C^0((0, \infty))$

$$a(s^2)s \in C^0([0, \infty)) \cap C^1((0, \infty)), \quad \left(a(s^2)s\right)' \geq 0 \text{ for } s > 0;$$

(ii) there exist constants $\tau > 1$, $\mu > 1$, $\beta \geq 0$, $C_a > 0$, $C_\mu > 0$ such that for all $s \geq 0$

$$C_a s^{\frac{\tau}{2}} \leq a(s)s \leq C_\mu \left(s^{\frac{\mu}{2}} + s\right);$$

(A.2) Let $q(x,t) : Q_T \to (1, \infty)$ be a measurable function and $q(\cdot, t) \in C^0(\bar{\Omega})$ such that there exist constants $q^{\pm}$ such that

$$1 < q^{-} \leq q(x,t) \leq q^{+} \text{ a.e. in } Q_T, \qquad \begin{array}{ll} q^{+} < \frac{2(N-1)}{N-2} & \text{if } N > 2, \\ q^{+} \in (q^{-}, \infty) & \text{if } N = 2; \end{array}$$

(A.3) (i) if $C_b \in \mathbb{R}\backslash\{0\}$, then the parameters τ, β, $q^{\pm}$ satisfy the inequalities

$$\beta + q^{+} < \max\{2, \tau\}, \quad 2(\beta + q^{+} - 1) < \tau < \min\{3, \beta + 2\},$$

(ii) if $C_b = 0$, then $1 < \tau < 3$.

The strong solution of problem (1.1) is understood in the following way:

Definition (Strong solution)**.** A function $u : Q_T \to \mathbb{R}$ is called a strong solution of problem (1.1) if

(i) $u \in \mathbf{V} \equiv \{u(z) : u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^0([0, T]; H_0^1(\Omega)),\ u_t \in L^2(Q_T)\}$;

(ii) for every test-function $\eta \in \mathbf{V}$

$$(2.4)\qquad \int_{Q_T} \left(\eta u_t + a(\|\nabla u\|_2^2)\nabla u \cdot \nabla \eta\right) dz = C_b \int_{Q_T} \|u\|_2^{\beta} |u|^{q(z)-2} u \log |u| \eta \, dz + \int_{Q_T} f\eta \, dz;$$

(iii) for every $\zeta \in L^2(\Omega)$

$$\int_{\Omega} (u(x,t) - u_0(x))\zeta(x)\, dx \to 0 \quad \text{as } t \to 0.$$

The conditions of global in time existence of strong solutions to problem (1.1) are given in the following theorem.

Theorem 2.3. *Assume that conditions* (A.1)-(A.3) *are fulfilled. Then for every* $u_0 \in H_0^1(\Omega)$ *and* $f \in L^2(0,T; H_0^1(\Omega))$ *problem* (1.1) *has a strong solution* $u(z)$ *which possesses the following properties*:

$$(2.5)\qquad \operatorname*{ess\,sup}_{(0,T)} \|u(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t u\|_{L^2(Q_T)} + \|\Delta u\|_{L^2(Q_T)}^2 + \int_{Q_T} a^2(\|\nabla u\|_2^2)\left(|\nabla u|^2 + (\Delta u)^2\right) dz \le C$$

with a constant C *depending on* C_a, C_b, β, τ, $q^{\pm}$, $\|u_0\|_{H_0^1(\Omega)}$, $\|f\|_{L^2(0,T;H_0^1(\Omega))}$.

3. Regularized problem

Given a parameter $\epsilon > 0$, let us consider the family of regularized nondegenerate parabolic problems:

$$(3.1)\qquad \begin{cases} u_t - a_\epsilon(\|\nabla u\|_2^2)\Delta u = C_b \|u\|_2^{\beta} |u|^{q(z)-2} u \log |u| + f & \text{in } Q_T, \\ u(x,0) = u_0(x), \quad u = 0 \text{ on } \partial\Omega \times (0,T), \\ a_\epsilon(s) := a(s+\epsilon). \end{cases}$$

A strong solution of problem (1.1) can be obtained as the limit as $\epsilon \to 0$ of a family of solutions of the regularized problem (3.1). By a solution of problem (3.1), we mean a function $u_\epsilon \in \mathbf{V}$ which satisfies the integral identity

$$(3.2)\qquad \int_{Q_T} \left(\partial_t u_\epsilon \xi + a_\epsilon(\|\nabla u_\epsilon\|_2^2)\nabla u_\epsilon \cdot \nabla \xi\right) dz = C_b \int_{Q_T} \|u_\epsilon\|_2^{\beta} |u_\epsilon|^{q(x,t)-2} u_\epsilon \log |u_\epsilon| \xi \, dz + \int_{Q_T} f\xi \, dz$$

with any the test-function $\xi \in \mathbf{V}$ and takes the initial data by continuity. The goal of this section is to prove the following assertion.

Lemma 3.1. *Under the conditions of Theorem 2.3, for every* $\epsilon \in (0,1)$ *problem* (3.1) *has at least one strong solution* $u = u_\epsilon$.

3.1. Galerkin's approximations

Let $\{\varphi_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ and $\{\lambda_k\}$ be the eigenfunctions and eigenvalues of the Dirichlet problem for the Laplace equation, respectively: for $\varphi_k \in H_0^1(\Omega)$

$$(\nabla\varphi_k, \nabla\varphi)_{2,\Omega} = \lambda_k(\varphi_k, \varphi)_{2,\Omega} \quad \forall\varphi \in H_0^1(\Omega).$$

The system $\{\varphi_k\} \subset H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$, $\left\{\frac{1}{\sqrt{\lambda_i}}\varphi_i\right\}$ is an orthonormal basis of $H_0^1(\Omega)$. The solution of problem (3.1) is sought as the limit of the sequence of functions

$$u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\varphi_i(x),$$

where the coefficients $g_{im}(t)$ are to be defined. Since the set $\{\varphi_k\}$ is dense in $H_0^1(\Omega)$, for every $u_0 \in H_0^1(\Omega)$

$$u_0^m = \sum_{k=1}^{m}(u_0, \varphi_k)_{2,\Omega}\varphi_k \overset{H_0^1(\Omega)}{\longrightarrow} u_0 \quad \text{as } m \to \infty.$$

The functions g_{km} satisfy the initial conditions

$$g_{km}(0) = (u_0, \varphi_k)_{2,\Omega}, \quad k = 1, \dots, m.$$

The functions $g_{km}(t)$ are defined from the system of nonlinear ordinary differential equations

$$\begin{aligned} (3.3) \qquad & (\partial_t u_m, \varphi)_{2,\Omega} + a_\epsilon\left(\|\nabla u_m\|_2^2\right)(\nabla u_m, \nabla\varphi)_{2,\Omega} \\ &= C_b \int_\Omega \|u_m\|_2^\beta |u_m|^{q(z)-2} u_m \log|u_m| \varphi\, dx + (f, \varphi)_{2,\Omega}. \end{aligned}$$

Taking $\varphi = \varphi_j$ in (3.3) we find that (3.3) is equivalent to the Cauchy problem for the system of m ordinary differential equations for the functions $g_{jm}(t)$:

$$(3.4) \quad \begin{cases} g'_{jm}(t) = & -\lambda_j a_\epsilon\left(\|\nabla u_m\|_2^2\right) g_{jm}(t) \\ & + C_b\|u_m\|_2^\beta \int_\Omega \left(\left|\sum_{i=1}^m g_{im}(t)\varphi_i\right|^{q(x,t)-2} \sum_{i=1}^m g_{im}(t)\varphi_i\right) \log|\sum_{i=1}^m g_{im}(t)\varphi_i|\varphi_j\, dx \\ & + (f, \varphi_j)_{2,\Omega}, \\ g_{jm}(0) = (u_0, \varphi_j)_{2,\Omega}, \ dj = 1, \dots, m. \end{cases}$$

Set $s = \|\nabla u_m\|_2 \equiv \left(\sum_{i=1}^m \lambda_i g_{im}^2(t)\right)^{\frac{1}{2}}$ and notice that because of conditions (A.1)(i)-(ii) with $\tau > 1$, $\mu > 1$ there exists $\lim_{s\to 0} a(s^2)s = 0$, whence

$$a_\epsilon(\|\nabla u_m\|_2^2) g_{jm}(t) = \left(a_\epsilon(s^2)s\right) \frac{g_{jm}(t)}{\left(\sum_{j=1}^m \lambda_j g_{jm}^2(t)\right)^{\frac{1}{2}}} \to 0 \quad \text{as } s \to 0^+.$$

It follows that under assumptions (A.1)-(A.2) on a and q the right-hand sides of (3.4) are continuous with respect to $g_{im}(t)$, and the existence of a solution of system (3.4) on an interval $(0, T_m)$ is guaranteed by Peano's theorem.

3.2. A priori estimates

In all estimates of this section we consider separately the cases $\tau > 2$ and $1 < \tau \leq 2$. Also, we tacitly assume that $C_b \neq 0$, the estimates in the case $C_b = 0$ are formulated in the end of the section as a corollary.

Lemma 3.2. *If $u_0 \in L^2(\Omega)$ and $q^+ + \beta < \max\{\tau, 2\}$, then for all $t \in (0,T)$*

$$\|u_m(t)\|_{2,\Omega}^2 + \int_0^t a_\epsilon(\|\nabla u_m(s)\|_2^2)\|\nabla u_m(s)\|_2^2\, ds \leq C\left(1 + \|u_0\|_2^2 + \|f\|_{2,Q_T}^2\right) \tag{3.5}$$

with a constant C depending on β, τ, $q^\pm$, T, C_a, C_b and the constant in the Poincaré inequality

$$\|v\|_2 \leq C^* \|\nabla v\|_2 \quad \forall v \in H_0^1(\Omega). \tag{3.6}$$

Proof. Let us multiply each of equations (3.4) by g_{jm} and sum up the results. By Lemma 2.1 we obtain the energy inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_m\|_2^2) + a_\epsilon(\|\nabla u_m\|_2^2)\|\nabla u_m\|_2^2 \\
&= C_b\|u_m\|_2^\beta \int_\Omega |u_m|^{q(x,t)} \log|u_m|\, dx + \int_\Omega f u_m\, dx \\
&\leq |C_b|\|u_m\|_2^\beta \left(\int_{\Omega_1} |u_m|^{q(x,t)}|\log|u_m||\, dx + \int_{\Omega_2} |u_m|^{q(x,t)} \log|u_m|\, dx\right) \\
&\quad + \int_\Omega |f||u_m|\, dx \\
&\leq |C_b|\|u_m\|_2^\beta \int_{\Omega_1} |u_m|^{q^-}|\log|u_m||\, dx + |C_b|\|u_m\|_2^\beta \int_{\Omega_2} |u_m|^{q^+}\log|u_m|\, dx \\
&\quad + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 \\
&\leq e^{-1}|\Omega||C_b|\|u_m\|_2^\beta + C|C_b|\|u_m\|_2^\beta \int_{\Omega_2} |u_m|^{q^+ + \sigma}\, dx + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 \\
&\leq C\|u_m\|_2^\beta (\|u_m\|_{q^+ + \sigma}^{q^+ + \sigma} + 1) + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2,
\end{aligned} \tag{3.7}$$

where

$\sigma \in (0,1)$ arbitrary, $\Omega_1 := \{x \in \Omega : |u_m| \leq 1\}$ and $\Omega_2 := \{x \in \Omega : |u_m| > 1\}$.

Let $\tau > 2$. Using (A.2), (A.3), the Poincaré, Cauchy and Young's inequalities and taking $\sigma > 0$ such that $q^+ + \sigma \leq \frac{2N}{N-2}$ and $\beta + q^+ + \sigma < \tau$, we transform (3.7) as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_m\|_2^2) + a_\epsilon(\|\nabla u_m\|_2^2)\,\|\nabla u_m\|_2^2 \\
&\leq C\,\|\nabla u_m\|_2^\beta\,(\|\nabla u_m\|_2^{q^+ + \sigma} + 1) + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2
\end{aligned}$$

$$\le C' \, \|\nabla u_m\|_2^{\beta+q^++\sigma} + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 + C$$

with a constant C' depending on C_b, β, q^+ and the constant C^* in the embedding inequality (3.6). It follows from Young's inequality that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_m\|_2^2) + a_\epsilon(\|\nabla u_m\|_2^2)\,\|\nabla u_m\|_2^2 \\ &\le \frac{C_a}{2}\|\nabla u_m\|_2^\tau + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 + C \end{aligned} \tag{3.8}$$

with a constant $C = C(C_a, C_b, \beta, \tau, q^\pm, C')$.

For $\tau \ge 2$

$$C_a\|\nabla u_m\|_2^\tau \le C_a(\epsilon + \|\nabla u_m\|_2^2)^{\frac{\tau}{2}-1}\|\nabla u_m\|_2^2 \le a_\epsilon(\|\nabla u_m\|_2^2)\|\nabla u_m\|_2^2.$$

Plugging this inequality into the right-hand sides of (3.8) and simplifying we obtain

$$\frac{d}{dt}(\|u_m\|_2^2) + a_\epsilon(\|\nabla u_m\|_2^2)\,\|\nabla u_m\|_2^2 \le C + \|u_m\|_2^2 + \|f\|_2^2$$

whence

$$\frac{d}{dt}\left(\|u_m\|_2^2\,\mathrm{e}^{-t}\right) \le (C + \|f\|_2^2)\mathrm{e}^{-t}.$$

Integration of this inequality over the interval $(0,t) \subset (0,T)$ gives estimate (3.5) in the case $\tau > 2$.

Let $1 < \tau \le 2$ and $\beta + q^+ + \sigma \le 2$. In this case (3.7) leads to the inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|u_m\|_2^2) &\le \frac{1}{2}\frac{d}{dt}(\|u_m\|_2^2) + a_\epsilon(\|\nabla u_m\|_2^2)\,\|\nabla u_m\|_2^2 \\ &\le C\|u_m\|_2^{\beta+q^++\sigma} + C\|u_m\|_2^\beta + \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 \\ &\le C'' + C'\|u_m\|_2^2 + \frac{1}{2}\|f\|_2^2 \end{aligned} \tag{3.9}$$

with constants C', C'' depending on β, $q^\pm$ but independent of m and ϵ. Inequality (3.9) can be written in the form

$$\frac{d}{dt}\left(\|u_m(t)\|_2^2\mathrm{e}^{-2C't}\right) \le \left(2C'' + \|f\|_2^2\right)\mathrm{e}^{-2C't}, \quad t \in (0,T),$$

and integrated:

$$\|u_m(t)\|_2^2\mathrm{e}^{-2C't} \le \|u_0^m\|_2^2 + \frac{C''}{C'}\left(1 - \mathrm{e}^{-2C't}\right) + \|f\|_{2,Q_T}^2.$$

It follows that

$$\|u_m(t)\|_2^2 \le C\left(1 + \|u_0\|_2^2\right)\mathrm{e}^{2C'T} + \mathrm{e}^{2C'T}\|f\|_{2,Q_T}^2.$$

Gathering this estimate with (3.9) and integrating we obtain the inequality

$$\int_0^t a_\epsilon(\|\nabla u_m\|_2^2)\|\nabla u_m\|_2^2\,ds \le C$$

with a constant C independent of m and ϵ. □

Remark 3.3. In the case $\tau \geq 2$ estimate (3.5) leads to the inequality

$$\|u_m(t)\|_{2,\Omega}^2 + \int_0^t \|\nabla u_m(s)\|_2^\tau \, ds \leq C\left(1 + \|u_0\|_2^2 + \|f\|_{2,Q_T}^2\right). \tag{3.10}$$

Lemma 3.4. *Let in the conditions of Lemma 3.2, $u_0 \in H_0^1(\Omega)$. If $2(q^+ - 1) < \tau$, then*

$$\|\partial_t u_m\|_{2,Q_T} + \int_0^{\|\nabla u_m\|_2^2} a_\epsilon(s) \, ds \leq C \tag{3.11}$$

with a constant C depending on $q^\pm$, β, τ, C_b, $\|u_0\|_{H_0^1(\Omega)}$, $\|f\|_{2,Q_T}$, T and

$$\int_0^{\|\nabla u_0\|_2^2} a(s) \, ds.$$

Proof. Multiplying each of equations (3.4) by $g'_{jm}(t)$ and summing the results we arrive at the equality

$$\begin{aligned}&\|\partial_t u_m\|_2^2 + a_\epsilon(\|\nabla u_m\|_2^2) \int_\Omega \nabla u_m \cdot \nabla(\partial_t u_m) \, dx \\ &= C_b \|u_m\|_2^\beta \int_\Omega |u_m|^{q(z)-2} u_m \log|u_m| \partial_t u_m \, dx + \int_\Omega f \partial_t u_m \, dx.\end{aligned}$$

Using assumptions (A.1)-(A.3), the inequalities of Hölder and Poincaré, and the Young inequality we find that

$$\begin{aligned}&\|\partial_t u_m\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\int_0^{\|\nabla u_m\|_2^2(t)} a_\epsilon(s) ds\right) \\ &\leq |C_b| \|u_m\|_2^\beta \int_\Omega |u_m|^{q(z)-1} |\log|u_m| |\partial_t u_m| \, dx + C\|f\|_2^2 + \frac{1}{4}\|\partial_t u_m\|_2^2 \\ &\leq |C_b| \|u_m\|_2^\beta \||u_m|^{q(x,t)-1} \log|u_m|\|_2 \, \|\partial_t u_m\|_2 + C\|f\|_2^2 + \frac{1}{4}\|\partial_t u_m\|_2^2 \\ &\leq \frac{1}{2}\|\partial_t u_m\|_2^2 + C' \|u_m\|_2^{2\beta} \||u_m|^{q(z)-1} \log|u_m|\|_2^2 + C''\|f\|_2^2.\end{aligned}$$

Simplifying and using (3.5) we obtain

$$\|\partial_t u_m\|_2^2 + \frac{d}{dt}\left(\int_0^{\|\nabla u_m\|_2^2(t)} a_\epsilon(s) ds\right) \leq C \||u_m|^{q-1} \log|u_m|\|_2^2 + 2C''\|f\|_2^2 \tag{3.12}$$

with a constant C independent of m and ϵ.

Let $\tau > 2$. By assumption (A.2) and choosing σ such that $2(q(z) - 1) < 2(q(z) - 1) + \sigma \leq \frac{2N}{N-2}$, which yields $\|u_m\|_{2(q(\cdot)-1)+\sigma} \leq C\|\nabla u_m\|_2$. On the other hand, by virtue of (2.1) and Lemma 2.2 for a.e. $t \in (0, T)$

$$\begin{aligned}&\||u_m|^{q-1} \log|u_m|\|_2^2 \\ &= \int_\Omega |u_m|^{2(q(z)-1)} |\log|u_m||^2 \, dx\end{aligned} \tag{3.13}$$

$$\leq C_1 \int_\Omega |u_m|^{2(q(z)-1)+\sigma}\,dx + C_2$$
$$\leq C_1 \max\left\{ \|u_m\|^{2(q^+-1)+\sigma}_{2(q(\cdot)-1)+\sigma}, \|u_m\|^{2(q^--1)+\sigma}_{2(q(\cdot)-1)+\sigma} \right\} + C_2$$
$$\leq C \max\left\{ \|\nabla u_m\|_2^{2(q^+-1)+\sigma}, \|\nabla u_m\|_2^{2(q^--1)+\sigma} \right\} + C_2.$$

Thus,

$$\|\partial_t u_m\|_2^2 + \frac{d}{dt}\left(\int_0^{\|\nabla u_m\|_2^2(t)} a_\epsilon(s)ds\right) \leq C\left(1 + \|\nabla u_m\|_2^{2(q^+-1)+\sigma}\right) + 2C''\|f\|_2^2$$

with an independent of m and ϵ constant C. Taking into account the assumption $2(q^+-1) < 2(q^+-1)+\sigma \leq \tau$ and applying the Young inequality, we find that

$$\|\partial_t u_m\|_2^2 + \frac{d}{dt}\left(\int_0^{\|\nabla u_m\|_2^2(t)} a_\epsilon(s)ds\right) \leq C\left(1 + \|\nabla u_m\|_2^{\tau}\right) + 2C''\|f\|_2^2.$$

Integrating this inequality in t and using (3.5) together with (3.10) we obtain (3.11) in the case $\tau > 2$.

Let $\tau \in (1,2]$ and $\beta + q^+ < 2$ which yields $2(q-1) < 2$. Then by Lemma 2.2 and (3.5) for a.e $t \in \Omega$

$$\||u_m|^{q-1}\log|u_m|\|_2^2 = \int_\Omega |u_m|^{2(q(\cdot,t)-1)}|\log|u_m||^2\,dx$$
$$\leq C_1 \int_\Omega |u_m|^{2(q(\cdot,t)-1)+\sigma}\,dx + C_2$$
$$\leq C(|\Omega|, q^\pm) \max\left\{ \|u_m\|_2^{2(q^+-1+\sigma)}, \|u_m\|_2^{2(q^--1)+\sigma} \right\} + C_2$$
$$\leq C$$

and (3.12) can be continued in the following way:

$$\|\partial_t u_m\|_2^2 + \frac{d}{dt}\left(\int_0^{\|\nabla u_m\|_2^2(t)} a_\epsilon(s)ds\right) \leq C + 2C''\|f\|_2^2.$$

Integrating in t and plugging the estimates of Lemma 3.2 we obtain (3.11). □

Lemma 3.5. *Let us assume that one of the following conditions is fulfilled:*

(1) $\tau > 2$ *and* $\beta + q^+ < \tau \leq 2(\beta + q^-)$,

(2) $\tau \in (1,2]$ *and* $2(q^+ + \beta - 1) < \tau$.

Then

$$\sup_{(0,T)} \|\nabla u_m(t)\|_2^2 + \int_0^T a_\epsilon(\|\nabla u_m\|_2^2)\|\Delta u_m\|_2^2\,ds \leq C \tag{3.14}$$

with a constant C depending on $\|u_0\|_{H_0^1(\Omega)}$, $\|\nabla f\|_{2,Q_T}$, β, τ, $q^\pm$, T, but independent of m and ϵ.

Proof. Multiplying each of equations (3.4) by $\lambda_j g_{jm}(t)$, summing up, integrating by parts, and applying Young's inequality we obtain the inequality

$$
\begin{aligned}
(3.15)\qquad & \frac{1}{2}\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right) + a_\epsilon\left(\|\nabla u_m\|_2^2\right)\|\Delta u_m\|_2^2 \\
&\leq |C_b|\|u_m\|_2^\beta \int_\Omega |u_m|^{q(x,t)-1}|\Delta u_m||\log|u_m||\,dx + \int_\Omega |\nabla u_m||\nabla f|\,dx \\
&\leq \frac{1}{2}a_\epsilon\left(\|\nabla u_m\|_2^2\right)\|\Delta u_m\|_2^2 + \frac{1}{2}\frac{C_b^2\|u_m\|_2^{2\beta}}{a_\epsilon(\|\nabla u_m\|_2^2)}\||u_m|^{q(x,t)-1}\log|u_m|\|_2^2 \\
&\quad + \frac{1}{2}\|\nabla u_m\|_2^2 + \frac{1}{2}\|\nabla f\|_2^2 \\
&\equiv \frac{1}{2}a_\epsilon\left(\|\nabla u_m\|_2^2\right)\|\Delta u_m\|_2^2 + \frac{1}{2}\|\nabla u_m\|_2^2 + \frac{1}{2}\|\nabla f\|_2^2 + J.
\end{aligned}
$$

Let $\tau > 2$. Then $\beta + q^+ < \tau$,

$$a_\epsilon(\|\nabla u_m\|_2^2) \geq C_a(\epsilon + \|\nabla u_m\|_2^2)^{\frac{\tau}{2}-1} \geq C_a\|\nabla u_m\|_2^{\tau-2},$$

and by (3.13)

$$
\begin{aligned}
(3.16)\quad J &= \frac{C_b^2\|u_m\|_2^{2\beta}}{a_\epsilon(\|\nabla u_m\|_2^2)}\||u_m|^{q(x,t)-1}\log|u_m|\|_2^2 \\
&\leq \frac{C\|\nabla u_m\|_2^{2\beta}}{C_a\|\nabla u_m\|_2^{\tau-2}}\left(C_1\max\left\{\|\nabla u_m\|_2^{2(q^--1)+\sigma}, \|\nabla u_m\|_2^{2(q^+-1)+\sigma}\right\} + C_2\right) \\
&\leq C_3\max\left\{\|\nabla u_m\|_2^{2(\beta+q^-)-\tau+\sigma}, \|\nabla u_m\|_2^{2(\beta+q^+)-\tau+\sigma}\right\} \\
&\quad + C_4\|\nabla u_m\|_2^{2(\beta+1)-\tau} \\
&\leq C'\|\nabla u_m\|_2^\tau + C'',
\end{aligned}
$$

with constants C, C', C'' depending on τ, β, C_a, C_b, $q^\pm$. Then

$$\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right) + a_\epsilon(\|\nabla u_m\|_2^2)\|\Delta u_m\|_2^2 \leq C\left(1 + \|\nabla u_m\|_2^\tau\right) + \|\nabla f\|_2^2,$$

and (3.14) follows after integration in t and application of (3.10).

Let $\tau \in (1, 2]$. For every $\epsilon \in (0, 1)$

$$a_\epsilon(\|\nabla u_m\|_2^2) \geq C_a(\epsilon + \|\nabla u_m\|_2^2)^{\frac{\tau}{2}-1} \geq C_a\left(1 + \|\nabla u_m\|_2^2\right)^{\frac{\tau}{2}-1}$$

and

$$
\begin{aligned}
\frac{1}{a_\epsilon(\|\nabla u_m\|_2^2)} \leq C\left(1 + \|\nabla u_m\|_2^2\right)^{1-\frac{\tau}{2}} &\leq C\left(1 + \|\nabla u_m\|_2^{2-\tau}\right) \\
&\leq C\left(1 + \|\nabla u_m\|_2^2\right).
\end{aligned}
$$

By assumption (A.3), in this case $2(q^+ + \beta - 1) < \tau$, which is equivalent to the inequality $2(q^+ + \beta) - \tau < 2$ and yields $2(\beta + 1) - \tau < 2$. Then taking σ such that $2(q^+ + \beta) - \tau + \sigma < 2$ and employing Young's inequality we have

$$\begin{aligned}
J &\le C\left(1 + \|\nabla u_m\|_2^{2-\tau}\right) C_5 \max\left\{\|\nabla u_m\|_2^{2(q^- + \beta - 1)+\sigma}, \|\nabla u_m\|_2^{2(q^+ + \beta - 1)+\sigma}\right\} \\
&\quad + C_6 \|\nabla u_m\|_2^{2\beta} \\
&\le C\left(1 + \|\nabla u_m\|_2^{2-\tau}\right)\left(1 + \|\nabla u_m\|_2^{2(q^+ + \beta - 1)+\sigma} + \|\nabla u_m\|_2^{2\beta}\right) \\
&\le C\left(1 + \|\nabla u_m\|_2^{2(q^+ + \beta) - \tau + \sigma} + \|\nabla u_m\|_2^{2(\beta+1)-\tau}\right) \\
&\le C\left(1 + \|\nabla u_m\|_2^2\right)
\end{aligned}$$

with a constant depending on τ, $q^\pm$, β, but independent of m and ϵ. It follows now from (3.15) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right) \le C\left(1 + \|\nabla u_m\|_2^2\right) + C'\|\nabla f\|_2^2, \\
&\|\nabla u_m(0)\|_2^2 \le \|\nabla u_0\|_2^2.
\end{aligned}$$

Then

$$\|\nabla u_m\|_2^2 \mathrm{e}^{-Ct} \le \|\nabla u_0\|_2^2 + \left(1 - \mathrm{e}^{-Ct}\right) + C'\|\nabla f\|_{2,Q_T}^2,$$

whence the uniform estimate

$$\|\nabla u_m(t)\|_2^2 \le C.$$

Returning to (3.15) and using this estimate we obtain the inequality

$$\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right) + a_\epsilon(\|\nabla u_m\|_2^2)\|\Delta u_m\|_2^2 \le C,$$

whence (3.14) in the case (2). □

Lemma 3.6. *Let us assume that the conditions of Lemma 3.5 are fulfilled and*

$$\textit{either } 2 \le \tau < \min\{3, \beta + 2\} \textit{ or } \tau \in (1, 2).$$

Then

$$\|\Delta u_m\|_{L^2(Q_T)} \le C \tag{3.17}$$

with a constant C independent of m and ϵ.

Proof. Let us rewrite inequality (3.15) in the form

$$\begin{aligned}
\frac{1}{a_\epsilon(\|\nabla u_m\|_2^2)}\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right) + \|\Delta u_m\|_2^2 &\le \frac{J}{a_\epsilon(\|\nabla u_m\|_2^2)} + \frac{\|\nabla u_m\|_2\|\nabla f\|_2}{a_\epsilon(\|\nabla u_m\|_2^2)} \\
&= I + \frac{\|\nabla u_m\|_2\|\nabla f\|_2}{a_\epsilon(\|\nabla u_m\|_2^2)}
\end{aligned}$$

with J defined in (3.16).

Let $2 \le \tau < 3$. Since $\beta + q^- + 1 - \tau > \beta + 2 - \tau > 0$ by assumption, we have

$$\begin{aligned}
I &\le C\|\nabla u_m\|_2^{2\beta + 2(2-\tau)}\left(\max\left\{\|\nabla u_m\|_2^{2(q^+-1)+\sigma}, \|\nabla u_m\|_2^{2(q^- - 1)+\sigma}\right\} + 1\right) \\
&\le C\max\left\{\|\nabla u_m\|_2^{2(\beta + q^- + 1 - \tau)+\sigma}, \|\nabla u_m\|_2^{2(\beta + q^+ + 1 - \tau)+\sigma}\right\}
\end{aligned}$$

$$+ C\|\nabla u_m\|_2^{2\beta+2(2-\tau)}$$
$$\leq C\left(1 + \|\nabla u_m\|_2^{2(\beta+q^++1-\tau)+\sigma} + \|\nabla u_m\|_2^{2(\beta+2-\tau)}\right),$$

where $C = C(C_a, C_b, \beta, \tau, q^\pm)$,

$$\frac{\|\nabla u_m\|_2\|\nabla f\|_2}{a_\epsilon(\|\nabla u_m\|_2^2)} \leq \frac{1}{C_a}(\epsilon + \|\nabla u_m\|_2^2)^{\frac{2-\tau}{2}}\|\nabla u_m\|_2\|\nabla f\|_2$$
$$\leq \frac{1}{C_a}(\epsilon + \|\nabla u_m\|_2^2)^{\frac{3-\tau}{2}}\|\nabla f\|_2.$$

By Lemma 3.5, $\|\nabla u_m\|_2^2 \leq C$ uniformly with respect to m and ϵ, whence

$$\frac{d}{dt}\left(\int_0^{\|\nabla u_m(t)\|_2^2}\frac{ds}{a_\epsilon(s)}\right) + \|\Delta u_m(t)\|_2^2 \leq C' \tag{3.18}$$

and (3.17) follows after integration in t:

$$\int_0^{\|\nabla u_m(t)\|_2^2}\frac{ds}{a_\epsilon(s)} + \|\Delta u_m\|_{2,Q_T}^2 \leq C + \int_0^{\|\nabla u_0\|_2^2}\frac{ds}{a_\epsilon(s)}.$$

The integral on the right-hand side converges because

$$\int_0^{\|\nabla u_0\|_2^2}\frac{ds}{a_\epsilon(s)} \leq \frac{1}{C_a}\int_0^{\|\nabla u_0\|_2^2}\frac{ds}{(\epsilon+s^2)^{\frac{\tau}{2}-1}} \leq \frac{1}{C_a}\int_0^{\|\nabla u_0\|_2^2}\frac{ds}{s^{\tau-2}}$$

and $\tau < 3$ by assumption.

In the case $\tau \in (1,2)$ the estimate (3.17) follows directly from (3.14) and (A.1) because

$$a_\epsilon(\|\nabla u_m\|_2^2) \geq C_a(\epsilon + \|\nabla u_m\|_2^2)^{\frac{\tau}{2}-1} \geq C_a(1+K)^{\frac{\tau}{2}-1}, \quad K = \sup_{(0,T)}\|\nabla u_m(t)\|_2^2.$$

□

Lemma 3.7. *Under the conditions of Lemma 3.6*

$$\int_0^{\|\nabla u_m(t)\|_2^2} a_\epsilon(s)\,ds + \int_{Q_T} a_\epsilon^2(\|\nabla u_m\|_2^2)\left(|\nabla u_m|^2 + (\Delta u_m)^2\right)dz \leq C \tag{3.19}$$

with a constant C depending on data but independent of m and ϵ.

Proof. Let us multiply each of equations (3.4) by $a_\epsilon(\|\nabla u_m\|_2^2)\lambda_j g_{jm}(t)$ and sum up:

$$a_\epsilon(\|\nabla u_m\|_2^2)\int_\Omega \partial_t u_m \Delta u_m\,dx + a_\epsilon^2(\|\nabla u_m\|_2^2)\int_\Omega \nabla u_m\cdot\nabla(\Delta u_m)\,dx$$
$$= C_b\|u_m\|_2^\beta a_\epsilon(\|\nabla u_m\|_2^2)\int_\Omega |u_m|^{q(z)-2}u_m\log|u_m|\Delta u_m\,dx$$
$$+ a_\epsilon(\|\nabla u_m\|_2^2)\int_\Omega \Delta u_m f\,dx.$$

Integrating by parts in Ω each term in the right-hand side and applying the inequalities of Cauchy and Young to the left-hand side, we arrive at the inequality

$$\begin{aligned}&\frac{1}{2}a_\epsilon(\|\nabla u_m\|_2^2)\frac{d}{dt}\left(\|\nabla u_m\|_2^2\right)+a_\epsilon^2(\|\nabla u_m\|_2^2)\|\Delta u_m\|_2^2\\&\leq C\|u_m\|_2^{2\beta}\left(1+\|\nabla u_m\|_2^{2(q^+-1)+\sigma}\right)+\frac{1}{2}a_\epsilon^2(\|\nabla u_m\|_2^2)\|\Delta u_m\|_2^2+C'\|f\|_2^2.\end{aligned}$$

Simplifying, integrating the resulting inequality in t, and plugging the estimates of Lemmas 3.2, 3.5 on $\|u_m(t)\|_2^2$, $\|\nabla u_m(t)\|_2^2$ we obtain

$$\|a_\epsilon(\|\nabla u_m\|_2^2)\Delta u_m\|_{2,Q_T}^2\leq C.$$

To obtain (3.19) we notice that since $\|\Delta v\|_2$ is an equivalent norm of the space $H^2(\Omega)\cap H_0^1(\Omega)$, it follows from the embedding theorem that for $v\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$

$$\|\nabla v(t)\|_{2,\Omega}\leq C\|\Delta v(t)\|_{2,\Omega}\quad\text{a.e. in }(0,T)$$

and

$$\begin{aligned}\|a_\epsilon(\|\nabla u_m\|_2^2)\nabla u_m\|_{2,Q_T}^2&=\int_0^T a_\epsilon^2(\|\nabla u_m(t)\|_2^2)\|\nabla u_m(t)\|_2^2\,dt\\&\leq C\|a_\epsilon(\|\nabla u_m\|_2^2)\Delta u_m\|_{2,Q_T}^2.\end{aligned}$$ □

Remark 3.8. The a priori estimates of this section are summarized as follows: let us assume that the exponents τ, β, $q^\pm$ satisfy the conditions

$$\text{(3.20)}\qquad \begin{aligned}&\text{(a)}\quad \begin{cases} q^+<\frac{2(N-1)}{N-2} & \text{if } N>2,\\ q^+=\text{any number from }(1,\infty) & \text{if } N=2,\end{cases}\\&\text{(b)}\quad q^++\beta<\max\{2,\tau\},\quad 2(\beta+q^+-1)<\tau<\min\{3,\beta+2\}.\end{aligned}$$

Then for every $u_0\in H_0^1(\Omega)$, $f\in L^2(0,T;H_0^1(\Omega))$ the solutions u_m of problem (3.3) satisfy the estimate

$$\text{(3.21)}\qquad \begin{aligned}&\sup_{(0,T)}\|u_m(t)\|_{H_0^1(\Omega)}^2+\|\partial_t u_m\|_{L^2(Q_T)}+\|\Delta u_m\|_{L^2(Q_T)}^2\\&+\int_{Q_T}a_\epsilon^2(\|\nabla u_m\|_2^2)\left(|\nabla u_m|^2+(\Delta u_m)^2\right)\,dz\leq C\end{aligned}$$

with a constant C depending on C_a, C_b, β, τ, $q^\pm$, T, $\|u_0\|_{H_0^1(\Omega))}$, $\|f\|_{2,Q_T}$ and $\|\nabla f\|_{2,Q_T}$, but independent of m and ϵ. Since $\|\Delta v\|_{L^2(\Omega)}$ is equivalent to the norm of $H^2(\Omega)\cap H_0^1(\Omega)$, we also have the uniform estimate

$$\text{(3.22)}\qquad \|u_m\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}\leq C.$$

Remark 3.9. If $C_b=0$, a revision of the derivation of estimate (3.21) shows that it remains valid under the assumption (A.3)(ii): $1<\tau<3$.

3.3. Solution of the regularized problem: proof of Lemma 3.1

By convention, we denote by $u \equiv u_\epsilon$ the solution of the regularized problem (3.1) with a fixed $\epsilon > 0$ and by $u_m \equiv u_{\epsilon,m}$ the finite-dimensional approximations of u. Because of the uniform estimates (3.21) the sequence $\{u_m\}$ contains a subsequence, which we assume coinciding with the whole sequence, and a function $u \equiv u_\epsilon$ such that:

$$\begin{cases} u_m \rightharpoonup u \text{ weakly-star in } L^\infty(0,T;H_0^1(\Omega)), \\ \partial_t u_m \rightharpoonup \partial_t u \text{ in } L^2(Q_T), \\ u_m \rightharpoonup u \text{ in } L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)). \end{cases} \tag{3.23}$$

Since $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow H_0^1(\Omega) \subset L^2(\Omega)$, it follows from (3.21), (3.22) that $\{u_m\}$ is precompact in $L^2(0,T;H_0^1(\Omega))$ [30, Th. 5]. Thus,

$$\begin{cases} u_m \to u \text{ in } W := L^2(0,T;H_0^1(\Omega)), \\ \|u_m\|_W \to \|u\|_W \text{ and } \|u_m(t)\|_{H_0^1(\Omega)} \to \|u(t)\|_{H_0^1(\Omega)} \text{ a.e. in } (0,T). \end{cases} \tag{3.24}$$

By continuity

$$\begin{cases} a_\epsilon(\|\nabla u_m(t)\|_2^2) \to a_\epsilon(\|\nabla u(t)\|_2^2), \\ \|u_m(t)\|_2^\beta \to \|u(t)\|_2^\beta \text{ a.e. in } (0,T). \end{cases} \tag{3.25}$$

Moreover, the inclusions $u_t \in L^2(Q_T)$ and $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ yield

$$u \in C^0([0,T];H_0^1(\Omega))$$

(after possible redefining on a set of zero measure in $(0,T)$).

By virtue of (3.24) $u_m \to u$ a.e. in Q_T. On the other hand by Lemma 2.2, condition (A.2) and (3.21) for a.e $t \in \Omega$ we have

$$\begin{aligned} \int_\Omega ||u_m|^{q-2}u_m \log|u_m||^{q'}\,dx &= \int_\Omega |u_m|^q|\log|u_m||^{q'}\,dx \\ &\le C_1\int_\Omega |u_m|^{q+\sigma}\,dx + C_2 \\ &\le C\|\nabla u_m\|_2^{q^++\sigma} + C' \le C, \end{aligned}$$

where $q'(z) = \frac{q(z)}{q(z)-1}$.

Hence using the uniform in m estimate $\||u_m|^{q-2}u_m\log|u_m|\|_{L^{q'(\cdot)}(Q_T)} \le C$ and an imitation of the proof of [22, Lemma 1.3, Ch. 1] shows that

$$|u_m|^{q(z)-2}u_m\log|u_m| \rightharpoonup |u|^{q(z)-2}u\log|u| \text{ in } L^{q'(\cdot)}(Q_T). \tag{3.26}$$

Then

$$\|u_m(t)\|_2^\beta|u_m|^{q(z)-2}u_m\log|u_m| \rightharpoonup \|u(t)\|_2^\beta|u|^{q(z)-2}u\log|u| \text{ in } L^{q'(\cdot)}(Q_T).$$

It follows from (3.24), (3.25) and the estimate $\|a_\epsilon(\|\nabla u_m\|_2^2)\nabla u_m\|_2 \le C$ that

$$a_\epsilon(\|\nabla u_m\|_2^2)\nabla u_m \rightharpoonup a_\epsilon(\|\nabla u\|_2^2)\nabla u \text{ in } (L^2(Q_T))^n.$$

By the method of construction, the functions u_m satisfy the identity

$$\int_{Q_T}\Big(\partial_t u_m\varphi + a_\epsilon(\|\nabla u_m\|_2^2)\nabla u_m\cdot\nabla\varphi - C_b\|u_m\|_2^\beta|u_m|^{q(z)-2}u_m\log|u_m|\varphi\Big)\,dz$$
$$= \int_{Q_T} f\varphi\,dz$$

for every $\varphi\in\mathcal{P}_k=\operatorname{span}\{\psi_1,\ldots,\psi_k\}$, $k\le m$. Letting $m\to\infty$ we find that $u=\lim u_m$ satisfies the same identity with any $\varphi\in\mathcal{P}_k$. Since $\mathcal{P}_k$ are dense in $V=\{v\in L^2(0,T;W_0^{1,2}(\Omega))\,|\,\partial_t v\in L^2(Q_T)\}$, the same is true for every $\psi\in V$. The initial condition for $u\in C([0,T];L^2(\Omega))$ is fulfilled by continuity.

4. Proof of Theorem 2.3: existence of a strong solution of problem (1.1)

For the proof we need several auxiliary technical assertions.

Proposition 4.1. *For all $\xi,\zeta\in(L^2(\Omega))^n$ and $\epsilon\ge 0$*

$$(a_\epsilon(\|\xi\|_2^2)\xi - a_\epsilon(\|\zeta\|_2^2)\zeta,\xi-\zeta)_{2,\Omega}$$
$$\ge (a_\epsilon(\|\xi\|_2^2)\|\xi\|_2 - a_\epsilon(\|\zeta\|_2^2)\|\zeta\|_2)(\|\xi\|_2-\|\zeta\|_2).$$

Proof. It is straightforward to check that

$$\begin{aligned}
&(a_\epsilon(\|\xi\|_2^2)\xi - a_\epsilon(\|\zeta\|_2^2)\zeta,\xi-\zeta)_{2,\Omega}\\
&= a_\epsilon(\|\xi\|_2^2)\|\xi\|_2^2 - \big(a_\epsilon(\|\xi\|_2^2)+a_\epsilon(\|\zeta\|_2^2)\big)(\xi,\zeta)_2 + a_\epsilon(\|\zeta\|_2^2)\|\zeta\|_2^2\\
&\ge a_\epsilon(\|\xi\|_2^2)\|\xi\|_2^2 - \big(a_\epsilon(\|\xi\|_2^2)+a_\epsilon(\|\zeta\|_2^2)\big)\|\xi\|_2\|\zeta\|_2 + a_\epsilon(\|\zeta\|_2^2)\|\zeta\|_2^2\\
&= (a_\epsilon(\|\xi\|_2^2)\|\xi\|_2 - a_\epsilon(\|\zeta\|_2^2)\|\zeta\|_2)(\|\xi\|_2-\|\zeta\|_2).
\end{aligned}$$ □

Proposition 4.2. *If $a(\cdot)$ satisfies condition* (A.1)*, then for all ξ, $\zeta\in(L^2(\Omega))^n$ and $\epsilon\ge 0$*

$$\big(a_\epsilon(\|\xi\|_2^2)\xi - a_\epsilon(\|\zeta\|_2^2)\zeta,\xi-\zeta\big)_{2,\Omega}\ge 0.$$

Proof. Let $\epsilon>0$. Due to Proposition 4.1 it is sufficient to show that

$$(a_\epsilon(s^2)s - a_\epsilon(r^2)r)(s-r)\ge 0\quad \forall s,r\in\mathbb{R}_+ \text{ and } \epsilon\ge 0.$$

Set $z=\sqrt{\epsilon+s^2}$. By the definition of a_ϵ and assumption (A.1)

$$\begin{aligned}
(a_\epsilon(s^2)s)' &= (a(\epsilon+s^2)s)' = \left(a(\epsilon+s^2)\sqrt{\epsilon+s^2}\cdot\frac{s}{\sqrt{\epsilon+s^2}}\right)'\\
&= \left(a(z^2)z\left(\frac{s}{\sqrt{\epsilon+s^2}}\right)\right)'_s = \big(a(z^2)z\big)'_z\,\frac{s^2}{\epsilon+s^2} + \epsilon\frac{a(z^2)z}{(\epsilon+s^2)^{\frac32}}\ge 0.
\end{aligned}$$

By the Lagrange mean value theorem, for every $s,r>0$ there exists $\mu=\theta s+(1-\theta)r$ with $\theta\in(0,1)$ such that

$$(a_\epsilon(s^2)s - a_\epsilon(r^2)r)(s-r) = (a_\epsilon(\mu^2)\mu)'(s-r)^2\ge 0.$$

If $\epsilon=0$, the assertion is an immediate consequence of the Lagrange theorem. □

Let $\{u_{\epsilon_k}\}$ be a sequence of solutions of the regularized problem (3.1), $\epsilon_k \to 0$. The functions u_{ϵ_k} satisfy the uniform estimates (3.21), which allow one to extract a subsequence (we assume that it coincides with $\{u_{\epsilon_k}\}$) such that

$$(4.1)\quad \begin{cases} \partial_t u_{\epsilon_k} \rightharpoonup \partial_t u \text{ in } L^2(Q_T), \\ u_{\epsilon_k} \to u \text{ weakly in } L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \text{ and} \\ \text{strongly in } L^2(0,T;H_0^1(\Omega)), \\ a_{\epsilon_k}(\|\nabla u_{\epsilon_k}(t)\|_2^2)\nabla u_{\epsilon_k} \rightharpoonup U \text{ in } (L^2(Q_T))^n, \\ \|u_{\epsilon_k}(t)\|_2^{\beta}|u_{\epsilon_k}|^{q(z)-2}u_{\epsilon_k}\log|u_{\epsilon_k}| \rightharpoonup \|u(t)\|_2^{\beta}|u|^{q(z)-2}u\log|u| \text{ in } L^{q'(\cdot)}(Q_T) \end{cases}$$

for some $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $U \in (L^2(Q_T))^n$. For every test-function $\varphi \in \mathbf{V}$ the solution of the regularized problem u_{ϵ_k} satisfies the identity

$$(4.2)\quad \begin{aligned} &\int_{Q_T} \left(\partial_t u_{\epsilon_k}\varphi + a_{\epsilon_k}(\|\nabla u_{\epsilon_k}\|_2^2)\nabla u_{\epsilon_k}\cdot\nabla\varphi\right)\,dz \\ &= \int_{Q_T}\left(C_b\|u_{\epsilon_k}\|_2^{\beta}|u_{\epsilon_k}|^{q(z)-2}u_{\epsilon_k}\log|u_{\epsilon_k}|\varphi + f\varphi\right)\,dz. \end{aligned}$$

Let us take in (4.2) $\varphi = u_{\epsilon_k}$. Since $u_{\epsilon_k}, \partial_t u_{\epsilon_k} \in L^2(Q_T)$, the formula of integration by parts holds: $\forall t, t+h \in [0,T]$

$$(4.3)\quad \int_t^{t+h}\int_\Omega u_{\epsilon_k}\partial_t u_{\epsilon_k}\,dxds = \frac{1}{2}\,\|u_{\epsilon_k}(s)\|_2^2\Big|_{s=t}^{s=t+h}.$$

Using (4.3) in (4.2) and then passing to the limit as $\epsilon_k \to 0$ we arrive at the equality

$$(4.4)\quad \frac{1}{2}\,\|u(s)\|_2^2\Big|_{s=0}^{s=T} + \int_{Q_T} U\cdot\nabla u\,dz - C_b\int_{Q_T}\|u\|_2^{\beta}|u|^q\log|u|\,dz - \int_{Q_T} fu\,dz = 0.$$

By Proposition 4.1, for every $\varphi \in \mathbf{V}$

$$\begin{aligned} &\int_{Q_T} a_{\epsilon_k}(\|\nabla u_{\epsilon_k}\|_2^2)\nabla u_{\epsilon_k}\cdot\nabla u_{\epsilon_k}\,dz \\ &= \int_{Q_T} a_{\epsilon_k}(\|\nabla u_{\epsilon_k}\|_2^2)\nabla u_{\epsilon_k}\cdot\nabla(u_{\epsilon_k}-\varphi)\,dz + \int_{Q_T} a_{\epsilon_k}(\|\nabla u_{\epsilon_k}\|_2^2)\nabla u_{\epsilon_k}\cdot\nabla\varphi\,dz \\ &\geq \int_{Q_T} a(\|\nabla\varphi\|_2^2)\nabla\varphi\cdot\nabla(u_{\epsilon_k}-\varphi)\,dz + \int_{Q_T} a_{\epsilon_k}(\|\nabla u_{\epsilon_k}\|_2^2)\nabla u_{\epsilon_k}\cdot\nabla\varphi\,dz \\ &\quad + \int_{Q_T}\left(a_{\epsilon_k}(\|\nabla\varphi\|_2^2) - a(\|\nabla\varphi\|_2^2)\right)\nabla\varphi\cdot\nabla(u_{\epsilon_k}-\varphi)\,dz \\ &= J_1 + J_2 + J_3. \end{aligned}$$

By (4.1)

$$J_1 \to \int_{Q_T} a(\|\nabla\varphi\|_2^2)\nabla\varphi\cdot\nabla(u-\varphi)\,dz, \quad J_2 \to \int_{Q_T} U\cdot\nabla\varphi\,dz \quad \text{as } \epsilon_k \to 0.$$

According to assumption (A.1) there exists $\mu \geq 1$ such that

$$\begin{aligned} a(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2 &\leq C_\mu \left(\|\nabla\varphi\|_2^{\mu-1} + \|\nabla\varphi\|_2\right), \\ a_{\epsilon_k}(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2 &= \left(a(\epsilon_k + \|\nabla\varphi\|_2^2)(\epsilon_k + \|\nabla\varphi\|_2^2)^{\frac{1}{2}}\right) \frac{\|\nabla\varphi\|_2}{(\epsilon_k + \|\nabla\varphi\|_2^2)^{\frac{1}{2}}} \\ &\leq C_\mu \left((\epsilon_k + \|\nabla\varphi\|_2^2)^{\frac{\mu-1}{2}} + (\epsilon_k + \|\nabla\varphi\|_2^2)^{\frac{1}{2}}\right) \\ &\leq C\left(1 + \|\nabla\varphi\|_2^{\mu-1} + \|\nabla\varphi\|_2\right), \quad C = C(C_\mu, \mu). \end{aligned}$$

Let us represent

$$J_3 = \int_0^T h_{\epsilon_k}(t)\,dt, \quad h_{\epsilon_k} = \left(a_{\epsilon_k}(\|\nabla\varphi\|_2^2) - a(\|\nabla\varphi\|_2^2)\right)(\nabla\varphi, \nabla(u_{\epsilon_k} - \varphi))_{2,\Omega}.$$

The sequence $\{h_{\epsilon_k}\}$ is uniformly bounded: for every $t \in [0,T]$

$$\begin{aligned} |h_{\epsilon_k}| &\leq \left(a_{\epsilon_k}(\|\nabla\varphi\|_2^2) + a(\|\nabla\varphi\|_2^2)\right) |(\nabla\varphi, \nabla(u_{\epsilon_k} - \varphi))_{2,\Omega}| \\ &\leq \left(a_{\epsilon_k}(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2 + a(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2\right) \|\nabla(u_{\epsilon_k} - \varphi)\|_2 \\ &\leq C\left(1 + \|\nabla\varphi\|_2^{\mu-1} + \|\nabla\varphi\|_2\right) \|\nabla(u_{\epsilon_k} - \varphi)\|_2 \\ &\leq C\left(\sup_{(0,T)} \|u_{\epsilon_k}(t)\|_{H_0^1(\Omega)} + \sup_{(0,T)} \|\varphi(t)\|_{H_0^1(\Omega)}\right)\left(2 + \|\nabla\varphi\|_2^{\mu-1} + \|\nabla\varphi\|_2\right). \end{aligned}$$

It follows from Hölder's inequality and the inclusion $\varphi \in C^0([0,T]; H_0^1(\Omega))$ that

$$|h_{\epsilon_k}| \leq C'(1 + \|\nabla\varphi(t)\|_2^2) \in C^0[0,T].$$

Since for a.e. $t \in (0,T)$

$$|h_{\epsilon_k}| \leq \left|a_{\epsilon_k}(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2 - a(\|\nabla\varphi\|_2^2)\|\nabla\varphi\|_2\right| \|\nabla(u_{\epsilon_k} - \varphi)\|_2 \to 0 \text{ as } \epsilon_k \to 0,$$

it follows from the dominated convergence theorem that $J_3 \to 0$ as $\epsilon_k \to 0$.

Combining now (4.2) with (4.4) and letting $\epsilon_k \to 0$ we arrive at the inequality

$$\int_{Q_T} (a(\|\nabla\varphi\|_2^2)\nabla\varphi - U)\nabla(u - \varphi)\,dz \geq 0$$

with the arbitrary $\varphi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, $\varphi_t \in L^2(Q_T)$. Choosing $\varphi = u + \lambda\psi$ with $\lambda > 0$ and $\psi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, $\psi_t \in L^2(Q_T)$, simplifying the result and letting $\lambda \to 0^+$ leads to the inequality

$$\int_{Q_T} (a(\|\nabla u\|_2^2)\nabla u - U)\nabla\psi\,dz \leq 0 \quad \forall \psi \in \mathbf{V},$$

which is possible only if the previous relation is the equality. It follows now from (4.2) as $\epsilon_k \to 0$ that the function $u = \lim u_{\epsilon_k}$ is a strong solution of equation (1.1). The regularity properties of the limit $u(z)$ follow from the uniform estimates (3.21) for the approximations $u_{\epsilon,m}$.

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