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TWO-SIDED ESTIMATES FOR TRANSITION PROBABILITIES OF SYMMETRIC MARKOV CHAINS ON $\mathbb{Z}^d$

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Abstract. In this paper, we are mainly concerned with two-sided estimates for transition probabilities of symmetric Markov chains on $\mathbb{Z}^d$, whose one-step transition probability is comparable to $|x-y|^{-d}\phi_j(|x-y|)^{-1}$ with ϕ_j being a positive regularly varying function on $[1,\infty)$ with index $\alpha \in [2,\infty)$. For upper bounds, we directly apply the comparison idea and the Davies method, which considerably improves the existing arguments in the literature; while for lower bounds the relation with the corresponding continuous time symmetric Markov chains are fully used. In particular, our results answer one open question mentioned in the paper by Murugan and Saloff-Coste (2015).

1. Introduction

Recently symmetric Markov chains with infinite range jumps have been received a lot of interest. In particular, two-sided estimates for transition probabilities of α-stable-like symmetric Markov chains on $\mathbb{Z}^d$, whose one-step transition probability is comparable to $|x-y|^{-d-\alpha}$ with $\alpha \in (0,2)$, were first established in [4]. Since then, there have been substantially excellent works on the extensions of [4]. One direction is to consider the corresponding results for continuous time symmetric jump processes; for example, see [7] for symmetric α-stable-like processes on d-sets with $\alpha \in (0,2)$ and [8–10] for the mixture of symmetric stable-like processes on metric measure spaces. The other direction is devoted to establishing explicit estimates for transition probabilities of various discrete time Markov chains in different settings; for example, see [15] for symmetric α-stable-like processes on graphs and [13] for symmetric subordinated Markov chains on $\mathbb{Z}^d$.

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Among them, two-sided estimates for transition probabilities were presented in [15] for discrete time Markov chains on a uniformly discrete metric measure space whose one-step transition probability is comparable to

$$(V(d(x,y))\phi(d(x,y)))^{-1},$$

where $d(x,y)$ is the distance for the underlying space, ϕ is a positive continuous regularly varying function with index $\alpha \in (0,2)$, and $r \mapsto V(r)$ is the homogeneous volume growth function.

The following question was put forward in [15, (A), page 727]:

What happens if $\alpha \geq 2$? Even in the simplest setting of $\mathbb{Z}$ or $\mathbb{R}$, no sharp two-sided time-space estimates are available for the iterated kernel h_n when $\alpha \geq 2$ (especially, when $\alpha = 2$!).

Some comments are further highlighted in [15, page 727] to show that the question above is very difficult to obtain. The purpose of this paper is to address this question for symmetric Markov chains on $\mathbb{Z}^d$ completely.

Below we first describe the assumptions and the setting of our paper, and then present the main result. Two concrete examples closely related to the question above are provided to illustrate the power of our main result.

1.1. Assumptions

Throughout this paper, we use ":=" as a way of definition, which is read as "is defined by". The symbol $|\cdot|$ denotes the Euclidean norm, and $B(x,r) := \{y \in \mathbb{Z}^d : |x-y| < r\}$ for any $x \in \mathbb{Z}^d$ and $r > 0$. For two real numbers a and b, $a \wedge b := \min\{a,b\}$. For functions f and g, the notation $f \asymp g$ means that there exist constants $c_1, c_2, c_3, c_4 > 0$ such that $c_1 f(c_2 r) \leq g(r) \leq c_3 f(c_4 r)$, and the notation $f \simeq g$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 f(r) \leq g(r) \leq c_2 f(r)$.

Let $J(x,y)$ be a symmetric function on $\mathbb{Z}^d \times \mathbb{Z}^d$ such that $J(x,y) = J(y,x)$ for all $x, y \in \mathbb{Z}^d$, and that there exists a constant $C_J \geq 1$ so that for all $x, y \in \mathbb{Z}^d$ with $x \neq y$,

$$\frac{C_J^{-1}}{|x-y|^d \phi_j(|x-y|)} \leq J(x,y) \leq \frac{C_J}{|x-y|^d \phi_j(|x-y|)}. \tag{1.1}$$

Here $\phi_j : [1,\infty) \to [1,\infty)$ is a non-decreasing function such that $\phi_j(1) = 1$ and that there are constants $c_2 \geq c_1 > 0$ and $\alpha_2 \geq \alpha_1 \geq 2$ so that for all $R \geq r \geq 1$,

$$c_1 \Big(\frac{R}{r}\Big)^{\alpha_1} \leq \frac{\phi_j(R)}{\phi_j(r)} \leq c_2 \Big(\frac{R}{r}\Big)^{\alpha_2}. \tag{1.2}$$

Below we extend ϕ_j to $\mathbb{R}_+ := [0,\infty)$ by setting $\phi_j(r) = r^{3/2}$ for all $r \in [0,1]$. In particular, ϕ_j is increasing on $\mathbb{R}_+$ with $\phi_j(0) = 0$. Correspondingly, we can define

$$\phi_c(r) := \frac{r^2}{\int_0^r \frac{s}{\phi_j(s)} \mathrm{d}s}, \quad r \geq 0. \tag{1.3}$$

The definition of ϕ_c is taken from [1, (1.10)]. Actually, as indicated by [1], ϕ_j is the scaling function of the transition kernel, and ϕ_c will play a role in the scaling function for the process. Note that, by (2.2) and Lemma 2.2 below, $2\phi_j(r) \geq \phi_c(r)$ for all $r \geq 0$, and

$$\lim_{r\to\infty} \frac{\phi_j(r)}{\phi_c(r)} = \infty.$$

Throughout the paper, the following Assumption **(H)** is imposed on the scaling functions ϕ_j and ϕ_c in force.

(H) *For any* $\gamma > 0$*, there exist constants* $l \in (0, 1)$ *and* $C > 0$ *such that for all* $r \geq 1$*,*

$$\int_0^{lr} \exp\left\{\gamma \frac{s}{M(r)}\right\} \frac{s}{\phi_j(s)}\, \mathrm{d}s \leq C \frac{M(r)^2}{\phi_c(M(r))}, \tag{1.4}$$

where

$$M(r) := r/\log\left(\frac{4\phi_j(r)}{\phi_c(r)}\right).$$

Remark 1.1. Since

$$\int_0^{M(r)} \exp\left\{\gamma \frac{s}{M(r)}\right\} \frac{s}{\phi_j(s)}\, \mathrm{d}s \leq e^{\gamma} \int_0^{M(r)} \frac{s}{\phi_j(s)}\, \mathrm{d}s = e^{\gamma} \frac{M(r)^2}{\phi_c(M(r))},$$

(1.4) is satisfied, when for any $\gamma > 0$ there exist $l \in (0, 1)$ and $A > 0$ such that for all $r \geq 1$ with $lr \geq M(r)$,

$$\int_{M(r)}^{lr} \exp\left\{\gamma \frac{s}{M(r)}\right\} \frac{s}{\phi_j(s)}\, \mathrm{d}s \leq A \frac{M(r)^2}{\phi_c(M(r))}. \tag{1.5}$$

1.2. Main result

Now, we consider a symmetric Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ with a one-step transition probability given by

$$p(x, y) = \frac{J(x, y)}{\sum_{z\neq x} J(x, z)}, \tag{1.6}$$

where $J(x, y)$ is given above and satisfies (1.1). We are interested in two-sided estimates for the transition probabilities $p_n(x, y) := \mathbb{P}^x(X_n = y)$ of the process $(X_n)_{n\geq 0}$, where $\mathbb{P}^x$ is denoted by the probability of the process $(X_n)_{n\geq 0}$ with $X_0 = x$. According to the Chapman-Kolmogorov equations, for any $1 \leq k \leq n$ and $x, y \in \mathbb{Z}^d$,

$$p_n(x, y) = \sum_{z\in\mathbb{Z}^d} p_k(x, z)p_{n-k}(z, y),$$

where $p_0(x, y) = \delta_x(y)$ and $p_1(x, y) = p(x, y)$.

The main result of the paper is as follows.

Theorem 1.2. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then there are constants* $c_1, c_2 > 0$ *such that for all* $x, y \in \mathbb{Z}^d$ *and* $n \geq 1$,

$$p_n(x,y) \asymp \begin{cases} [\phi_c^{-1}(n)]^{-d}, & n \geq c_1\phi_c(|x-y|), \\ [\phi_c^{-1}(n)]^{-d} \exp\left\{ -\frac{|x-y|}{\bar{\phi}_c^{-1}(\frac{n}{|x-y|})} \right\}, & c_2 n_*(|x-y|) \leq n \leq c_1\phi_c(|x-y|), \\ \frac{n}{|x-y|^d \phi_j(|x-y|)}, & n \leq c_2 n_*(|x-y|), \end{cases} \tag{1.7}$$

where $\bar{\phi}_c(r) = \phi_c(r)/r$, $\phi_c^{-1}(r) = \inf\{s \geq 0 : \phi_c(s) \geq r\}$, $\bar{\phi}_c^{-1}(r) = \inf\{s \geq 0 : \bar{\phi}_c(s) \geq r\}$ *and*

$$n_*(r) = r\bar{\phi}_c\left(\frac{r}{\log \frac{4\phi_j(r)}{\phi_c(r)}}\right) = \phi_c\left(\frac{r}{\log \frac{4\phi_j(r)}{\phi_c(r)}}\right) \log \frac{4\phi_j(r)}{\phi_c(r)}, \qquad r \geq 1.$$

Remark 1.3. We make three comments on Theorem 1.2 and its proof.

(i) Theorem 1.2 is analogous to [1, Theorems 1.2 and 1.4], where sharp two-sided estimates of the transition densities (heat kernel) for symmetric jump processes on $\mathbb{R}^d$ whose weak scaling index is not necessarily strictly less than 2 (that is, jump processes in [1] may be allowed to have light tails of jumping kernels with any polynomial decay at infinity) were established. The readers can refer to [2,11] for further study on two-sided heat kernel estimates and their stabilities properties for symmetric non-local Dirichlet forms of pure jump type on metric measure space.

(ii) As done in [4, 15], in order to prove the upper bounds of transition probabilities stated in Theorem 1.2 for discrete time Markov chain $(X_n)_{n\geq 0}$ we will partly make use of heat kernel estimates for the continuous time Markov chain $(Y_t)_{t\geq 0}$ associated with $(X_n)_{n\geq 0}$; see the end of Section 2 for the definition and properties of $(Y_t)_{t\geq 0}$. Instead of directly apply the results in [1], in the present paper we will prove upper bounds of transition probabilities for $(Y_t)_{t\geq 0}$ via the so-called Davies method. We mention that the argument in [1] is based on some self-improving arguments starting from rough estimates for the exit time of the process $(Y_t)_{t\geq 0}$. As we see below, the Davies method adopt here considerably simplifies the proofs in [1], and the approach should be interesting and useful of its own.

(iii) For lower bounds of transition probabilities in Theorem 1.2, the common method in the literature is to establish parabolic Harnack inequalities for the Markov chain $(X_n)_{n\geq 0}$, whose proof is lengthy; see [4, 15]. However, here we directly use the probabilistic relation between the processes $(X_n)_{n\geq 0}$ and $(Y_t)_{t\geq 0}$, which immediately yields near-diagonal lower bounds for transition probabilities of $(X_n)_{n\geq 0}$; see Proposition 4.3.

1.3. Two examples

In this part, we take two examples which indicate that the function $\phi_j(r) = r^\beta$ for $r \geq 1$ with $\beta \geq 2$ satisfies Assumption **(H)**. Therefore, Theorem 1.2

applies. In particular, this answers the question in [15, (A), page 727] affirmatively for symmetric Markov chains on $\mathbb{Z}^d$ with this special scale function $\phi_j(r) = r^\beta$ for all $r \geq 1$ with $\beta \geq 2$.

Example 1.4. Suppose that ϕ_j satisfies (1.2) with $\alpha_1 > 2$. Then, $\phi_c(r) \simeq r^2$ for $r \geq 1$. In particular, $\bar{\phi}_c(r) \simeq r$, $\bar{\phi}_c^{-1}(r) \simeq r$, and so for all $r > 1$ large enough,

$$M(r) = r \Big/ \log \frac{4\phi_j(r)}{\phi_c(r)} \simeq r(\log r)^{-1},$$

thanks to (1.2).

In order to verify Assumption **(H)**, we only need to consider $r \geq 1$ large enough. First, define

$$F_1(s) = \exp\left\{\gamma \frac{s}{M(r)}\right\}, \quad F_2(s) = C_0(e+s)^\theta, \quad s \in [M(r), lr]$$

with $\gamma, C_0 > 0$ and $0 < \theta < 1$. We will claim that there are $l, C_0 > 0$ and $\theta \in (0, 1 \wedge (\alpha_1 - 2))$ such that for all $r \geq 1$ large enough with $lr \geq 1$ and for all $s \in [M(r), lr]$, $F_1(s) \leq F_2(s)$. Indeed, set $H(s) := F_2(s) - F_1(s)$. By choosing $C_0 > 0$ large enough, we find that for all $r \geq 1$,

$$H(M(r)) := F_2(M(r)) - F_1(M(r)) = C_0(e + M(r))^\theta - e^\gamma \geq C_0 e^\theta - e^\gamma \geq 0.$$

On the other hand, for all $r \geq 1$ large enough with $lr \geq 1$ and for all $s \in [M(r), lr]$,

$$\begin{aligned} H'(s) = F_2'(s) - F_1'(s) &= \theta C_0 (e+s)^{\theta-1} - \frac{\gamma}{M(r)} \exp\left\{\gamma \frac{s}{M(r)}\right\} \\ &\geq \theta C_0 (e+s)^{\theta-1} - \frac{c_1 \gamma \log r}{r} r^{\frac{c_1 \gamma s}{r}} \\ &\geq \theta C_0 (e+1)^{\theta-1} l^{\theta-1} r^{\theta-1} - c_1 \gamma \log r \cdot r^{c_1 l \gamma - 1}, \end{aligned}$$

where in the first inequality we used $M(r) \simeq r(\log r)^{-1}$ for $r \geq 1$ large enough and $c_1 > 0$ is independent of r. Then, for fixed $\theta \in (0, 1 \wedge (\alpha_1 - 2))$, we can choose $l = \theta/(2c_1\gamma)$ so that $H'(s) \geq 0$ for all $r \geq l^{-1}$ large enough and $s \in [M(r), lr]$. Hence, with the choices of C_0 and l above, we can obtain that $F_1(s) \leq F_2(s)$ for all $r \geq l^{-1}$ large enough and $s \in [M(r), lr]$. Therefore, due to $\theta \in (0, 1 \wedge (\alpha_1 - 2))$, for all $r \geq l^{-1}$ large enough,

$$\int_{M(r)}^{lr} \exp\left\{\gamma \frac{s}{M(r)}\right\} \frac{1}{s^{\alpha_1 - 1}} \mathrm{d}s \leq \int_1^{+\infty} C_0 (e+s)^\theta \frac{1}{s^{\alpha_1 - 1}} \,\mathrm{d}s := C < \infty,$$

which, along with the fact $\phi_j(r) \geq c_* r^{\alpha_1}$ for all $r \geq 1$ with some constant $c_* > 0$ (due to (1.2) again), yields that (1.5) is satisfied for all $r > 1$ large enough, and so Assumption **(H)** holds true.

Thus, according to Theorem 1.2, the associated transition probabilities $p_n(x, y)$ satisfy the following estimates: there are constants $c_2, c_3 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \ge 1$,

$$p_n(x,y) \asymp \begin{cases} n^{-d/2}, & n \ge c_2|x-y|^2, \\ n^{-d/2}\exp\left\{-\frac{|x-y|^2}{n}\right\}, & c_3|x-y|^2\log^{-1}(1+|x-y|) \le n \le c_2|x-y|^2, \\ \frac{n}{|x-y|^d\phi_j(|x-y|)}, & n \le c_3|x-y|^2\log^{-1}(1+|x-y|). \end{cases}$$

Example 1.5. Suppose that $\phi_j(r) = r^2$ for all $r \ge 1$. Then, for all $r \ge 1$ large enough, $\phi_c(r) \simeq \frac{r^2}{\log r}$, $\bar\phi_c(r) \simeq \frac{r}{\log r}$ and $\bar\phi_c^{-1}(r) \simeq r\log r$; moreover, for all $r \ge 1$ large enough,

$$M(r) = r \Big/ \log\frac{4\phi_j(r)}{\phi_c(r)} \simeq r(\log\log r)^{-1}, \quad \frac{M(r)^2}{\phi_c(M(r))} \simeq \log r.$$

For any $\gamma > 0$, $l \in (0,1)$ and $r \ge 1$ large enough,

$$\begin{aligned} \int_{M(r)}^{lr} \exp\left\{\gamma\frac{s}{M(r)}\right\}\frac{1}{s}\mathrm{d}s &\le \exp\left\{\gamma\frac{lr}{M(r)}\right\}\frac{lr}{M(r)} \\ &\le c_1\exp\Big\{c_2\gamma l\log\log r\Big\}l(\log\log r) \\ &\le c_1 l(\log r)^{c_2\gamma l}(\log\log r), \end{aligned}$$

where c_1, c_2 are independent of r. Then, by taking $l = 1/(2(c_2\gamma \vee 1))$ in the inequality above, we know that (1.5) is also true for all $r \ge l^{-1}$ large enough. Hence, Assumption **(H)** is satisfied for this example.

Therefore, according to Theorem 1.2, the associated transition probabilities $p_n(x,y)$ satisfy the following estimates: there are constants $c_3, c_4 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \ge 1$,

$$p_n(x,y) \asymp \begin{cases} [n\log(1+n)]^{-d/2}, & n \ge c_3 n_1(|x-y|), \\ [n\log(1+n)]^{-d/2}\exp\left\{-\frac{|x-y|^2}{n\log(1+n/|x-y|)}\right\}, & c_4 n_2(|x-y|) \le n \\ & \le c_3 n_1(|x-y|), \\ \frac{n}{|x-y|^{d+2}}, & n \le c_4 n_2(|x-y|), \end{cases}$$

where

$$n_1(r) = r^2\log^{-1}(1+r), \quad n_2(r) = r^2\log^{-1}(1+r)\log^{-1}\log(e+r).$$

The rest of the paper is arranged as follows. In Section 2, we give preliminaries on the scale functions ϕ_j and ϕ_c as well as some properties for $J(x,y)$. Section 3 and Section 4 are devoted to the proofs of upper bounds and lower bounds for the transition probabilities $p_n(x,y)$, respectively.

2. Preliminaries

In this section, we present some preliminary results that will be frequently used in the sequel. Recall that

$$\phi_c(r) = \frac{r^2}{\int_0^r \frac{s}{\phi_j(s)}\,\mathrm{d}s}, \quad r \geq 0.$$

Since $\phi_j(r) = r^{3/2}$ for all $r \in [0,1]$, we know that $\phi_c(r) = \frac{1}{2}r^{3/2}$ for $r \in [0,1]$. Then, by (1.2), there is a constant $c_0 \geq 1$ such that for all $R \geq r > 0$,

$$c_0^{-1}\left(\frac{R}{r}\right)^{3/2} \leq \frac{\phi_j(R)}{\phi_j(r)} \leq c_0\left(\frac{R}{r}\right)^{\alpha_2}, \tag{2.1}$$

where $\alpha_2 \geq 2$ is given in (1.2). On the other hand, because ϕ_j is non-decreasing on $[0,\infty)$,

$$\phi_c(r) = \frac{r^2}{\int_0^r \frac{s}{\phi_j(s)}\,\mathrm{d}s} \leq \frac{r^2}{\int_0^r \frac{s}{\phi_j(r)}\,\mathrm{d}s} = 2\phi_j(r), \quad r \geq 0. \tag{2.2}$$

Furthermore, it is easy to see that for all $0 < r \leq R$,

$$\frac{\phi_c(R)}{\phi_c(r)} = \frac{R^2/\int_0^R s/\phi_j(s)\,\mathrm{d}s}{r^2/\int_0^r s/\phi_j(s)\,\mathrm{d}s} \leq \frac{R^2}{r^2}. \tag{2.3}$$

With the aid of all the discussions above, we have the following statements for $\phi_c(r)$ and $\bar{\phi}_c(r) := \phi_c(r)/r$.

Lemma 2.1. *Under* (1.2)*, there are constants* $c_2 \geq c_1 > 0$ *such that for all* $0 < r \leq R$*,*

$$c_1\Big(\frac{R}{r}\Big)^{3/2} \leq \frac{\phi_c(R)}{\phi_c(r)} \leq c_2\Big(\frac{R}{r}\Big)^2. \tag{2.4}$$

In particular, for all $0 < r \leq R$*,*

$$c_1\Big(\frac{R}{r}\Big)^{1/2} \leq \frac{\bar{\phi}_c(R)}{\bar{\phi}_c(r)} \leq c_2\frac{R}{r} \tag{2.5}$$

and there is a constant $c_3 > 0$ *such that for all* $0 < r \leq R$*,*

$$R/\bar{\phi}_c^{-1}(R) \leq c_3 r/\bar{\phi}_c^{-1}(r). \tag{2.6}$$

Proof. By (2.3), we only need to verify the first inequality in (2.4). For any $R \geq r \geq 0$,

$$\int_0^R \frac{s}{\phi_j(s)}\,\mathrm{d}s = \left(\frac{R}{r}\right)^2 \int_0^r \frac{u}{\phi_j(Ru/r)}\,\mathrm{d}u \leq c_0\left(\frac{R}{r}\right)^{1/2}\int_0^r \frac{u}{\phi_j(u)}\,\mathrm{d}u,$$

where in the inequality we used (2.1). This along with the definition of ϕ_c can yield the desired assertion. □

Next, we present two lemmas related to (1.2).

Lemma 2.2. *Under* (1.2), *it holds that*

$$\lim_{r\to\infty}\frac{\phi_j(r)}{\phi_c(r)}=\infty.$$

Proof. Note that

$$\frac{\phi_j(r)}{\phi_c(r)}=\frac{\phi_j(r)}{r^2}\int_0^r\frac{s}{\phi_j(s)}\,\mathrm{d}s$$

and, by (1.2),

$$c_1r^{\alpha_1}\le\phi_j(r)\le c_2r^{\alpha_2},\quad r\ge 1$$

for some $\alpha_2\ge\alpha_1\ge 2$. It is easy to see that the desired assertion holds when $\int_0^\infty s/\phi_j(s)\,\mathrm{d}s=\infty$.

On the other hand, if $\int_0^\infty s/\phi_j(s)\,\mathrm{d}s<\infty$, then there is a constant $c_3>0$ so that for all $r\ge 1$,

$$\frac{\phi_j(r)}{\phi_c(r)}\ge c_3\frac{\phi_j(r)}{r^2}.$$

We claim that $\lim_{r\to\infty}\phi_j(r)/r^2=\infty$, and so the desired assertion holds as well. If it is not true, then there is a constant $c_4>0$ such that for all $\phi_j(r)/r\le c_4r$ for all $r\ge 1$, and so $\int_0^\infty s/\phi_j(s)\,\mathrm{d}s=\infty$, which is a contradiction. Next, we verify that $\lim_{r\to\infty}\phi_j(r)/r^2=\infty$. Indeed, since $\int_1^\infty s/\phi_j(s)\,\mathrm{d}s=\sum_{n=0}^\infty\int_{2^n}^{2^{n+1}}s/\phi_j(s)\,\mathrm{d}s<\infty$, it follows from (1.2) that $\sum_{n=0}^\infty\frac{2^{2n}}{\phi_j(2^n)}<\infty$ and so

$$\lim_{n\to\infty}\frac{2^{2n}}{\phi_j(2^n)}=0,$$

which along with (1.2) again yields that $\lim_{r\to\infty}r^2/\phi_j(r)=0$; that is,

$$\lim_{r\to\infty}\phi_j(r)/r^2=\infty.$$

The proof is complete. □

Lemma 2.3. *Under* (1.2), *there is a constant* $c>0$ *such that for all* $r\ge 1$,

$$\log\frac{4\phi_j(r)}{r\bar\phi_c(M(r))}\le c\log\frac{4\phi_j(r)}{\phi_c(r)},$$

where $\bar\phi_c(r)=\phi_c(r)/r$, *and* $M(r)$ *is defined in Assumption* **(H)**.

Proof. For any $r\ge 1$, by the definition of $M(r)$, we can write

$$\begin{aligned}\log\frac{4\phi_j(r)}{r\bar\phi_c(M(r))}&=\log\frac{4\phi_j(r)}{\phi_c(M(r))}-\log\log\frac{4\phi_j(r)}{\phi_c(r)}\\&=\log\frac{\phi_c(r)}{\phi_c(M(r))}+\log\frac{4\phi_j(r)}{\phi_c(r)}-\log\log\frac{4\phi_j(r)}{\phi_c(r)}.\end{aligned}$$

So, it suffices to verify that there is a constant $c_0>0$ so that for any $r\ge 1$,

$$\log\frac{\phi_c(r)}{\phi_c(M(r))}\le c_0\log\frac{4\phi_j(r)}{\phi_c(r)}.$$

Indeed, according to the definition $\phi_c(r)$,

$$\begin{aligned}\log \frac{\phi_c(r)}{\phi_c(M(r))} =&\log \frac{r^2}{M(r)^2} + \log\left(\int_0^{M(r)} s/\phi_j(s)\,\mathrm{d}s \Big/ \int_0^r s/\phi_j(s)\,\mathrm{d}s\right)\\ \le& 2\log\log \frac{4\phi_j(r)}{\phi_c(r)} + \log\left(\int_0^{r/\log 2} s/\phi_j(s)\,\mathrm{d}s \Big/ \int_0^r s/\phi_j(s)\,\mathrm{d}s\right)\\ \le& c_1 + 2\log\log \frac{4\phi_j(r)}{\phi_c(r)},\end{aligned}$$

where in the first inequality we used the fact that $M(r) \le r/\log 2$ (thanks to $2\phi_j(r) \ge \phi_c(r)$ for all $r \ge 1$), and the last inequality follows from the scaling property (2.1) of $\phi_j(r)$. Hence, we can obtain the desired assertion. □

The following lemma is a direct consequence of (1.1) on $J(x, y)$.

Lemma 2.4. *Under* (1.1) *and* (1.2), *there exist constants* $C_1, C_2 > 0$ *such that for all* $x \in \mathbb{Z}^d$ *and* $r \ge 1$,

$$\sum_{y\in B(x,r)^c} J(x,y) \le \frac{C_1}{\phi_j(r)} \tag{2.7}$$

and

$$\sum_{y\in B(x,r)} |x-y|^2 J(x,y) \le C_2 \int_0^r \frac{s}{\phi_j(s)}\,\mathrm{d}s. \tag{2.8}$$

Proof. According to (1.1) and (1.2),

$$\begin{aligned}\sum_{y\in B(x,r)^c} J(x,y) \le& C_J \sum_{y\in B(x,r)^c} \frac{1}{|x-y|^d \phi_j(|x-y|)}\\ =& C_J \sum_{n=1}^{\infty} \sum_{2^{n-1}r\le |x-y| < 2^n r} \frac{1}{|x-y|^d\phi_j(|x-y|)}\\ \le& c_1 \sum_{n=1}^{\infty} \frac{(2^n r)^d}{(2^{n-1}r)^d \phi_j(2^{n-1}r)} \le \frac{c_2}{\phi_j(r)} \sum_{n=1}^{\infty} 2^{-n\alpha_1} \le \frac{c_3}{\phi_j(r)}\end{aligned}$$

and

$$\sum_{y\in B(x,r)} |x-y|^2 J(x,y) \le C_J \sum_{y\in B(x,r)} \frac{1}{|x-y|^{d-2}\phi_j(|x-y|)} \le c_4 \int_0^r \frac{s}{\phi_j(s)}\,\mathrm{d}s.$$

The proof is completed. □

Recall that $(X_n)_{n\ge 0}$ is a symmetric Markov chain on $\mathbb{Z}^d$ with one-step transition probability $p(x,y) = \mu_x^{-1} J(x,y)$, where

$$\mu_x = \sum_{z\ne x} J(x,z). \tag{2.9}$$

According to (1.1), there is a constant $c_0 \ge 1$ such that for all $x \in \mathbb{Z}^d$, $c_0^{-1} \le \mu_x \le c_0$; that is, $\mu(\mathrm{d}z) = \sum_{x\in\mathbb{Z}^d} \mu_x \delta_x(\mathrm{d}z)$ is comparable with the counting measure on $\mathbb{Z}^d$, where $\delta_x(\cdot)$ is the Dirac measure. For the proof of our main result, we also need the corresponding continuous time Markov chain associated with $(X_n)_{n\ge 0}$, which is defined by $Y_t := X_{N(t)}$ and $N(t)$ is a standard Poisson process independent of $(X_n)_{n\ge 0}$. For the details of the construction of the process $(Y_t)_{t\ge 0}$, one can refer to [4, Section 2] or [15, Section 1]. Furthermore, we can see that the infinitesimal generator of the process $(Y_t)_{t\ge 0}$ is given by

$$L_Y f(x) = \mu_x^{-1} \sum_{y\neq x} (f(y) - f(x)) J(x, y).$$

In particular, L_Y is symmetric on $L^2(\mathbb{Z}^d;\mu)$ and, by (1.1) again, the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} (f(x) - f(y))^2 J(x, y), \quad f \in \mathcal{F}, \tag{2.10}$$
$$\mathcal{F} = \left\{ f \in L^2(\mathbb{Z}^d;\mu) : \mathcal{E}(f, f) < \infty \right\} = L^2(\mathbb{Z}^d;\mu).$$

Indeed, $(\mathcal{E}, \mathcal{F})$ is regular; e.g., see (the first statement in) [5, Theorem 3.2]. Denote the transition probability density with respect to μ of $(Y_t)_{t\ge 0}$ by $q(t, \cdot, \cdot)$. It holds that

$$q(t, x, y) := \frac{\mathbb{P}^x(Y_t = y)}{\mu_x} = \sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} p_n(x, y), \quad t \ge 0,\ x, y \in \mathbb{Z}^d.$$

3. Upper bounds for transition probabilities

In this section, we establish upper bounds for the transition probabilities $p_n(x, y)$ of the Markov chain $(X_n)_{n\ge 0}$. For this aim, we will study the upper bounds for heat kernel $q(t, x, y)$ of the continuous time Markov chain $(Y_t)_{t\ge 0}$, which consists of two steps. First, we apply the comparison idea and the Nash-type inequality to obtain on-diagonal upper bounds for $q(t, x, y)$, and then adopt the Davies method to derive off-diagonal upper bounds for $q(t, x, y)$. With the upper bounds of $q(t, x, y)$ at hand, we can present the corresponding results for $p_n(x, y)$ by using the comparison idea as well as the $\Delta(\alpha)$ condition originated from [14].

3.1. On-diagonal upper bounds for $q(t, x, y)$

To establish on-diagonal upper bounds for $q(t, x, y)$, we will compare the process $(Y_t)_{t\ge 0}$ with a Lévy process $(Z_t)_{t\ge 0}$, which takes the value on $\mathbb{Z}^d$ and whose Lévy measure is given by

$$n(\mathrm{d}x) = \sum_{z\in\mathbb{Z}^d : z\neq 0} \frac{1}{|z|^d \phi_j(|z|)} \delta_z(\mathrm{d}x).$$

We begin with the following lemma.

Lemma 3.1. *The Lévy process* $(Z_t)_{t\geq 0}$ *has the transition density* $q^Z(t,x,y)$, *which satisfies that there is a constant* $C_0 > 0$ *so that for all* $x, y \in \mathbb{Z}^d$ *and* $t > 0$,

$$q^Z(t,x,y) \leq C_0[\phi_c^{-1}(t)]^{-d}. \tag{3.1}$$

Proof. Let $\varphi_t^Z(u)$ be the characteristic function of the process $(Z_t)_{t\geq 0}$. Since $(Z_t)_{t\geq 0}$ takes value on $\mathbb{Z}^d$, $\varphi_t^Z(u)$ is periodic with period 2π. According to the Lévy-Khintchine formula and the symmetry of the Lévy measure $n(\mathrm{d}z)$, for all $t > 0$,

$$\varphi_t^Z(u) = \exp\left\{ -t \sum_{z\in\mathbb{Z}^d} (1-\cos\langle u,z\rangle)\frac{1}{|z|^d\phi_j(|z|)}\right\}, \quad |u| \leq \pi.$$

When $|u| \leq 1/2$,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}(1-\cos\langle u,z\rangle)\frac{1}{|z|^d\phi_j(|z|)} &\geq \sum_{z\in\mathbb{Z}^d:|z|\leq 1/|u|}(1-\cos\langle u,z\rangle)\frac{1}{|z|^d\phi_j(|z|)}\\ &\geq \frac{\cos 1}{2}\sum_{z\in\mathbb{Z}^d:|z|\leq 1/|u|}\langle u,z\rangle^2\frac{1}{|z|^d\phi_j(|z|)}\\ &= \frac{\cos 1\cdot|u|^2}{2}\sum_{z\in\mathbb{Z}^d:|z|\leq 1/|u|}\langle u/|u|,z\rangle^2\,\frac{1}{|z|^d\phi_j(|z|)},\end{aligned}$$

where the second inequality follows from the fact that $1-\cos r \geq \frac{\cos 1}{2}r^2$ for all $|r| \leq 1$. For any $x \in \mathbb{R}^d$, let $x = (x_1, x_2, \ldots, x_d)$. By the rotational invariance property, without loss of generality we assume that $u/|u| = e_1 := (1,0,0,\ldots)$. Then

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}(1-\cos\langle u,z\rangle)\frac{1}{|z|^d\phi_j(|z|)} &\geq \frac{\cos 1\cdot|u|^2}{2}\sum_{z\in\mathbb{Z}^d:|z|\leq 1/|u|}\frac{z_1^2}{|z|^d\phi_j(|z|)}\\ &= \frac{\cos 1\cdot|u|^2}{2d}\sum_{z\in\mathbb{Z}^d:1\leq|z|\leq 1/|u|}\frac{1}{|z|^{d-2}\phi_j(|z|)}\\ &= c_1|u|^2\sum_{1\leq r\leq 1/|u|}\frac{r}{\phi_j(r)} \geq c_2\frac{1}{\phi_c(1/|u|)},\end{aligned}$$

where the last equality follows from the definition of the scale function $\phi_c(r)$.

When $1/2 \leq |u| \leq \pi$, without loss of generality we may and can assume that $|u_1| \geq \frac{1}{2\sqrt{d}}$. Choosing $z_0 = e_1$,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}(1-\cos\langle u,z\rangle)\frac{1}{|z|^d\phi_j(|z|)} &\geq \frac{1}{|z_0|^d\phi_j(|z_0|)}(1-\cos\langle u,z_0\rangle)\\ &\geq \frac{1}{\phi_j(1)}\left(1-\cos\frac{1}{2\sqrt{d}}\right) \geq c_3.\end{aligned}$$

Hence, for all $t > 0$,

$$
\begin{aligned}
q^Z(t,x,y) &= \frac{1}{(2\pi)^d}\int_{\{|u|\le\pi\}} e^{\mathrm{i}\langle x-y,u\rangle}\varphi_t^Z(u)\,\mathrm{d}u \le \frac{1}{(2\pi)^d}\int_{\{|u|\le\pi\}}\varphi_t^Z(u)\,\mathrm{d}u\\
&\le c_4\left\{\int_{\{1/2\le|u|\le\pi\}} e^{-c_3t}\,\mathrm{d}u + \int_{\{|u|\le 1/2\}}\exp\Big(-\frac{c_2t}{\phi_c(1/|u|)}\Big)\,\mathrm{d}u\right\}\\
&=: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}
$$

Note that

$$\mathrm{I}_1 \le c_5 e^{-c_3t}.$$

On the other hand,

$$
\begin{aligned}
\mathrm{I}_2 &\le c_6\int_0^\infty r^{d-1}\exp\Big(-\frac{c_2t}{\phi_c(1/r)}\Big)\,\mathrm{d}r = \frac{c_6}{d}\int_0^\infty e^{-c_2s}\,\mathrm{d}_s\left[\frac{1}{\phi_c^{-1}(t/s)}\right]^d\\
&= \frac{c_6}{d}\Big\{\int_0^1+\sum_{n=0}^\infty\int_{2^n}^{2^{n+1}}\Big\}e^{-c_2s}\,\mathrm{d}_s\left[\frac{1}{\phi_c^{-1}(t/s)}\right]^d\\
&\le \frac{c_6}{d}[\phi_c^{-1}(t)]^{-d}+\frac{c_6}{d}\sum_{n=0}^\infty e^{-c_22^n}[\phi_c^{-1}(t/2^{n+1})]^{-d}\le c_7[\phi_c^{-1}(t)]^{-d},
\end{aligned}
$$

where the last inequality is a consequence of (2.4).

By (2.4) again, there is a constant $c_8 > 0$ so that $e^{-c_3t} \le c_8[\phi_c^{-1}(t)]^{-d}$ for all $t > 0$. Combining with all the estimates above, we finish the proof. □

We now can obtain on-diagonal upper bounds for the transition densities of $(Y_t)_{t\ge0}$.

Proposition 3.2. *Suppose that* (1.1) *and* (1.2) *hold. Then the continuous time Markov chain* $(Y_t)_{t\ge0}$ *has the transition density* $q(t,x,y)$*, and there exists a constant* $C_1 > 0$ *such that for all* $x,y\in\mathbb{Z}^d$ *and* $t>0$,

$$q(t,x,y)\le C_1[\phi_c^{-1}(t)]^{-d}. \tag{3.2}$$

Proof. Let μ be the positive measure on $\mathbb{Z}^d$ so that $\mu(\{x\}) = \mu_x$ for all $x\in\mathbb{Z}^d$, where μ_x is given by (2.9). Let $(\mathcal{E},\mathcal{F})$ (with $\mathcal{F} = L^2(\mathbb{Z}^d;\mu)$) be the Dirichlet form on $L^2(\mathbb{Z}^d;\mu)$ associated with the continuous time Markov chain $(Y_t)_{t\ge0}$; see (2.10). Denote by $(\mathcal{E}^Z,\mathcal{F}^Z)$ the Dirichlet form corresponding to the Lévy process $(Z_t)_{t\ge0}$. It is easy to see that $(\mathcal{E},\mathcal{F})$ is comparable with $(\mathcal{E}^Z,\mathcal{F}^Z)$ thanks to (1.1), and $(\mathcal{E},\mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(\mathbb{Z}^d;\mu)$,

Set

$$\psi(t) := C_1[\phi_c^{-1}(t)]^{-d},\quad t>0.$$

We can see that the function $r\mapsto 1/\psi(r)$ satisfies the doubling property; that is, there is a constant $c_0 > 0$ such that for all $r > 0$, $\psi(r)/\psi(2r)\le c_0$.

With those two conclusions above at hand and Lemma 3.1, we can verify the desired assertion. Indeed, according to [12, Proposition II.1] or [6, Theorem

3.4 and Remark 3.5(i)], the following generalized Nash-type inequality holds for $(\mathcal{E}, \mathcal{F})$:

$$\theta(\|f\|_2^2) \le c_1 \mathcal{E}(f,f), \quad f \in \mathcal{F} \text{ with } \|f\|_1 = 1, \tag{3.3}$$

where $\theta(r) = r/\psi^{-1}(r)$. □

3.2. Off-diagonal upper bounds for $q(t, x, y)$

Proposition 3.3. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exist positive constants* c_0, c_*, C_1 *and* C_2 *such that for all* $x, y \in \mathbb{Z}^d$ *and* $t \ge 1$,

$$q(t,x,y) \le \begin{cases} C_1[\phi_c^{-1}(t)]^{-d}, & t \ge c_0\phi_c(|x-y|), \\ C_1[\phi_c^{-1}(t)]^{-d} \exp\left\{ -\frac{C_2|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)} \right\}, & t_* \le t \le c_0\phi_c(|x-y|), \\ \frac{C_1 t}{|x-y|^d \phi_j(|x-y|)}, & 1 \le t \le t_*, \end{cases} \tag{3.4}$$

where $t_* := c_*|x-y|\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)$.

Proof. According to Proposition 3.2, we only need to verify the desired assertion for the case that $c_0\phi_c(|x-y|) \ge t$ with any fixed $c_0 > 0$ (small enough). Since we are concerned on (3.4) for $t \ge 1$, without loss of generality it suffices to consider the case that both $|x-y|$ and t are large enough, also thanks to Proposition 3.2.

(i) We first claim that for any $c_0, c_* > 0$ there are constants $c_1, c_2 > 0$ such that for all $c_0\phi_c(|x-y|) \ge t \ge t_*$,

$$c_1[\phi_c^{-1}(t)]^{-d} \exp\left\{ -\frac{c_2|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)} \right\} \ge \frac{t}{|x-y|^d\phi_j(|x-y|)},$$

where

$$t_* := c_*|x-y|\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right).$$

Indeed, by (2.5) and (2.2), we can take $c_3 > 0$ (which depends on c_* only) such that $t_* \ge t_0$ for all $x, y \in \mathbb{Z}^d$, where

$$t_0 = |x-y|\bar{\phi}_c\left(\frac{c_3|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right).$$

Then, for any $c_0\phi_c(|x-y|) \ge t \ge t_* \ge t_0$,

$$\exp\left\{ -\frac{c_3|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)} \right\} \ge \exp\left\{ -\frac{c_3|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t_0}{|x-y|}\right)} \right\} = \frac{\phi_c(|x-y|)}{4\phi_j(|x-y|)},$$

which is equivalent to saying that

$$4[\phi_c^{-1}(t)]^{-d}\frac{[\phi_c^{-1}(t)]^d}{|x-y|^d}\frac{t}{\phi_c(|x-y|)}\exp\left\{-\frac{c_3|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}$$
$$-\frac{t}{\phi_c(|x-y|)}\frac{\phi_c(|x-y|)}{|x-y|^d\phi_j(|x-y|)}\geq 0.$$

This yields the desired assertion, thanks to the fact that there are constants $c_4, c_5 > 0$ so that

$$\frac{[\phi_c^{-1}(t)]^d}{|x-y|^d}\frac{t}{\phi_c(|x-y|)}\exp\left\{-\frac{c_3|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}\leq c_4\exp\left\{-\frac{c_5|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\},$$

which in turn is due to $c_0\phi_c(|x-y|)\geq t$.

(ii) For any $K\geq 1$, set $J_K(x,y)=J(x,y)\mathbb{1}_{\{|x-y|\leq K\}}$ and

$$\mathcal{E}_K(f,f)=\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d}|f(x)-f(y)|^2J_K(x,y).$$

Note that

$$\begin{aligned}\mathcal{E}(f,f)-\mathcal{E}_K(f,f)&=\frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|>K}}|f(x)-f(y)|^2J(x,y)\\&\leq\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|>K}}(f(x)^2+f(y)^2)J(x,y)\\&\leq c_6\|f\|_2^2[\phi_j(K)]^{-1},\end{aligned}\tag{3.5}$$

where the last inequality follows from (2.7). In particular, this implies that $(\mathcal{E}_K,\mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(\mathbb{Z}^d,\mu)$. Furthermore, denote by $q^K(t,x,y)$ the heat kernel associated with $(\mathcal{E}_K,\mathcal{F})$. According to (3.5), the Nash-type inequality (3.3) and [6, Theorem 1.2], there exists a constant $c_7>0$ such that

$$q^K(t,x,y)\leq c_7[\phi_c^{-1}(t)]^{-d}\exp\left\{c_6\frac{t}{\phi_j(K)}-E_K(2t,x,y)\right\},\quad x,y\in\mathbb{Z}^d,\ t>0,$$

where

$$\Gamma_K(\varphi)(x)=\sum_{y\in\mathbb{Z}^d}(e^{\varphi(x)-\varphi(y)}-1)^2J_K(x,y),\quad \Lambda_K(\varphi)^2=\|\Gamma_K(\varphi)\|_\infty\vee\|\Gamma_K(-\varphi)\|_\infty$$

and

$$E_K(t,x,y)=\sup\{|\varphi(x)-\varphi(y)|-t\Lambda_K(\varphi)^2:\varphi\text{ has compact support}\}.$$

In the following, we fix $l\in(0,1)$ small enough, $c_8>0$ large enough, $t\geq 1$ and $x,y\in\mathbb{Z}^d$ with $|x-y|\geq\max\{c_8\phi_c^{-1}(t),l^{-1}\}$. We take $K=l|x-y|$. In particular, $K\geq 1$. Define $\varphi(z):=s(|x-y|-|x-z|)_+$, where $s>0$ is chosen later. Using the facts that $|e^t-1|^2 \le t^2 e^{2|t|}$ and $|\varphi(z)-\varphi(w)| \le s|z-w|$, as well as (1.1), we find that

$$\begin{aligned}\Gamma_K(\varphi)(z) &= \sum_{w\in B(z,K)} (e^{\varphi(z)-\varphi(w)}-1)^2 J_K(z,w)\\ &\le \sum_{w\in B(z,K)} (\varphi(z)-\varphi(w))^2 e^{2|\varphi(z)-\varphi(w)|} J_K(z,w)\\ &\le s^2 \sum_{w\in B(z,K)} |z-w|^2 e^{2s|z-w|} J_K(z,w) \le c_9 s^2 \int_0^K e^{2sr} \frac{r}{\phi_j(r)}\, \mathrm{d}r.\end{aligned}$$

The same estimate holds for $\Gamma_K(-\varphi)$. Hence,

$$\Lambda_K(\varphi)^2 \le c_9 s^2 \int_0^K e^{2sr}\frac{r}{\phi_j(r)}\, \mathrm{d}r.$$

Then

$$\begin{aligned}q^K(t,x,y) &\le c_{10}[\phi_c^{-1}(t)]^{-d} \exp\left\{-|\varphi(x)-\varphi(y)| + 2c_9 t s^2 \int_0^K e^{2sr}\frac{r}{\phi_j(r)}\, \mathrm{d}r\right\}\\ &\le c_{10}[\phi_c^{-1}(t)]^{-d} \exp\left\{s|x-y|\left[-1+\frac{2c_9 ts}{|x-y|}\int_0^K e^{2sr}\frac{r}{\phi_j(r)}\, \mathrm{d}r\right]\right\},\end{aligned}$$

where in the first inequality we used the fact that $\phi_j(K) \ge c_{11} t$ when $|x-y| \ge c_8\phi_c^{-1}(t)$ and $K = l|x-y|$, thanks to the fact that $2\phi_j(r) \ge \phi_c(r)$ for all $r \ge 1$. Here we note that the constant c_9 is independent of l.

(iii) Recall that we set

$$t_* = c_*|x-y|\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right),$$

where $c_* > 0$ is small that will be fixed in the next part. Suppose that $t_* \le t \le c_0\phi_c(|x-y|)$, where c_0 is small enough such that $|x-y| \ge c_8\phi_c^{-1}(t)$ with the constant c_8 (large) above. (This is guaranteed by (2.5).) Let $s = \frac{a}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}$, where $a \in (0,1/2)$ is small enough and is chosen later. Then,

$$\begin{aligned}&\frac{2c_9 ts}{|x-y|}\int_0^K e^{2sr}\frac{r}{\phi_j(r)}\, \mathrm{d}r\\ &= \frac{2c_9 t}{|x-y|}\frac{a}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\int_0^K \exp\left\{\frac{2ar}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}\frac{r}{\phi_j(r)}\, \mathrm{d}r\\ &\le \frac{c_{12} t_*}{|x-y|}\frac{a}{\bar{\phi}_c^{-1}\left(\frac{t_*}{|x-y|}\right)}\int_0^K \exp\left\{\frac{2ar}{\bar{\phi}_c^{-1}\left(\frac{t_*}{|x-y|}\right)}\right\}\frac{r}{\phi_j(r)}\, \mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&\le c_{12}c_*\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)\frac{a}{\bar{\phi}_c^{-1}\left(c_*\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)\right)}\\
&\quad\times\int_0^K\exp\left\{r\Big/\bar{\phi}_c^{-1}\left(c_*\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)\right)\right\}\frac{r}{\phi_j(r)}\,\mathrm{d}r\\
&\le c_{12}c_*\frac{a}{c_{13}}\frac{\phi_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)}{\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)^2}\int_0^K\exp\left\{\frac{r}{c_{13}}\frac{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}{|x-y|}\right\}\frac{r}{\phi_j(r)}\,\mathrm{d}r\\
&\le c_{12}c_*\frac{a}{c_{13}}A\le\frac{1}{2}.
\end{aligned}
$$

The first inequality above follows from (2.6); in the second inequality we used $a < 1/2$; the third inequality is a consequence of (2.5), and c_{13} depends on c_* only; in the last second inequality we used (1.4) by choosing l small enough and the constant A here is independent of a; and in the last inequality we take a small enough. Therefore,

$$
\begin{aligned}
q^K(t,x,y)&\le c_{10}[\phi_c^{-1}(t)]^{-d}\exp\left\{-\frac{s}{2}|x-y|\right\}\\
&\le c_{10}[\phi_c^{-1}(t)]^{-d}\exp\left\{-\frac{a|x-y|}{2\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}.
\end{aligned}
$$

Let $J_{K'}(x,y) := J(x,y) - J_K(x,y) = J(x,y)\mathbb{1}_{\{|x-y|>K\}}$. Then, it follows from [3, Lemma 3.1(c)] and (1.1) that

$$
\begin{aligned}
q(t,x,y)&\le q_t^K(x,y)+t\|J_{K'}(x,y)\|_\infty\\
&\le c_{10}[\phi_c^{-1}(t)]^{-d}\exp\left\{-\frac{a|x-y|}{2\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}+\frac{c_{14}t}{|x-y|^d\phi_j(|x-y|)},
\end{aligned}
$$

which along with the assertion in part (i) yields that

$$
q(t,x,y)\le c_{15}[\phi_c^{-1}(t)]^{-d}\exp\left\{-\frac{c_{16}|x-y|}{\bar{\phi}_c^{-1}\left(\frac{t}{|x-y|}\right)}\right\}.
$$

(iv) Suppose that $t \le t_*$, where $t_* = c_*|x-y|\bar{\phi}_c\left(\frac{|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)$ and c_* is small enough that is fixed later. Now, we choose $s = \frac{b}{|x-y|}\log\frac{\phi_j(|x-y|)}{c_*^{-1}t}$, where

$b = 3 + 4d/3$. For convenience, we set

$$M = M(|x-y|) := |x-y| \left(\log \frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)} \right)^{-1},$$

and

$$D = D(|x-y|) := \left(\log \frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)} \right) \left(\log \frac{\phi_j(|x-y|)}{|x-y|\bar{\phi}_c(M)} \right)^{-1}.$$

Then, taking l small enough so that $lb \le 1/4$,

$$\begin{aligned}
&\frac{2c_9ts}{|x-y|} \int_0^K e^{2sr} \frac{r}{\phi_j(r)}\, \mathrm{d}r \\
&= \frac{2c_9tb}{|x-y|^2} \log \frac{\phi_j(|x-y|)}{c_*^{-1}t} \int_0^K \exp\left\{ \frac{2br}{|x-y|} \log \frac{\phi_j(|x-y|)}{c_*^{-1}t} \right\} \frac{r}{\phi_j(r)}\, \mathrm{d}r \\
&\le \frac{2c_9bt_*}{|x-y|^2} \log \frac{\phi_j(|x-y|)}{c_*^{-1}t_*} \int_0^K \exp\left\{ \frac{2br}{|x-y|} \log \frac{\phi_j(|x-y|)}{c_*^{-1}t_*} \right\} \frac{r}{\phi_j(r)}\, \mathrm{d}r \\
&= \frac{2c_9bc_*\phi_c(M)}{M^2} \log \frac{\phi_j(|x-y|)}{|x-y|\bar{\phi}_c(M)} \left(\log \frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)} \right)^{-1} \\
&\quad \times \int_0^K \exp\left\{ \frac{2br}{|x-y|} \log \frac{\phi_j(|x-y|)}{|x-y|\bar{\phi}_c(M)} \right\} \frac{r}{\phi_j(r)}\, \mathrm{d}r \\
&\le \frac{2c_9bc_*}{D} \frac{\phi_c(M)}{M^2} \int_0^K \exp\left\{ \frac{2b}{D} \frac{r}{M} \right\} \frac{r}{\phi_j(r)}\, \mathrm{d}r \\
&\le c_{17}c_* \frac{\phi_c(M)}{M^2} \int_0^K \exp\left\{ \frac{c_{18}r}{M} \right\} \frac{r}{\phi_j(r)}\, \mathrm{d}r \le c_{17}c_*A \le \frac{1}{2}.
\end{aligned}$$

Here, in the first inequality we used the facts that $lb \le 1/4$ and the function

$$t \mapsto t^{1-2bl} \log(\phi_j(|x-y|)/(c_*^{-1}t))$$

is increasing for large $|x-y|$; in the third inequality we used the facts that D is bounded from below by a positive constant (thanks to Lemma 2.3) and c_{17}, c_{18} are independent of c_*; in the fourth inequality we used Assumption **(H)** and the constant A can be independent of c_* by taking l small enough; and in the last inequality we take c_* small enough. Hence,

$$\begin{aligned}
q^K(t,x,y) &\le c_{19}[\phi_c^{-1}(t)]^{-d} \exp\left\{ -\frac{s}{2}|x-y| \right\} \\
&\le c_{19}[\phi_c^{-1}(t)]^{-d} \exp\left\{ -\frac{b}{2} \log \frac{\phi_j(|x-y|)}{c_*^{-1}t} \right\} \\
&= c_{19}[\phi_c^{-1}(t)]^{-d} \left(\frac{t}{\phi_j(|x-y|)} \right)^{b/2} \le c_{20} \frac{t}{|x-y|^d \phi_j(|x-y|)},
\end{aligned}$$

where the last inequality follows from Lemma 2.1 and the fact that

$$\left(\frac{\phi_c(|x-y|)}{t}\right)^{2d/3} \le c_{21}\left(\frac{\phi_j(|x-y|)}{t}\right)^{b/2-1}.$$

With the estimate above, we then can apply [3, Lemma 3.1(c)] and get that

$$q(t,x,y) \le \frac{c_{22}t}{|x-y|^d\phi_j(|x-y|)}.$$

Therefore, the proof is complete. □

3.3. Upper bounds for $p_n(x,y)$

Now, we can state the main statement about upper bounds for transition functions of the discrete time Markov chain $(X_n)_{n\ge 0}$.

Theorem 3.4. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exist c_0, c_*, $\widetilde{C}_1$ and $\widetilde{C}_2 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \ge 1$,*

$$p_n(x,y) \le \begin{cases} \widetilde{C}_1[\phi_c^{-1}(n)]^{-d}, & n \ge c_0\phi_c(|x-y|),\\ \widetilde{C}_1[\phi_c^{-1}(n)]^{-d}\exp\left\{-\frac{\widetilde{C}_2|x-y|}{\bar{\phi}_c^{-1}(\frac{n}{|x-y|})}\right\}, & n_* \le n \le c_0\phi_c(|x-y|),\\ \widetilde{C}_1\frac{n}{|x-y|^d\phi_j(|x-y|)}, & n \le n_*, \end{cases}$$

where $n_* := |x-y|\bar{\phi}_c\left(\frac{c_*|x-y|}{\log\frac{4\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)$.

Proof. We first note that for any $x \in \mathbb{Z}^d$,

$$p_2(x,x) = \sum_{z\in\mathbb{Z}^d} p(x,z)p(z,x) \ge \sum_{z\in\mathbb{Z}^d:|x-z|=1} p(x,z)p(z,x) \ge \alpha_0 > 0, \tag{3.6}$$

where α_0 is independent of $x \in \mathbb{Z}^d$. On the other hand, there is a constant $c_0 \ge 1$ such that for any $x, y \in \mathbb{Z}^d$ with $|x-y| \ge 2$,

$$p(x,y) \le c_0 p_2(x,y). \tag{3.7}$$

Indeed, for any $x, y \in \mathbb{Z}^d$ with $|x-y| \ge 2$,

$$\begin{aligned} p_2(x,y) = \sum_{z\in\mathbb{Z}^d} p(x,z)p(z,y) &\ge \sum_{z\in\mathbb{Z}^d:|z-x|=1} p(x,z)p(z,y)\\ &\ge c_1 p(x,y)\sum_{z\in\mathbb{Z}^d:|z-x|=1} p(x,z) \ge c_2 p(x,y), \end{aligned}$$

where in the second inequality we used (1.6) and (1.1).

Regarding $p_{2n}(x,y)$ as the transition probabilities of the Markov chain $(X_{2n})_{n\ge 0}$, the jumping kernel of the continuous time Markov chain $(\tilde{Y}_t)_{t\ge 0}$ corresponding to $(X_{2n})_{n\ge 0}$ is comparable with $p_2(x,y)$ and satisfies (1.1) (possibly with different constants). Denote by $\tilde{q}(t,x,y)$ the heat kernel of the process $(\tilde{Y}_t)_{t\ge 0}$. According to Proposition 3.3, $\tilde{q}(t,x,y)$ enjoys the same upper bounds as those for $q(t,x,y)$ that are stated in (3.4) (possibly with different constants). Furthermore, it follows from (3.6) and [14, Theorem 3.6] that there exists a constant $C(\alpha_0) > 0$ such that for all $n \geq 1$ and $x, y \in \mathbb{Z}^d$,

$$p_{2n}(x,y) \leq C(\alpha_0)\tilde{q}(2n,x,y).$$

Therefore, the desired assertion holds for even n.

Clearly, the assertion holds with $n = 1$. Next, we consider the estimate for $p_{2k+1}(x, y)$ for any $k \geq 1$. When $|x - y| \leq 4$,

$$p_{2k+1}(x,y) = \sum_{z\in\mathbb{Z}^d} p(x,z)p_{2k}(z,y) \leq c_3[\phi_c^{-1}(2k)]^{-d}. \tag{3.8}$$

When $|x - y| > 4$, by (3.7),

$$\begin{aligned} p_{2k+1}(x,y) &= \sum_{z\in\mathbb{Z}^d} p(x,z)p_{2k}(z,y) \\ &= \sum_{z\in\mathbb{Z}^d:|x-z|\geq 2} p(x,z)p_{2k}(z,y) + \sum_{z\in\mathbb{Z}^d:|x-z|<2} p(x,z)p_{2k}(z,y) \\ &\leq c_0 \sum_{z\in\mathbb{Z}^d} p_2(x,z)p_{2k}(z,y) + \sup_{z\in\mathbb{Z}^d:|x-z|<2} p_{2k}(z,y) \\ &= c_0 p_{2k+2}(x,y) + \sup_{z\in\mathbb{Z}^d:|x-z|<2} p_{2k}(z,y), \end{aligned}$$

which along with (3.8) yields the desired assertion for odd n by adjusting the constants involved if necessary. □

Remark 3.5. According to Theorem 3.4, under (1.1), (1.2) and Assumption **(H)**, there exist $c_1, c_2 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \geq 1$,

$$p_n(x,y) \leq c_1 \left([\phi_c^{-1}(n)]^{-d} \exp\left\{ -\frac{c_2|x-y|}{\bar{\phi}_c^{-1}(\frac{n}{|x-y|})} \right\} + \frac{n}{|x-y|^d \phi_j(|x-y|)} \right).$$

4. Lower bounds for transition probabilities

In this section, we consider lower bounds for the transition probabilities $p_n(x, y)$. The section is split into three parts.

4.1. On-diagonal lower bounds

For any $B \subset \mathbb{Z}^d$, set $\tau_B := \inf\{n \geq 1 : X_n \notin B\}$.

Lemma 4.1. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exists a constant $A \geq 1$ such that for all $n \geq 1$ and $x \in \mathbb{Z}^d$,*

$$\mathbb{P}^x(X_n \in B(x, A\phi_c^{-1}(n))^c) \leq 1/4. \tag{4.1}$$

Consequently,

$$\mathbb{P}^x(\tau_{B(x,A\phi_c^{-1}(n))} \leq n) \leq 1/2.$$

Proof. For fixed $x \in \mathbb{Z}^d$ and $n \geq 1$, set $r = A\phi_c^{-1}(n)$ and $B = B(x,r)$ for some $A > 0$. Then, according to Theorem 3.4 and Remark 3.5,

$$\begin{aligned}
&\mathbb{P}^x(X_n \in B^c)\\
&= \sum_{y\in B^c} p_n(x,y) \leq \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} p_n(x,y)\\
&\leq c_1 \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} \left[[\phi_c^{-1}(n)]^{-d} \exp\left(-\frac{c_0|x-y|}{\bar{\phi}_c^{-1}(\frac{n}{|x-y|})} \right) + \frac{n}{|x-y|^d \phi_j(|x-y|)} \right]\\
&=: c_1(I_1 + I_2).
\end{aligned}$$

Note that

$$\begin{aligned}
I_1 &\leq \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -\frac{c_0 2^{i-1} A\phi_c^{-1}(n)}{\bar{\phi}_c^{-1}\left(\frac{n}{2^{i-1}A\phi_c^{-1}(n)}\right)} \right\}\\
&= \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -c_0 \frac{\bar{\phi}_c^{-1}\left[\bar{\phi}_c(2^{i-1}A\phi_c^{-1}(n))\right]}{\bar{\phi}_c^{-1}\left(\frac{n}{2^{i-1}A\phi_c^{-1}(n)}\right)} \right\}\\
&\leq \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -c_2 \frac{\bar{\phi}_c\left(2^{i-1}A\phi_c^{-1}(n)\right)}{\frac{n}{2^{i-1}A\phi_c^{-1}(n)}} \right\}\\
&= \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -c_2 \frac{\phi_c\left(2^{i-1}A\phi_c^{-1}(n)\right)}{\phi_c(\phi_c^{-1}(n))} \right\}\\
&\leq \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -c_3(2^{i-1}A)^{3/2} \right\}\\
&\leq c_4 \sum_{i=1}^{\infty} [\phi_c^{-1}(n)]^{-d} \exp\left\{ -c_3(2^{i-1}A)^{3/2} \right\} 2^{id} A^d [\phi_c^{-1}(n)]^d\\
&\leq c_5 \exp\left(-c_6 A^{3/2} \right),
\end{aligned}$$

where the second inequality follows from (2.6), in the second equality we used $\phi_c(r) = \bar{\phi}_c(r)r$, and in the third inequality we used (2.4). Here, the constants c_5 and c_6 are independent of A. Letting A large enough, we obtain that $c_1 I_1 \leq 1/8$. On the other hand,

$$\begin{aligned}
I_2 &\leq \sum_{i=1}^{\infty} \sum_{2^{i-1}r\leq|x-y|<2^i r} \frac{n}{(2^{i-1}A\phi_c^{-1}(n))^d \phi_j(2^{i-1}A\phi_c^{-1}(n))}\\
&\leq c_7 \sum_{i=1}^{\infty} \frac{n}{\phi_j(2^{i-1}A\phi_c^{-1}(n))} \leq 2c_7 \sum_{i=1}^{\infty} \frac{n}{\phi_c(2^{i-1}A\phi_c^{-1}(n))}
\end{aligned}$$

$$\leq c_8 \sum_{i=1}^{\infty} (2^{i-1}A)^{-3/2} \leq c_9 A^{-3/2},$$

where in the third inequality we used the fact that $2\phi_j(r) \geq \phi_c(r)$ for all $r > 0$, the fourth inequality follows from (2.4) and the constant c_9 is independent of A. Similarly, we can see that $c_1 I_2 \leq 1/8$ by taking A large enough. Thus, the proof of (4.1) is complete.

Furthermore, according to (4.1) (by taking A larger if necessary) and the strong Markov property,

$$\begin{aligned}
\mathbb{P}^x(\tau_B \leq n) &= \mathbb{P}^x\left(\tau_B \leq n, |X_{2n} - x| \leq r/2\right) + \mathbb{P}^x\left(\tau_B \leq n, |X_{2n} - x| > r/2\right) \\
&\leq \mathbb{P}^x\left(\tau_B \leq n, |X_{2n} - x| \leq r/2\right) + \mathbb{P}^x\left(|X_{2n} - x| > r/2\right) \\
&\leq \sup_{\substack{z \in B(x, r/2)^c \\ m \leq n}} \mathbb{P}^z\left(X_{2n-m} \in B(z, r/2)^c\right) + \frac{1}{4} \leq \frac{1}{2}.
\end{aligned}$$

The proof is complete. □

For any subset $B \subset \mathbb{Z}^d$, set $p_n^B(x, y) = \mathbb{P}^x(X_n = y, \tau_B > n)$ for any $x, y \in B$; and $p_n^B(x, y) = 0$ when $x \notin B$ or $y \notin B$. It is clear that $p_n^B(x, y) \leq p_n(x, y)$ for all $x, y \in \mathbb{Z}^d$ and $n \geq 1$.

Proposition 4.2. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exists $C_1 > 0$ such that for all $x \in \mathbb{Z}^d$ and $n \geq 1$,*

$$p_{2n}(x, x) \geq C_1[\phi_c^{-1}(n)]^{-d}.$$

Proof. Let A be the constant in Lemma 4.1. According to Lemma 4.1, for all $x \in \mathbb{Z}^d$ and $n \geq 1$,

$$\mathbb{P}^x(\tau_{B(x, A\phi_c^{-1}(n))} \leq n) \leq 1/2.$$

Then, by the symmetry of $p_n(x, y)$ and the Cauchy-Schwarz inequality, for all $x \in \mathbb{Z}^d$ and $n \geq 1$,

$$\begin{aligned}
p_{2n}(x, x) &= \sum_{y \in \mathbb{Z}^d} p_n(x, y) p_n(y, x) = \sum_{y \in \mathbb{Z}^d} p_n(x, y)^2 \\
&\geq \sum_{y \in B(x, A\phi_c^{-1}(n))} p_n^{B(x, A\phi_c^{-1}(n))}(x, y)^2 \\
&\geq \frac{1}{\sum_{y \in B(x, A\phi_c^{-1}(n))}} \left(\sum_{y \in B(x, A\phi_c^{-1}(n))} p_n^{B(x, A\phi_c^{-1}(n))}(x, y) \right)^2 \\
&= \frac{1}{\sum_{y \in B(x, A\phi_c^{-1}(n))}} [\mathbb{P}^x(\tau_{B(x, A\phi_c^{-1}(n))} > n)]^2 \\
&\geq \frac{c_1}{[\phi_c^{-1}(n)]^d} \Big(1 - \mathbb{P}^x(\tau_{B(x, A\phi_c^{-1}(n))} \leq n)\Big)^2 \geq c_2[\phi_c^{-1}(n)]^{-d}.
\end{aligned}$$

The proof is complete. □

4.2. Near-diagonal lower bounds

Proposition 4.3. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exist constants* $c_0, c_1 > 0$ *such that for all* $x, y \in \mathbb{Z}^d$ *and* $n \geq 1$ *with* $c_0\phi_c^{-1}(n) \geq |x-y|$,

$$p_n(x,y) \geq c_1[\phi_c^{-1}(n)]^{-d}.$$

Proof. For any $x \in \mathbb{Z}^d$, $n \geq 1$ and $l > 0$, set $B := B(x, l\phi_c^{-1}(n))$. Since $p_n(x,y) \geq p_n^B(x,y)$, it suffices to show that there are constants $c_0, c_1, l > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \geq 1$ with $c_0\phi_c^{-1}(n) \geq |x-y|$,

$$p_n^B(x,y) \geq c_1[\phi_c^{-1}(n)]^{-d}.$$

For any subset $B \subset \mathbb{Z}^d$, let $q_t^B(x,y)$ be the (Dirichlet) heat kernel of the process $(Y_t)_{t\geq 0}$ with the Dirichlet boundary B^c. Then, according to [11, Definition 1.10(vi)] and [11, Proposition 4.1], there exist $\varepsilon \in (0,1)$ and $c_2 > 0$ such that for any $x \in \mathbb{Z}^d$, $0 < t \leq \phi_c(\varepsilon r)$ and $y \in B\left(x, \varepsilon\phi_c^{-1}(t)\right)$,

$$q_t^{B(x,r)}(x,y) \geq c_2[\phi_c^{-1}(t)]^{-d}. \tag{4.2}$$

Indeed, for the continuous time process $(Y_t)_{t\geq 0}$ of the present paper, it is clear that the upper bound for the jumping kernel is satisfied (see [11, Definition 1.3]). On the one hand, it follows from the argument of [11, Example 5.3], the cut-off Sobolev inequality holds; on the other hand, according to [1, Proposition 3.2] and the standard discretization method, we can see that the Poincaré inequality is also fulfilled. Thus, [11, Proposition 4.1] applies to the process $(Y_t)_{t\geq 0}$.

Recall that $Y_t := X_{N(t)}$ for any $t \geq 0$, where $N(t)$ is a standard Poisson process independent of $(X_n)_{n\geq 0}$. Then, for all $y \in B(x, \varepsilon\phi_c^{-1}(n/2))$,

$$\begin{aligned}
p_n^B(x,y) &= \mathbb{P}^x(X_n = y, \tau_B > n)\\
&= \mathbb{P}^x\left(\sup_{k\leq n}|X_k - x| \leq l\phi_c^{-1}(n), X_n = y\right)\\
&= \frac{1}{\int_{n/2}^n \mathbb{P}(N(t)=n)\,\mathrm{d}t}\\
&\quad\times \int_{n/2}^n \mathbb{P}^x\left(\sup_{k\leq n}|X_k - x| \leq l\phi_c^{-1}(n), X_n = y, N(t) = n\right)\,\mathrm{d}t\\
&= \frac{1}{\int_{\frac{n}{2}}^n \mathbb{P}(N(t)=n)\,\mathrm{d}t}\int_{n/2}^n \mathbb{P}^x\left(\sup_{s\leq t}|Y_s - x| \leq l\phi_c^{-1}(n), Y_t = y\right)\,\mathrm{d}t\\
&= \frac{1}{\int_{n/2}^n \mathbb{P}(N(t)=n)\,\mathrm{d}t}\int_{n/2}^n q_t^{B(x,l\phi_c^{-1}(n))}(x,y)\,\mathrm{d}t\\
&\geq \frac{2}{n}\int_{n/2}^n q_t^{B(x,l\phi_c^{-1}(n))}(x,y)\,\mathrm{d}t.
\end{aligned}$$

Now, we take $l \geq \varepsilon^{-1}$ and $c_0 \leq \varepsilon/2$. Then, by (4.2), for all $x, y \in \mathbb{Z}^d$ and $n \geq 1$ with $c_0\phi_c^{-1}(n) \geq |x - y|$,

$$p_n^B(x, y) \geq \frac{2}{n} \int_{n/2}^{n} c_2[\phi_c^{-1}(t)]^{-d} \,\mathrm{d}t \geq c_3[\phi_c^{-1}(n)]^{-d}.$$

The proof is complete. □

4.3. Off-diagonal lower bounds

In this part, we will establish off-diagonal lower bounds for $p_n(x, y)$.

Proposition 4.4. *Suppose that* (1.1)*,* (1.2) *and Assumption* **(H)** *hold. Then there are constants $C_0, c_1, c_2 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \geq 1$ with $|x - y| \geq C_0\phi_c^{-1}(n)$, it holds that*

$$p_n(x, y) \geq c_1[\phi_c^{-1}(n)]^{-d} \exp\left\{ -\frac{c_2|x - y|}{\bar{\phi}_c^{-1}\left(\frac{n}{|x-y|}\right)} \right\}.$$

Proof. Take $c^* = ((c_0/12) \wedge 1) \in (0, 1]$, where $c_0 > 0$ is the constant in Proposition 4.3. For any $x, y \in \mathbb{Z}^d$ and $n \geq 1$ so that $n \leq \phi_c(|x - y|/c^*)$, let ℓ be the positive integer such that

$$\phi_c\left(\frac{|x - y|}{c^*(\ell + 1)}\right) < n \leq \phi_c\left(\frac{|x - y|}{c^*\ell}\right). \tag{4.3}$$

In particular,

$$\frac{n\ell}{|x - y|} \simeq \bar{\phi}_c\left(\frac{|x - y|}{\ell}\right); \tag{4.4}$$

that is,

$$\ell \simeq \frac{|x - y|}{\bar{\phi}_c^{-1}\left(\frac{n\ell}{|x-y|}\right)}. \tag{4.5}$$

Next, let $\{x_i\}_{0 \leq i \leq \ell}$ be a sequence on $\mathbb{Z}^d$ joining $x_0 = x$ and $x_\ell = y$ such that

$$|x_i - x_{i-1}| \leq \frac{\sqrt{2}|x - y|}{\ell}, \quad 1 \leq i \leq \ell.$$

Thus, for any $0 \leq i \leq \ell$, and $y_i \in B(x_i, |x - y|/\ell)$,

$$\begin{aligned} |y_i - y_{i-1}| &\leq |y_i - x_i| + |x_i - x_{i-1}| + |x_{i-1} - y_{i-1}| \leq \frac{3\sqrt{2}|x - y|}{\ell} \\ &\leq 12c^*\phi_c^{-1}(n) \leq c_0\phi_c^{-1}(n) \end{aligned}$$

holds for all $1 \leq i \leq \ell$, where in the third inequality we used the fact that $|x - y|/\ell \leq 2c^*\phi_c^{-1}(n)$, thanks to (4.3). Hence, according to Proposition 4.3, there exist constants $C \in (0,1)$ and $C_0 > 0$ such that

$$\begin{aligned} p_n(y_{i-1}, y_i) &\geq C\left[\phi_c^{-1}(n)\right]^{-d}, \quad 1 \leq i \leq \ell - 1, \\ p_{n+k}(y_{\ell-1}, y_\ell) &\geq C\left[\phi_c^{-1}(n+k)\right]^{-d} \\ &\geq C_0 C\left[\phi_c^{-1}(n)\right]^{-d}, \quad 0 \leq k \leq n-1, \end{aligned} \tag{4.6}$$

where the last inequality follows from (2.4). Therefore, for any $k \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} &p_{\ell n+k}(x,y) \\ &= \sum_{y_1 \in \mathbb{Z}^d} \cdots \sum_{y_{\ell-1} \in \mathbb{Z}^d} p_n(x, y_1) p_n(y_1, y_2) \cdots p_{n+k}(y_{\ell-1}, y) \\ &\geq \sum_{y_1 \in B\left(x_1, \frac{|x-y|}{\ell}\right)} \cdots \sum_{y_{\ell-1} \in B\left(x_{\ell-1}, \frac{|x-y|}{\ell}\right)} p_n(x, y_1) p_n(y_1, y_2) \cdots p_{n+k}(y_{\ell-1}, y) \\ &\geq C_0 \sum_{y_1 \in B\left(x_1, \frac{|x-y|}{\ell}\right)} \cdots \sum_{y_{\ell-1} \in B\left(x_{\ell-1}, \frac{|x-y|}{\ell}\right)} C^\ell \left[\phi_c^{-1}(n)\right]^{-d\ell} \\ &\geq C_0 C^\ell \left[\phi_c^{-1}(n)\right]^{-d\ell} c_1 \left(\frac{|x-y|}{\ell}\right)^{d(\ell-1)} \geq c_2 C^\ell \left[\phi_c^{-1}(n)\right]^{-d} \\ &\geq c_3 \left[\phi_c^{-1}(\ell n)\right]^{-d} \exp\left\{-c_4 \frac{|x-y|}{\bar{\phi}_c^{-1}\left(\frac{n\ell}{|x-y|}\right)}\right\} \\ &\geq c_3 \left[\phi_c^{-1}(\ell n)\right]^{-d} \exp\left\{-c_5 \frac{|x-y|}{\bar{\phi}_c^{-1}\left(\frac{2n\ell}{|x-y|}\right)}\right\} \\ &\geq c_6 \left[\phi_c^{-1}(\ell n+k)\right]^{-d} \exp\left\{-c_7 \frac{|x-y|}{\bar{\phi}_c^{-1}\left(\frac{\ell n+k}{|x-y|}\right)}\right\}, \end{aligned}$$

where the second inequality follows from (4.6), in the fourth inequality we used (4.3), the sixth inequality follows from (2.6), and in the last inequality we used (4.5) and the constants c_6, c_7 are independent of n, k and ℓ.

Furthermore, according to (4.4), there are constants $c_8, c_9 > 0$ such that for any positive integer ℓ,

$$n\ell \leq c_8 |x-y| \bar{\phi}_c\left(\frac{|x-y|}{\ell}\right) \leq c_9 \phi_c(|x-y|).$$

This along with the estimate above yields the desired assertion. □

Proposition 4.5. *Suppose that* (1.1)*,* (1.2) *and Assumption* **(H)** *hold. Then, there are constants $c_1, C_0 > 0$ such that for any $x, y \in \mathbb{Z}^d$ and $n \geq 1$ with*

$|x-y| \ge C_0\phi_c^{-1}(n)$,

$$p_n(x,y) \ge \frac{c_1 n}{|x-y|^d \phi_j(|x-y|)}.$$

Proof. Let A and c_0 be the constants in Lemma 4.1 and Proposition 4.3, respectively. Take $l_0 \ge 1$ such that $2A\phi_c^{-1}(n) \le c_0\phi_c^{-1}(l_0 n)$ for all $n \ge 1$. Then, for any $x, y \in \mathbb{Z}^d$ with $|x-y| \ge A\phi_c^{-1}(n)$ and $k \in \{0, 1, \dots, n-1\}$,

$$\begin{aligned}
p_{(l_0+1)n+k}(x,y) &= \sum_{z\in\mathbb{Z}^d} p_n(x,z)p_{l_0 n+k}(z,y) \\
&\ge \sum_{|z-y|\le 2A\phi_c^{-1}(n)} p_n(x,z)p_{l_0 n+k}(z,y) \\
&\ge \left(\inf_{|z-y|\le 2A\phi_c^{-1}(n)} p_{l_0 n+k}(z,y)\right) \sum_{|z-y|\le 2A\phi_c^{-1}(n)} p_n(x,z) \\
&\ge \left(\inf_{|z-y|\le c_0\phi_c^{-1}(l_0 n+k)} p_{l_0 n+k}(z,y)\right) \sum_{|z-y|\le 2A\phi_c^{-1}(n)} p_n(x,z) \\
&\ge c_1[\phi_c^{-1}(l_0 n+k)]^{-d}\mathbb{P}^x\left(X_n \in B\left(y, 2A\phi_c^{-1}(n)\right)\right) \\
&\ge c_2[\phi_c^{-1}(n)]^{-d}\mathbb{P}^x\left(X_n \in B\left(y, 2A\phi_c^{-1}(n)\right)\right),
\end{aligned}$$

where the fourth inequality follows from Proposition 4.3 and c_1 is independent of n, l_0 and k.

On the other hand, set $B := B(y, A\phi_c^{-1}(n))$ and define $\sigma_B := \inf\{n \ge 1 : X_n \in B\}$. According to the strong Markov property, for all $n \ge 2$,

$$\begin{aligned}
&\mathbb{P}^x\left(X_n \in B(y, 2A\phi_c^{-1}(n))\right) \\
&\ge \mathbb{P}^x\left(\sigma_B \le [n/2], \sup_{\sigma_B\le k\le n} |X_k - X_{\sigma_B}| \le A\phi_c^{-1}(n)\right) \\
&= \mathbb{E}^x\left[\mathbb{1}_{\{\sigma_B\le[n/2]\}}\mathbb{E}^{X_{\sigma_B}}\left(\sup_{\sigma_B\le k\le n} |X_k - X_{\sigma_B}| \le A\phi_c^{-1}(n)\right)\right] \\
&\ge \mathbb{P}^x\left(\sigma_B \le [n/2]\right) \inf_{z\in B} \mathbb{P}^z\left(\tau_{B(z,A\phi_c^{-1}(n))} > n\right) \\
&\ge \frac{1}{2}\mathbb{P}^x\left(\sigma_B \le [n/2]\right),
\end{aligned}$$

where the last inequality follows from Lemma 4.1. Furthermore, by [4, Lemma 3.2],

$$\begin{aligned}
&\mathbb{P}^x\left(\sigma_B \le [n/2]\right) \\
&\ge \mathbb{P}^x\left(X_{[n/2]\wedge\tau_{B(x,A\phi_c^{-1}(n))}} \in B\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^x \left[\sum_{0 \le k \le \left([n/2] \wedge \tau_{B(x, A\phi_c^{-1}(n))}\right) - 1} \mathbb{1}_{\{X_k \in B\}} \right] \\
&= \mathbb{E}^x \left[\sum_{0 \le k \le \left([n/2] \wedge \tau_{B(x, A\phi_c^{-1}(n))}\right) - 1} \sum_{z \in B} J(X_k, z) \right] \\
&\ge c_3 \mathbb{E}^x \left[\sum_{0 \le k \le \left([n/2] \wedge \tau_{B(x, A\phi_c^{-1}(n))}\right) - 1} \sum_{z \in B} \frac{1}{|X_k - z|^d \phi_j(|X_k - z|)} \right] \\
&\ge c_4 [n/2] \mathbb{P}^x(\tau_{B(x, A\phi_c^{-1}(n))} \ge [n/2]) [\phi_c^{-1}(n)]^d \frac{1}{|x-y|^d \phi_j(|x-y|)} \\
&\ge c_5 [\phi_c^{-1}(n)]^d \frac{n}{|x-y|^d \phi_j(|x-y|)},
\end{aligned}
$$

where the fourth inequality we used the fact that for all $z \in B\left(y, A\phi_c^{-1}(n)\right)$ and $0 \le k \le \left([n/2] \wedge \tau_{B(x, A\phi_c^{-1}(n))}\right) - 1$,

$$
|X_k - z| \le |X_k - x| + |x - y| + |y - z| \le |x - y| + 2A\phi_c^{-1}(n) \le 3|x - y|.
$$

Thus, combining with all the estimates above, we obtain that for any $n \ge 2$, $k \in \{0, 1, \ldots, n-1\}$ and any $x, y \in \mathbb{Z}^d$ with $|x - y| \ge A\phi_c^{-1}(n)$,

$$
p_{(l_0+1)n+k}(x, y) \ge c_6 \frac{n}{|x-y|^d \phi_j(|x-y|)}.
$$

This immediately yields that there are constants $n_0 \ge 1$ and $c_7 > 0$ so that for any $x, y \in \mathbb{Z}^d$ and $n \ge n_0$ with $|x - y| \ge A\phi_c^{-1}(n)$,

$$
p_n(x, y) \ge c_7 \frac{n}{|x-y|^d \phi_j(|x-y|)}.
$$

On the other hand, for any odd n with $1 \le n \le n_0 - 1$ and any $x, y \in \mathbb{Z}^d$ with $|x - y| \ge c_8 \phi_c^{-1}(n)$,

$$
\begin{aligned}
p_n(x, y) = &\sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} p(x, z_1) \cdots p(z_{n-1}, y) \\
\ge &\sum_{|z_1 - x| = 1, z_2 = x, |z_3 - x| = 1, z_4 = x, \ldots, z_{n-1} = x} p(x, z_1) \cdots p(z_{n-1}, y) \\
\ge &c_9 p(x, y) \ge c_{10} \frac{n}{|x-y|^d \phi_j(|x-y|)}.
\end{aligned}
$$

Similarly, for any even n with $1 \le n \le n_0 - 1$ and any $x, y \in \mathbb{Z}^d$ with $|x-y| \ge c_8\phi_c^{-1}(n)$,

$$\begin{aligned}p_n(x,y) &= \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} p(x,z_1)\cdots p(z_{n-1},y)\\ &\ge \sum_{|z_1-x|=1,z_2=x,|z_3-x|=1,z_4=x,\dots,z_{n-2}=x,|z_{n-1}-x|=1} p(z_{n-1},y)\\ &\ge c_{11}p(x,y) \ge c_{12}\frac{n}{|x-y|^d\phi_j(|x-y|)}.\end{aligned}$$

Putting all the estimates together, we can prove the desired assertion. □

Finally, we can present the main statement about lower bounds for transition functions $p_n(x,y)$.

Theorem 4.6. *Suppose that* (1.1), (1.2) *and Assumption* **(H)** *hold. Then, there exist* c_0, c_*, C_1 *and* $C_2 > 0$ *such that for all* $x, y \in \mathbb{Z}^d$ *and* $n \ge 1$,

$$p_n(x,y) \ge \begin{cases} C_1[\phi_c^{-1}(n)]^{-d}, & n \ge c_0\phi_c(|x-y|),\\ C_1[\phi_c^{-1}(n)]^{-d}\exp\left\{-\frac{C_2|x-y|}{\bar\phi_c^{-1}(\frac{n}{|x-y|})}\right\}, & n_* \le n \le c_0\phi_c(|x-y|),\\ C_1\frac{n}{|x-y|^d\phi_j(|x-y|)}, & n \le n_*,\end{cases}$$

where $n_* := |x-y|\bar\phi_c\left(\frac{c_*|x-y|}{\log\frac{\phi_j(|x-y|)}{\phi_c(|x-y|)}}\right)$.

Proof. According to Propositions 4.3, 4.4 and 4.5, there are constants c_0, C_1, C_2, $C_3 > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $n \ge 1$ with $n \ge c_0\phi_c(|x-y|)$,

$$p_n(x,y) \ge C_1\phi_c^{-1}(n)^{-d};$$

and for all $x, y \in \mathbb{Z}^d$ and $n \ge 1$ with $n \le c_0\phi_c(|x-y|)$,

$$p_n(x,y) \ge C_2\left[\phi_c^{-1}(n)^{-d}\exp\left\{-\frac{C_3|x-y|}{\bar\phi_c^{-1}(\frac{n}{|x-y|})}\right\} + \frac{n}{|x-y|^d\phi_j(|x-y|)}\right].$$

This along with the conclusion in part (i) of the proof for Proposition 3.3 yields the desired assertion. □

Finally, Theorem 1.2 is a direct consequence of Theorems 3.4 and 4.6.

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References

[1] J. Bae, J. Kang, P. Kim, and J. Lee, *Heat kernel estimates for symmetric jump processes with mixed polynomial growths*, Ann. Probab. **47** (2019), no. 5, 2830–2868. https://doi.org/10.1214/18-AOP1323

[2] J. Bae, J. Kang, P. Kim, and J. Lee, *Heat kernel estimates and their stabilities for symmetric jump processes with general mixed polynomial growths on metric measure spaces*, See arXiv:1904.10189.

[3] M. T. Barlow, A. Grigor'yan, and T. Kumagai, *Heat kernel upper bounds for jump processes and the first exit time*, J. Reine Angew. Math. **626** (2009), 135–157. https://doi.org/10.1515/CRELLE.2009.005

[4] R. F. Bass and D. A. Levin, *Transition probabilities for symmetric jump processes*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2933–2953. https://doi.org/10.1090/S0002-9947-02-02998-7

[5] Z.-Q. Chen, P. Kim, and T. Kumagai, *Discrete approximation of symmetric jump processes on metric measure spaces*, Probab. Theory Related Fields **155** (2013), no. 3-4, 703–749. https://doi.org/10.1007/s00440-012-0411-x

[6] Z.-Q. Chen, P. Kim, T. Kumagai, and J. Wang, *Heat kernel upper bounds for symmetric Markov semigroups*, J. Funct. Anal. **281** (2021), no. 4, Paper No. 109074, 40 pp. https://doi.org/10.1016/j.jfa.2021.109074

[7] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Process. Appl. **108** (2003), no. 1, 27–62. https://doi.org/10.1016/S0304-4149(03)00105-4

[8] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields **140** (2008), no. 1-2, 277–317. https://doi.org/10.1007/s00440-007-0070-5

[9] Z.-Q. Chen, T. Kumagai, and J. Wang, *Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 11, 3747–3803. https://doi.org/10.4171/jems/996

[10] Z.-Q. Chen, T. Kumagai, and J. Wang, *Stability of heat kernel estimates for symmetric non-local Dirichlet forms*, Mem. Amer. Math. Soc. **271** (2021), no. 1330, v+89 pp. https://doi.org/10.1090/memo/1330

[11] Z.-Q. Chen, T. Kumagai, and J. Wang, *Heat kernel estimates for general symmetric pure jump Dirichlet forms*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **23** (2022), no. 3, 1091–1140.

[12] T. Coulhon, *Ultracontractivity and Nash type inequalities*, J. Funct. Anal. **141** (1996), no. 2, 510–539. https://doi.org/10.1006/jfan.1996.0140

[13] W. Cygan and S. Šebek, *Transition probability estimates for subordinate random walks*, Math. Nachr. **294** (2021), no. 3, 518–558. https://doi.org/10.1002/mana.201900065

[14] T. Delmotte, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Mat. Iberoam. **15** (1999), no. 1, 181–232. https://doi.org/10.4171/RMI/254

[15] M. Murugan and L. Saloff-Coste, *Transition probability estimates for long range random walks*, New York J. Math. **21** (2015), 723–757. http://nyjm.albany.edu:8000/j/2015/21_723.html

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