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ON UNIFORMLY S-ABSOLUTELY PURE MODULES

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Abstract. Let R be a commutative ring with identity and S a multiplicative subset of R. In this paper, we introduce and study the notions of u-S-pure u-S-exact sequences and uniformly S-absolutely pure modules which extend the classical notions of pure exact sequences and absolutely pure modules. And then we characterize uniformly S-von Neumann regular rings and uniformly S-Noetherian rings using uniformly S-absolutely pure modules.

1. Introduction and preliminary

Throughout this paper, R is always a commutative ring with identity, all modules are unitary and S is always a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$.

The notion of absolutely pure modules was first introduced by Maddox [10] in 1967. An R-module E is said to be *absolutely pure* provided that E is a pure submodule of every module which contains E as a submodule. It is well-known that an R-module E is absolutely pure if and only if $\mathrm{Ext}_R^1(N, E) = 0$ for any finitely presented module N ([14, Proposition 2.6]). So absolutely pure modules are also studied with the terminology FP-injective modules (FP for finitely presented), see Stenström [14] and Jain [7] for example. The notion of absolutely pure modules is very attractive in that it is not only a generalization of that of injective modules but also an important tool to characterize some classical rings. For example, a ring R is semihereditary if and only if any homomorphic image of an absolutely pure R-module is absolutely pure ([11, Theorem 2]); a ring R is Noetherian if and only if any absolutely pure R-module is injective ([11, Theorem 3]); a ring R is von-Neumann regular if and only if any R-module is absolutely pure ([11, Theorem 5]); a ring R is coherent if and only if the class of absolutely pure R-modules is closed under direct limits if and only if the class of absolutely pure R-modules is a (pre)cover ([14, Theorem 3.2], [4, Corollary 3.5]).

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One of the most important methods to generalize the classical rings and modules is in terms of multiplicative subsets S of R (see [1–3,8,9] for example). In 2002, Anderson and Dumitrescu [1] introduced *S-Noetherian rings* R, that is, for any ideal I of R, there exists a finitely generated sub-ideal K of I such that $sI \subseteq K$. Cohen's Theorem, Eakin-Nagata Theorem and Hilbert Basis Theorem for S-Noetherian rings are given in [1]. However, the choice of $s \in S$ such that $sI \subseteq K$ in the definition of S-Noetherian rings as above is not uniform. Hence, Qi et al. [12] introduced the notion of uniformly S-Noetherian rings and obtained the Eakin-Nagata-Formanek Theorem and Cartan-Eilenberg-Bass Theorem for uniformly S-Noetherian rings. Recently, the author of the paper [17] introduced the notions of u-S-flat modules and uniformly S-von Neumann regular rings which can be seen as uniformly S-versions of flat modules and von Neumann regular rings. In this paper, we generalized the classical pure exact sequences and absolutely pure modules to u-S-pure u-S-exact sequences and u-S-absolutely pure modules, and then obtain uniformly S-versions of some classical characterizations of pure exact sequences and absolutely pure modules (see Theorem 2.2 and Theorem 3.2). Finally, we characterize uniformly S-von Neumann regular rings and uniformly S-Noetherian rings using u-S-absolutely pure modules (see Theorem 3.5 and Theorem 3.7). As our work involves the uniformly S-torsion theory, we provide a quick review as below.

Recall from [17], an R-module T is said to be *u-S-torsion* (with respect to s) provided that there exists an element $s \in S$ such that $sT = 0$. An R-sequence

$$\cdots \to A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \to \cdots$$

is u-S-exact if for any n there is an element $s \in S$ such that

$$s\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n) \ \text{ and } \ s\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1}).$$

An R-sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short u-S-exact sequence (with respect to s) if $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ for some $s \in S$. An R-homomorphism $f : M \to N$ is a *u-S-monomorphism* (resp. *u-S-epimorphism*, *u-S-isomorphism*) (with respect to s) provided $0 \to M \xrightarrow{f} N$ (resp. $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact (with respect to s). Let M and N be R-modules. We say M is *u-S-isomorphic* to N if there exists a u-S-isomorphism $f : M \to N$. A family $\mathcal{C}$ of R-modules is said to be closed under u-S-isomorphisms if whenever M is u-S-isomorphic to N and M is in $\mathcal{C}$, we have N is also in $\mathcal{C}$. One can deduce from the following Proposition 1.1 that the existence of u-S-isomorphisms of two R-modules is actually an equivalence relation.

Proposition 1.1. *Let R be a ring and S a multiplicative subset of R. Suppose there is a u-S-isomorphism $f : M \to N$ for R-modules M and N. Then there is a u-S-isomorphism $g : N \to M$ and $t \in S$ such that $f \circ g = t\mathrm{Id}_N$ and $g \circ f = t\mathrm{Id}_M$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & M & & \xrightarrow{\quad f \quad} & & N & \longrightarrow & \mathrm{Coker}(f) \longrightarrow 0 \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & \mathrm{Im}(f) & & & & \end{array}$$

with $s\mathrm{Ker}(f) = 0$ and $sN \subseteq \mathrm{Im}(f)$ for some $s \in S$. Define $g_1 : N \to \mathrm{Im}(f)$ by $g_1(n) = sn$ for any $n \in N$. Then g_1 is a well-defined R-homomorphism since $sn \in \mathrm{Im}(f)$. Define $g_2 : \mathrm{Im}(f) \to M$ by $g_2(f(m)) = sm$. Then g_2 is a well-defined R-homomorphism. Indeed, if $f(m) = 0$, then $m \in \mathrm{Ker}(f)$ and so $sm = 0$. Set $g = g_2 \circ g_1 : N \to M$. We claim that g is a u-S-isomorphism. Indeed, let n be an element in $\mathrm{Ker}(g)$. Then $sn = g_1(n) \in \mathrm{Ker}(g_2)$. Note that $s\mathrm{Ker}(g_2) = 0$. Thus $s^2n = 0$. So $s^2\mathrm{Ker}(g) = 0$. On the other hand, let $m \in M$. Then $g(f(m)) = g_2 \circ g_1(f(m)) = g_2(f(sm)) = s^2m$. Set $t = s^2 \in S$. Then $g \circ f = t\mathrm{Id}_M$ and $tm \in \mathrm{Im}(g)$. So $tM \subseteq \mathrm{Im}(g)$. It follows that g is a u-S-isomorphism. It is also easy to verify that $f \circ g = t\mathrm{Id}_N$. □

Remark 1.2. Let R be a ring, S be a multiplicative subset of R, and M and N be R-modules. Then the condition "there is an R-homomorphism $f : M \to N$ such that $f_S : M_S \to N_S$ is an isomorphism" does not mean "there is an R-homomorphism $g : N \to M$ such that $g_S : N_S \to M_S$ is an isomorphism".

Indeed, let $R = \mathbb{Z}$ be the ring of integers, $S = R - \{0\}$ and $\mathbb{Q}$ the quotient field of integers. Then the embedding map $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ satisfies $f_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism. However, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, there does not exist any R-homomorphism $g : \mathbb{Q} \to \mathbb{Z}$ such that $g_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism.

The following two results state that a short u-S-exact sequence induces long u-S-exact sequences by the functors "Tor" and "Ext" as the classical cases.

Theorem 1.3. *Let R be a ring, S a multiplicative subset of R and N an R-module. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence of R-modules. Then for any $n \geq 1$ there is an R-homomorphism $\delta_n : \mathrm{Tor}_n^R(C, N) \to \mathrm{Tor}_{n-1}^R(A, N)$ such that the induced sequence*

$$\cdots \to \mathrm{Tor}_n^R(A, N) \to \mathrm{Tor}_n^R(B, N) \to \mathrm{Tor}_n^R(C, N) \xrightarrow{\delta_n} \mathrm{Tor}_{n-1}^R(A, N) \to$$

$$\mathrm{Tor}_{n-1}^R(B, N) \to \cdots \to \mathrm{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

is u-S-exact.

Proof. Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is u-S-exact at B. There are three exact sequences $0 \to \mathrm{Ker}(f) \xrightarrow{i_{\mathrm{Ker}(f)}} A \xrightarrow{\pi_{\mathrm{Im}(f)}} \mathrm{Im}(f) \to 0$, $0 \to \mathrm{Ker}(g) \xrightarrow{i_{\mathrm{Ker}(g)}} B \xrightarrow{\pi_{\mathrm{Im}(g)}} \mathrm{Im}(g) \to 0$ and $0 \to \mathrm{Im}(g) \xrightarrow{i_{\mathrm{Im}(g)}} C \xrightarrow{\pi_{\mathrm{Coker}(g)}} \mathrm{Coker}(g) \to 0$ with $\mathrm{Ker}(f)$ and $\mathrm{Coker}(g)$ u-S-torsion. There also exists $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. Denote $T = \mathrm{Ker}(f)$ and $T' = \mathrm{Coker}(g)$.

Firstly, consider the exact sequence

$$\mathrm{Tor}_{n+1}^R(T',N) \to \mathrm{Tor}_n^R(\mathrm{Im}(g),N) \xrightarrow{\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)},N)} \mathrm{Tor}_n^R(C,N) \to \mathrm{Tor}_n^R(T',N).$$

Since T' is u-S-torsion, $\mathrm{Tor}_{n+1}^R(T',N)$ and $\mathrm{Tor}_n^R(T',N)$ is u-S-torsion. Thus $\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)},N)$ is a u-S-isomorphism. So there is also a u-S-isomorphism $h_{\mathrm{Im}(g)}^n : \mathrm{Tor}_n^R(C,N) \to \mathrm{Tor}_n^R(\mathrm{Im}(g),N)$ by Proposition 1.1. Consider the exact sequence:

$$\mathrm{Tor}_{n-1}^R(T,N) \to \mathrm{Tor}_{n-1}^R(A,N) \xrightarrow{\mathrm{Tor}_{n-1}^R(\pi_{\mathrm{Im}(f)},N)} \mathrm{Tor}_{n-1}^R(\mathrm{Im}(f),N) \to \mathrm{Tor}_{n-2}^R(T,N).$$

Since T is u-S-torsion, we have $\mathrm{Tor}_{n-1}^R(\pi_{\mathrm{Im}(f)},N)$ is a u-S-isomorphism. So there is also a u-S-isomorphism $h_{\mathrm{Im}(f)}^{n-1} : \mathrm{Tor}_{n-1}^R(\mathrm{Im}(f),N) \to \mathrm{Tor}_{n-1}^R(A,N)$ by Proposition 1.1. We have two exact sequences

$$\mathrm{Tor}_{n+1}^R(T_1,N) \to \mathrm{Tor}_n^R(s\mathrm{Ker}(g),N) \xrightarrow{\mathrm{Tor}_n^R(i^1_{s\mathrm{Ker}(g)},N)} \mathrm{Tor}_n^R(\mathrm{Im}(f),N) \to \mathrm{Tor}_{n+1}^R(T_1,N)$$

and

$$\mathrm{Tor}_{n+1}^R(T_2,N) \to \mathrm{Tor}_n^R(s\mathrm{Ker}(g),N) \xrightarrow{\mathrm{Tor}_n^R(i^2_{s\mathrm{Ker}(g)},N)} \mathrm{Tor}_n^R(\mathrm{Ker}(g),N) \to \mathrm{Tor}_{n+1}^R(T_2,N),$$

where $T_1 = \mathrm{Im}(f)/s\mathrm{Ker}(g)$ and $T_2 = \mathrm{Im}(f)/s\mathrm{Im}(f)$ is u-S-torsion. So $\mathrm{Tor}_n^R(i^1_{s\mathrm{Ker}(g)},N)$ and $\mathrm{Tor}_n^R(i^2_{s\mathrm{Ker}(g)},N)$ are u-S-isomorphisms. Thus there is a u-S-isomorphism $h_{s\mathrm{Ker}(g)}^n : \mathrm{Tor}_n^R(\mathrm{Ker}(g),N) \to \mathrm{Tor}_n^R(s\mathrm{Ker}(g),N)$. Note that there is an exact sequence

$$\begin{aligned}\mathrm{Tor}_n^R(B,N) &\xrightarrow{\mathrm{Tor}_n^R(\pi_{\mathrm{Im}(g)},N)} \mathrm{Tor}_n^R(\mathrm{Im}(g),N) \xrightarrow{\delta_{\mathrm{Im}(g)}^n} \mathrm{Tor}_{n-1}^R(\mathrm{Ker}(g),N)\\ &\xrightarrow{\mathrm{Tor}_{n-1}^R(i_{\mathrm{Ker}(g)},N)} \mathrm{Tor}_{n-1}^R(B,N).\end{aligned}$$

Set $\delta_n = h_{\mathrm{Im}(g)}^n \circ \delta_{\mathrm{Im}(g)}^n \circ h_{s\mathrm{Ker}(g)}^n \circ \mathrm{Tor}_n^R(i^1_{s\mathrm{Ker}(g)},N) \circ h_{\mathrm{Im}(f)}^{n-1} : \mathrm{Tor}_n^R(C,N) \to \mathrm{Tor}_{n-1}^R(A,N)$. Since $h_{\mathrm{Im}(g)}^n, \delta_{\mathrm{Im}(g)}^n, h_{s\mathrm{Ker}(g)}^n$ and $h_{\mathrm{Im}(f)}^{n-1}$ are u-S-isomorphisms, we have the sequence

$$\mathrm{Tor}_n^R(B,N) \to \mathrm{Tor}_n^R(C,N) \xrightarrow{\delta^n} \mathrm{Tor}_{n-1}^R(A,N) \to \mathrm{Tor}_{n-1}^R(B,N)$$

is u-S-exact.

Secondly, consider the exact sequence:

$$\mathrm{Tor}_{n+1}^R(T,N) \to \mathrm{Tor}_n^R(A,N) \xrightarrow{\mathrm{Tor}_n^R(i_{\mathrm{Im}(f)},N)} \mathrm{Tor}_n^R(\mathrm{Im}(f),N) \to \mathrm{Tor}_n^R(T,N).$$

Since T is u-S-torsion, $\mathrm{Tor}_n^R(i_{\mathrm{Im}(f)},N)$ is a u-S-isomorphism. Consider the exact sequences:

$$\mathrm{Tor}_{n+1}^R(\mathrm{Im}(g),N) \to \mathrm{Tor}_n^R(\mathrm{Ker}(g),N) \xrightarrow{\mathrm{Tor}_n^R(i_{\mathrm{Ker}(g)},N)} \mathrm{Tor}_n^R(B,N) \to \mathrm{Tor}_n^R(\mathrm{Im}(g),N)$$

and

$$\mathrm{Tor}_{n+1}^R(T',N) \to \mathrm{Tor}_n^R(\mathrm{Im}(g),N) \xrightarrow{\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)},N)} \mathrm{Tor}_n^R(C,N) \to \mathrm{Tor}_n^R(T',N).$$

Since T' is u-S-torsion, we have $\mathrm{Tor}_n^R(i_{\mathrm{Im}(g)}, N)$ is a u-S-isomorphism. Since $\mathrm{Tor}_n^R(i^1_{s\mathrm{Ker}(g)}, N)$ and $\mathrm{Tor}_n^R(i^2_{s\mathrm{Ker}(g)}, N)$ are u-S-isomorphisms as above, $\mathrm{Tor}_n^R(A, N) \to \mathrm{Tor}_n^R(B, N) \to \mathrm{Tor}_n^R(C, N)$ is u-S-exact at $\mathrm{Tor}_n^R(B, N)$.

Iterating the above steps, we have the following u-S-exact sequence:

$$\cdots \to \mathrm{Tor}_n^R(A, N) \to \mathrm{Tor}_n^R(B, N) \to \mathrm{Tor}_n^R(C, N) \xrightarrow{\delta_n} \mathrm{Tor}_{n-1}^R(A, N) \to$$

$$\mathrm{Tor}_{n-1}^R(B, N) \to \cdots \to \mathrm{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

□

Similarly to the proof of Theorem 1.3, we can deduce the following result.

Theorem 1.4. *Let R be a ring, S be a multiplicative subset of R, and M and N be R-modules. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-exact sequence of R-modules. Then for any $n \geq 1$ there are R-homomorphisms $\delta_n : \mathrm{Ext}_R^{n-1}(M, C) \to \mathrm{Ext}_R^n(M, A)$ and $\delta^n : \mathrm{Ext}_R^{n-1}(A, N) \to \mathrm{Ext}_R^n(C, N)$ such that the induced sequences*

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \xrightarrow{\delta_0} \mathrm{Ext}_R^1(M, A) \to \cdots \to$$
$$\mathrm{Ext}_R^{n-1}(M, B) \to \mathrm{Ext}_R^{n-1}(M, C) \xrightarrow{\delta_n} \mathrm{Ext}_R^n(M, A) \to \mathrm{Ext}_R^n(M, B) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \xrightarrow{\delta^0} \mathrm{Ext}_R^1(C, N) \to \cdots \to$$
$$\mathrm{Ext}_R^{n-1}(B, N) \to \mathrm{Ext}_R^{n-1}(A, N) \xrightarrow{\delta^n} \mathrm{Ext}_R^n(C, N) \to \mathrm{Ext}_R^n(B, N) \to \cdots$$

are u-S-exact.

2. u-S-pure u-S-exact sequences

Recall from [13] that an exact sequence $0 \to A \to B \to C \to 0$ is said to be pure provided that for any R-module M, the induced sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is also exact. Now we introduce the uniformly S-version of pure exact sequences.

Definition 2.1. Let R be a ring and S a multiplicative subset of R. A short u-S-exact sequence $0 \to A \to B \to C \to 0$ is said to be *u-S-pure* provided that for any R-module M, the induced sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is also u-S-exact.

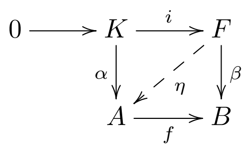
Obviously, any pure exact sequence is u-S-pure. In [16, 34.5], there are many characterizations of pure exact sequences. The next result generalizes some of these characterizations to u-S-pure u-S-exact sequences.

Theorem 2.2. *Let $0 \to A \xrightarrow{f} B \xrightarrow{f'} C \to 0$ be a short u-S-exact sequence of R-modules. Then the following statements are equivalent*:

(1) $0 \to A \xrightarrow{f} B \xrightarrow{f'} C \to 0$ *is a u-S-pure u-S-exact sequence*;

(2) *there exists an element $s \in S$ satisfying that if a system of equations $f(a_i) = \sum_{j=1}^{m} r_{ij}x_j$ $(i = 1, \dots, n)$ with $r_{ij} \in R$ and unknowns $x_1, \dots, x_m$ has a solution in B, then the system of equations $sa_i = \sum_{j=1}^{m} r_{ij}x_j$ $(i = 1, \dots, n)$ is solvable in A;*

(3) *there exists an element $s \in S$ satisfying that for any given commutative diagram with F finitely generated free and K a finitely generated submodule of F, there exists a homomorphism $\eta : F \to A$ such that $s\alpha = \eta i$;*



(4) *there exists an element $s \in S$ satisfying that for any finitely presented R-module N, the induced sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B)$ $\to \mathrm{Hom}_R(N, C) \to 0$ is u-S-exact with respect to s.*

Proof. (1) $\Rightarrow$ (2) Set $\Gamma = \{(K, R^n) \mid K$ is a finitely generated submodule of R^n and $n < \infty\}$. Define $M = \bigoplus_{(K,R^n)\in\Gamma} R^n/K$. Then $0 \to M \otimes_R A \xrightarrow{1\otimes f} M \otimes_R B \to M \otimes_R C \to 0$ is u-S-exact by (1). So there is an element $s \in S$ such that $s\mathrm{Ker}(1_M \otimes f) = 0$. Hence $s\mathrm{Ker}(1_{R^n/K} \otimes f) = 0$ for any $(K, R^n) \in \Gamma$. Now assume that there exists $b_j \in B$ such that $f(a_i) = \sum_{j=1}^{m} r_{ij}b_j$ for any $j = 1, \dots, m$. Let F be a free R-module with a basis $\{e_1, \dots, e_n\}$, and let $K \subseteq F$ be the submodule generated by m elements $\{\sum_{i=1}^{n} r_{ij}e_i \mid j = 1, \dots, m\}$. Then, F/K is generated by $\{e_1+K, \dots, e_n+K\}$. Note that $\sum_{i=1}^{n} r_{ij}(e_i+K) = \sum_{i=1}^{n} r_{ij}e_i + K = 0 + K$ in F/K. Hence, we have

$$\begin{aligned}\sum_{i=1}^{n}((e_i + K) \otimes f(a_i)) &= \sum_{i=1}^{n}((e_i + K) \otimes (\sum_{j=1}^{m} r_{ij}b_j)) \\ &= \sum_{j=1}^{m}((\sum_{i=1}^{n} r_{ij}(e_i + K)) \otimes b_j) = 0\end{aligned}$$

in $F/K\otimes B$. And so $\sum_{i=1}^{n}((e_i+K)\otimes a_i) \in \mathrm{Ker}(1_{F/K}\otimes f)$. Hence, $s\sum_{i=1}^{n}((e_i+K) \otimes a_i) = \sum_{i=1}^{n}((e_i + K) \otimes sa_i) = 0$ in $F/K \otimes_R A$. By [6, Chapter I, Lemma 6.1], there exist $d_j \in A$ and $t_{ij} \in R$ such that $sa_i = \sum_{k=1}^{t} l_{ik}d_k$ and $\sum_{i=1}^{n} l_{ik}(e_i + K) = 0$, and so $\sum_{i=1}^{n} l_{ik}e_i \in K$. Then there exists $t_{jk} \in R$ such that $\sum_{i=1}^{n} l_{ik}e_i = \sum_{j=1}^{m} t_{jk}(\sum_{i=1}^{n} r_{ij}e_i) = \sum_{i=1}^{n}(\sum_{j=1}^{m}(t_{jk}r_{ij})e_i)$. Since F is free, we have $l_{ik} = \sum_{j=1}^{m} r_{ij}t_{jk}$. Hence

$$sa_i = \sum_{k=1}^{t} l_{ik}d_k = \sum_{k=1}^{t}(\sum_{j=1}^{m} r_{ij}t_{jk})d_k = \sum_{j=1}^{m} r_{ij}(\sum_{k=1}^{t} t_{jk}d_k)$$

with $\sum_{k=1}^{t} t_{jk}d_k \in A$. That is, $sa_i = \sum_{j=1}^{m} r_{ij}x_j$ is solvable in A.

(2) $\Rightarrow$ (1) Let $s \in S$ satisfying (2) and M be an R-module. Then we have a u-S-exact sequence $M \otimes_R A \xrightarrow{1\otimes f} M \otimes_R B \to M \otimes_R C \to 0$ by Theorem 1.3. We will show that $\mathrm{Ker}(1 \otimes f)$ is u-S-torsion. Let $\{\sum_{i=1}^{n_\lambda} u_i^\lambda \otimes a_i^\lambda \mid \lambda \in \Lambda\}$ be the generators of $\mathrm{Ker}(1 \otimes f)$. Then $\sum_{i=1}^{n_\lambda} u_i^\lambda \otimes f(a_i^\lambda) = 0$ in $M \otimes_R B$ for each $\lambda \in \Lambda$. By [6, Chapter I, Lemma 6.1], there exist $r_{ij}^\lambda \in R$ and $b_j^\lambda \in B$ such that $f(a_i^\lambda) = \sum_{j=1}^{m_\lambda} r_{ij}^\lambda b_j^\lambda$ and $\sum_{i=1}^{n_\lambda} u_i^\lambda r_{ij}^\lambda = 0$ for each $\lambda \in \Lambda$. So $sa_i^\lambda = \sum_{j=1}^{m_\lambda} r_{ij}^\lambda x_j^\lambda$ have a solution, say a_j^λ in A by (2). Then

$$\begin{aligned} s\Big(\sum_{i=1}^{n_\lambda} u_i^\lambda \otimes a_i^\lambda\Big) &= \sum_{i=1}^{n_\lambda} u_i^\lambda \otimes sa_i^\lambda \\ &= \sum_{i=1}^{n_\lambda} u_i^\lambda \otimes \Big(\sum_{j=1}^{m_\lambda} r_{ij}^\lambda a_j^\lambda\Big) \\ &= \sum_{j=1}^{m_\lambda}\Big(\Big(\sum_{i=1}^{n_\lambda} r_{ij}^\lambda u_i^\lambda\Big) \otimes a_j^\lambda\Big) \\ &= 0 \end{aligned}$$

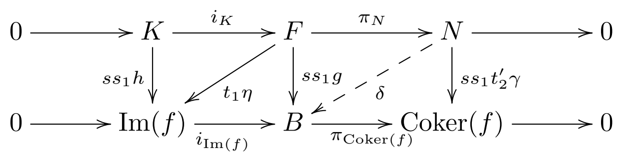
for each $\lambda \in \Lambda$. Hence $s\mathrm{Ker}(1 \otimes f) = 0$, and $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is u-S-exact.

(2) $\Rightarrow$ (3) Let $s \in S$ satisfying (2) and $\{e_1, \dots, e_n\}$ the basis of F. Suppose K is generated by $\{y_i = \sum_{j=1}^m r_{ij} e_j \mid i = 1, \dots, m\}$. Set $\beta(e_j) = b_j$ and $\alpha(y_i) = a_i$ for each i and j. Then $f(a_i) = \sum_{j=1}^m r_{ij} b_j$. By (2), we have $sa_i = \sum_{j=1}^m r_{ij} d_j$ for some $d_j \in A$. Let $\eta : F \to A$ be the R-homomorphism satisfying $\eta(e_j) = d_j$. Then $\eta i(y_i) = \eta i(\sum_{j=1}^m r_{ij} e_j) = \sum_{j=1}^m r_{ij} \eta(e_j) = \sum_{j=1}^m r_{ij} d_j = sa_i = s\alpha(y_i)$, and so we have $s\alpha = \eta i$.

(3) $\Rightarrow$ (4) Let $s \in S$ satisfy (3). Note that A is u-S-isomorphic to $\mathrm{Im}(f)$ and C is u-S-isomorphic to $\mathrm{Coker}(f)$. Thus, by Proposition 1.1, we have homomorphisms $t_1 : A \to \mathrm{Im}(f)$ with $t_1(a) = f(a)$ for any $a \in A$ and $t_1' : \mathrm{Im}(f) \to A$ such that $t_1 t_1' = s_1 \mathrm{Id}_{\mathrm{Im}(f)}$ and $t_1' t_1 = s_1 \mathrm{Id}_A$, and homomorphisms $t_2 : \mathrm{Coker}(f) \to C$ and $t_2' : C \to \mathrm{Coker}(f)$ such that $f' = t_2 \pi_{\mathrm{Coker}(f)}$, $t_2 t_2' = s_2 \mathrm{Id}_C$ and $t_2' t_2 = s_2 \mathrm{Id}_{\mathrm{Coker}(f)}$ for some $s_1, s_2 \in S$, where $\pi_{\mathrm{Coker}(f)} : B \twoheadrightarrow \mathrm{Coker}(f)$ is the natural epimorphism. Let N be a finitely presented R-module with $0 \to K \to F \to N \to 0$ exact, where F is finitely generated free and K finitely generated. Let γ be a homomorphism in $\mathrm{Hom}_R(N, C)$. Considering the exact sequence $0 \to \mathrm{Im}(f) \to B \to \mathrm{Coker}(f) \to 0$, we have the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_K} & F & \xrightarrow{\pi_N} & N & \longrightarrow & 0 \\ & & \big\downarrow h & & \big\downarrow g & & \big\downarrow t_2'\gamma & & \\ 0 & \longrightarrow & \mathrm{Im}(f) & \xrightarrow[i_{\mathrm{Im}(f)}]{} & B & \xrightarrow[\pi_{\mathrm{Coker}(f)}]{} & \mathrm{Coker}(f) & \longrightarrow & 0 \end{array}$$

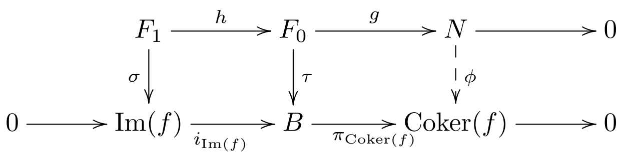
By (3), there exists an homomorphism $\eta : F \to A$ such that $st_1'h = \eta i_K$. So $ss_1h = st_1t_1'h = t_1\eta i_K$. So the following diagram is also commutative:



So by [15, Exercise 1.60], there is an R-homomorphism $\delta : N \to B$ such that $ss_1t_2'\gamma = \pi_{\mathrm{Coker}(f)}\delta$. So $ss_1s_2\gamma = ss_1t_2t_2'\gamma = t_2\pi_{\mathrm{Coker}(f)}\delta = f'\delta = f'^*(\delta)$. Hence $f'^* : \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C)$ is a u-S-epimorphism with respect to ss_1s_2. Consequently, one can verify the R-sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$ is u-S-exact with respect to ss_1s_2 by Theorem 1.4.

(4) $\Rightarrow$ (2) Let $s \in S$ satisfying (4) and $0 \to A \xrightarrow{f} B \xrightarrow{f'} C \to 0$ a short u-S-exact sequence of R-modules. Similarly to the proof of (3) $\Rightarrow$ (4), we have homomorphisms $t_1 : A \to \mathrm{Im}(f)$ with $t_1(a) = f(a)$ for any $a \in A$ and $t_1' : \mathrm{Im}(f) \to A$ such that $t_1t_1' = s_1\mathrm{Id}_{\mathrm{Im}(f)}$ and $t_1't_1 = s_1\mathrm{Id}_A$, and homomorphisms $t_2 : \mathrm{Coker}(f) \to C$ and $t_2' : C \to \mathrm{Coker}(f)$ such that $f' = t_2\pi_{\mathrm{Coker}(f)}$, $t_2t_2' = s_2\mathrm{Id}_C$ and $t_2't_2 = s_2\mathrm{Id}_{\mathrm{Coker}(f)}$ for some $s_1, s_2 \in S$, where $\pi_{\mathrm{Coker}(f)} : B \twoheadrightarrow \mathrm{Coker}(f)$ is the natural epimorphism.

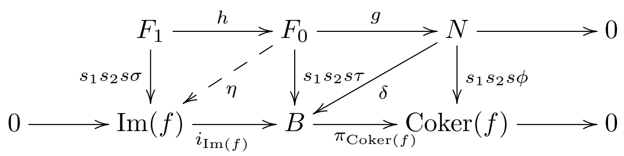
Suppose that $f(a_i) = \sum_{j=1}^m r_{ij}b_j$ $(i = 1, \dots, n)$ with $a_i \in A$, $b_j \in B$ and $r_{ij} \in R$. Let F_0 be a free module with a basis $\{e_1, \dots, e_m\}$ and F_1 a free module with basis $\{e_1', \dots, e_n'\}$. Then there are R-homomorphisms $\tau : F_0 \to B$ and $\sigma : F_1 \to \mathrm{Im}(f)$ satisfying $\tau(e_j) = b_j$ and $\sigma(e_i') = f(a_i)$ for each i, j. Define an R-homomorphism $h : F_1 \to F_0$ by $h(e_i') = \sum_{j=1}^m r_{ij}e_j$ for each i. Then $\tau h(e_i') = \sum_{j=1}^m r_{ij}\tau(e_j) = \sum_{j=1}^m r_{ij}b_j = f(a_i) = \sigma(e_i')$. Set $N = \mathrm{Coker}(h)$. Then N is finitely presented. Thus there exists a homomorphism $\phi : N \to \mathrm{Coker}(f)$ such that the following diagram commutative:



Note that the induced sequence

$$0 \to \mathrm{Hom}_R(N, \mathrm{Im}(f)) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, \mathrm{Coker}(f)) \to 0$$

is u-S-exact with respect to s_1s_2s by (4). Hence there exists a homomorphism $\delta : N \to \mathrm{Coker}(f)$ such that $s_1s_2s\phi = \pi_{\mathrm{Coker}(f)}\delta$. Consider the following commutative diagram:



We claim that there exists a homomorphism $\eta : F_0 \to \mathrm{Im}(f)$ such that $\eta f = s_1s_2s\sigma$. Indeed, since $\pi_{\mathrm{Coker}(f)}\delta g = s_1s_2s\phi g = \pi_{\mathrm{Coker}(f)}s_1s_2s\tau$, we have

$$\mathrm{Im}(s_1s_2s\tau - \delta g) \subseteq \mathrm{Ker}(\pi_{\mathrm{Coker}(f)}) = \mathrm{Im}(f).$$

Define $\eta : F_0 \to \mathrm{Im}(f)$ to be a homomorphism satisfying $\eta(e_i) = s_1s_2s\tau(e_i) - \delta g(e_i)$ for each i. So for each $e'_i \in F_1$, we have $\eta f(e'_i) = s_1s_2s\tau f(e'_i) - \delta g f(e'_i) = s_1s_2s\tau f(e'_i)$. Thus $i_{\mathrm{Im}(f)}(s_1s_2s\sigma) = s_1s_2si_{\mathrm{Im}(f)}\sigma = s_1s_2s\tau f = i_{\mathrm{Im}(f)}\eta f$. Therefore, $\eta f = s_1s_2s\sigma$. Hence $s_1s_2sf(a_i) = s_1s_2s\sigma(e'_i) = \eta f(e'_i) = \eta(\sum_{j=1}^m r_{ij}e_j) = \sum_{j=1}^m r_{ij}\eta(e_j)$ with $\eta(e_j) \in \mathrm{Im}(f)$. So we have $s_1^2s_2sa_i = s_1s_2st'_1f(a_i) = \sum_{j=1}^m r_{ij}t'_1\eta(e_j)$ with $t'_1\eta(e_j) \in A$ for each i. □

Recall from [18, Definition 2.1] that a short u-S-exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be u-S-split provided that there are $s \in S$ and an R-homomorphism $t : B \to A$ such that $tf(a) = sa$ for any $a \in A$, that is, $tf = s\mathrm{Id}_A$.

Proposition 2.3. *Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-split short u-S-exact sequence. Then ξ is u-S-pure.*

Proof. Let $t : B \to A$ be an R-homomorphism satisfying $tf = s\mathrm{Id}_A$. Let $f(a_i) = \sum_{j=1}^m r_{ij}x_j$ be a system of equations with $r_{ij} \in R$ and unknowns $x_1, \dots, x_m$ has a solution, say $\{b_j \mid j = 1, \dots, m\}$, in B. Then $sa_i = tf(a_i) = \sum_{j=1}^m r_{ij}t(b_j)$ with $t(b_j) \in A$. Thus $sa_i = \sum_{j=1}^m r_{ij}x_j$ is solvable in A. So ξ is u-S-pure by Theorem 2.2. □

Recall from [17, Definition 3.1] that an R-module F is called u-S-flat provided that for any u-S-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact. By [17, Theorem 3.2], an R-module F is u-S-flat if and only if $\mathrm{Tor}_1^R(M, F)$ is u-S-torsion for any R-module M.

Proposition 2.4. *An R-module F is u-S-flat if and only if every (u-S-)exact sequence $0 \to A \to B \to F \to 0$ is u-S-pure.*

Proof. Suppose F is a u-S-flat module. Let M be an R-module and $0 \to A \to B \to F \to 0$ a short u-S-exact sequence. Then by Theorem 1.3, there is a u-S-exact sequence $\mathrm{Tor}_1^R(M, F) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R F \to 0$. Since F is u-S-flat, $\mathrm{Tor}_1^R(M, F)$ is u-S-torsion by [17, Theorem 3.2]. Hence

$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R F \to 0$ is u-S-exact. So $0 \to A \to B \to F \to 0$ is u-S-pure.

On the other hand, considering the exact sequence $0 \to A \to P \to F \to 0$ with P projective, we have an exact sequence $0 \to \mathrm{Tor}_1^R(M, F) \to M \otimes_R A \to M \otimes_R P \to M \otimes_R F \to 0$ for any R-module M. Since $0 \to A \to P \to F \to 0$ is u-S-pure, $\mathrm{Tor}_1^R(M, F)$ is u-S-torsion. So F is u-S-flat □

Proposition 2.5. *Let $\xi : 0 \to A \to B \to C \to 0$ be a short u-S-exact sequence, where B is u-S-flat. Then C is u-S-flat if and only if ξ is u-S-pure.*

Proof. Suppose C is u-S-flat. Then ξ is u-S-pure by Proposition 2.4.

On the other hand, let M be an R-module. Then we have a u-S-exact sequence $\mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$. Since B is u-S-flat, $\mathrm{Tor}_1^R(M, B)$ is u-S-torsion by [17, Theorem 3.2]. Since ξ is u-S-pure by assumption, $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R F \to 0$ is u-S-exact. Then $\mathrm{Tor}_1^R(M, C)$ is also u-S-torsion. Thus C is u-S-flat by [17, Theorem 3.2] again. □

3. Uniformly S-absolutely pure modules

Recall from [10] that an R-module E is said to be absolutely pure provided that E is a pure submodule of every module which contains E as a submodule, that is, any short exact sequence $0 \to E \to B \to C \to 0$ beginning with E is pure. Now we define the uniformly S-analogue of absolutely pure modules.

Definition 3.1. Let R be a ring and S a multiplicative subset of R. An R-module E is said to be *u-S-absolutely pure* (abbreviates uniformly S-absolutely pure) provided that any short u-S-exact sequence $0 \to E \to B \to C \to 0$ beginning with E is u-S-pure.

Recall from [12, Definition 4.1] that an R-module E is called *u-S-injective* provided that the induced sequence

$$0 \to \mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(A, E) \to 0$$

is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$. Following from [12, Theorem 4.3], an R-module E is u-S-injective if and only if for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to \mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(A, E) \to 0$ is u-S-exact if and only if $\mathrm{Ext}_R^1(M, E)$ is u-S-torsion for any R-module M if and only if $\mathrm{Ext}_R^n(M, E)$ is u-S-torsion for any R-module M and $n \geq 1$. Next, we characterize u-S-absolutely pure modules in terms of u-S-injective modules.

Theorem 3.2. *Let R be a ring, S a multiplicative subset of R and E an R-module. Then the following statements are equivalent*:

(1) *E is u-S-absolutely pure*;
(2) *any short exact sequence $0 \to E \to B \to C \to 0$ beginning with E is u-S-pure*;

(3) *E is a u-S-pure submodule in every u-S-injective module containing E;*
(4) *E is a u-S-pure submodule in every injective module containing E;*
(5) *E is a u-S-pure submodule in its injective envelope;*
(6) *there exists an element $s \in S$ satisfying that for any finitely presented R-module N, $\mathrm{Ext}_R^1(N, E)$ is u-S-torsion with respect to s;*
(7) *there exists an element $s \in S$ satisfying that if P is finitely generated projective, K is a finitely generated submodule of P and $f : K \to E$ is an R-homomorphism, then there is an R-homomorphism $g : P \to E$ such that $sf = gi$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) It is obvious.

(5) $\Rightarrow$ (6) Let I be the injective envelope of E. Then we have a u-S-pure exact sequence $0 \to E \to I \to L \to 0$ by (5). Then, by Theorem 2.2, there is an element $s \in S$ such that $0 \to \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(N, L) \to 0$ is u-S-exact with respect to s for any finitely presented R-module N. Since $0 \to \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(N, L) \to \mathrm{Ext}_R^1(N, E) \to 0$ is exact. Hence $\mathrm{Ext}_R^1(N, E)$ is u-S-torsion with respect to s for any finitely presented R-module N.

(6) $\Rightarrow$ (1) Let $s \in S$ satisfy (6). Let N be a finitely presented R-module and $0 \to E \to B \to C \to 0$ a u-S-exact sequence with respect to $s_1 \in S$. Then, by Theorem 1.4, there is a u-S-exact sequence $0 \to \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to \mathrm{Ext}_R^1(N, E)$ with respect to s_1 for any finitely presented R-module N. By (6),

$$0 \to \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$$

is u-S-exact with respect to ss_1 for any finitely presented R-module N. Hence E is u-S-absolutely pure by Theorem 2.2.

(6) $\Rightarrow$ (7) Let $s \in S$ satisfy (6). Considering the exact sequence $0 \to K \xrightarrow{i} P \to P/K \to 0$, we have the following exact sequence

$$\mathrm{Hom}_R(P, E) \xrightarrow{i_*} \mathrm{Hom}_R(K, E) \to \mathrm{Ext}_R^1(P/K, E) \to 0.$$

Since P/K is finitely presented, $\mathrm{Ext}_R^1(P/K, E)$ is u-S-torsion with respect to s by (6). Hence i_* is a u-S-epimorphism, and so $s\mathrm{Hom}_R(K, E) \subseteq \mathrm{Im}(i_*)$. Let $f : K \to E$ be an R-homomorphism. Then there is an R-homomorphism $g : P \to E$ such that $sf = gi$.

(7) $\Rightarrow$ (6) Let $s \in S$ satisfy (7). Let N be a finitely presented R-module. Then we have an exact sequence $0 \to K \xrightarrow{i} P \to N \to 0$, where P is finitely generated projective and K is finitely generated. Consider the following exact sequence

$$\mathrm{Hom}_R(P, E) \xrightarrow{i_*} \mathrm{Hom}_R(K, E) \to \mathrm{Ext}_R^1(N, E) \to 0.$$

By (7), we have $s\mathrm{Hom}_R(K, E) \subseteq \mathrm{Im}(i_*)$. Hence $\mathrm{Ext}_R^1(N, E)$ is u-S-torsion with respect to s. ☐

Proposition 3.3. *Let R be a ring and S a multiplicative subset of R. Then the following statements hold.*

(1) *Any absolutely pure module and any u-S-injective module are u-S-absolutely pure.*
(2) *Any finite direct sum of u-S-absolutely pure modules is u-S-absolutely pure.*
(3) *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. If A and C are u-S-absolutely pure modules, so is B.*
(4) *The class of u-S-absolutely pure modules is closed under u-S-isomorphisms.*
(5) *Let $0 \to A \to B \to C \to 0$ be a u-S-pure u-S-exact sequence. If B is u-S-absolutely pure, so is B.*

Proof. (1) This follows from Theorem 3.2.

(2) Suppose $E_1, \dots, E_n$ are u-S-absolutely pure modules. Then there exists $s_i \in S$ such that $s_i \mathrm{Ext}_R^1(M, E_i) = 0$ for any finitely presented R-module M $(i = 1, \dots, n)$. Set $s = s_1 \cdots s_n$. Then

$$s\mathrm{Ext}_R^1(M, \bigoplus_{i=1}^{n} E_i) \cong \bigoplus_{i=1}^{n} s\mathrm{Ext}_R^1(M, E_i) = 0.$$

Thus $\bigoplus_{i=1}^{n} E_i$ is u-S-absolutely pure.

(3) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. Since A and C are u-S-absolutely pure modules, it follows by Theorem 3.2 that $\mathrm{Ext}_R^1(N, A)$ and $\mathrm{Ext}_R^1(N, C)$ are u-S-torsion with respect to some $s_1, s_2 \in S$, respectively, for any finitely presented R-module N. Considering the u-S-sequence $\mathrm{Ext}_R^1(N, A) \to \mathrm{Ext}_R^1(N, B) \to \mathrm{Ext}_R^1(N, C)$ by Theorem 1.4, we have $\mathrm{Ext}_R^1(N, B)$ is u-S-torsion with respect to $s_1 s_2$ for any finitely presented R-module N. Hence B is u-S-absolutely pure by Theorem 3.2 again.

(4) Considering the u-S-exact sequences $0 \to A \to B \to 0 \to 0$ and $0 \to 0 \to A \to B \to 0$, we have A is u-S-absolutely pure if and only if B is u-S-absolutely pure by (3).

(5) Let $0 \to A \to B \to C \to 0$ be a u-S-pure u-S-exact sequence with respect to some $s \in S$. Then, by Theorem 1.4, there exists a u-S-sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to \mathrm{Ext}_R^1(N, A) \to \mathrm{Ext}_R^1(N, B)$ with respect to s for any finitely presented R-module N. Note that the natural homomorphism $\mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C)$ is a u-S-epimorphism. Since B is u-S-absolutely pure, it follows that $\mathrm{Ext}_R^1(N, B)$ is u-S-torsion with respect to some $s_1 \in S$ for any finitely presented R-module N by Theorem 3.2. Then $\mathrm{Ext}_R^1(N, A)$ is u-S-torsion with respect to ss_1 for any finitely presented R-module N. Thus A is u-S-absolutely pure by Theorem 3.2 again. □

Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module E is u-$\mathfrak{p}$-*absolutely pure* shortly provided that E is u-$(R \setminus \mathfrak{p})$-absolutely pure.

Proposition 3.4. *Let R be a ring and E an R-module. Then the following statements are equivalent*:

(1) *E is absolutely pure*;
(2) *E is u-$\mathfrak{p}$-absolutely pure for any $\mathfrak{p} \in \mathrm{Spec}(R)$*;
(3) *E is u-$\mathfrak{m}$-absolutely pure for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Since E is $\mathfrak{m}$-absolutely pure for any $\mathfrak{m} \in \mathrm{Max}(R)$, we have $\mathrm{Ext}^1_R(N, E)$ is uniformly $(R \setminus \mathfrak{m})$-torsion for any finitely presented R-module N. Thus for any $\mathfrak{m} \in \mathrm{Max}(R)$, there exists $s_{\mathfrak{m}} \in S$ such that $s_{\mathfrak{m}}\mathrm{Ext}^1_R(N, E) = 0$ for any finitely presented R-module N. Since the ideal generated by all $s_{\mathfrak{m}}$ is R, $\mathrm{Ext}^1_R(N, E) = 0$ for any finitely presented R-module N. So E is absolutely pure. □

Recall from [17, Definition 3.12] a ring R is called *uniformly S-von Neumann regular* provided there exists an element $s \in S$ satisfying that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$. It was proved in [17, Theorem 3.13] that a ring R is uniformly S-von Neumann regular if and only if any R-module is u-S-flat.

Theorem 3.5. *A ring R is uniformly S-von Neumann regular if and only if any R-module is u-S-absolutely pure.*

Proof. Suppose R is a uniformly S-von Neumann regular ring. Let M be an R-module and I its injective envelope. Then I/M is u-S-flat by [17, Theorem 3.13]. Hence M is a u-S-pure submodule of I by Proposition 2.4. So M is u-S-absolutely pure by Theorem 3.2.

Conversely, assume that any R-module is u-S-absolutely pure and let M be an R-module and $\xi : 0 \to K \to P \to M \to 0$ an exact sequence with P projective. Then P is u-S-flat. Since K is u-S-absolutely pure, the exact sequence ξ is u-S-pure. By Proposition 2.5, M is also u-S-flat. Hence R is uniformly S-von Neumann regular by [17, Theorem 3.13]. □

It follows from Proposition 3.3 that every absolutely pure module is u-S-absolutely pure. The following example shows that the converse is not true in general.

Example 3.6 ([17, Example 3.18])**.** Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Then any element $a \in T$ satisfies $a^2 = a$ and $2a = 0$. Let $R = T[x]/\langle sx, x^2\rangle$ with x an indeterminate and $S = \{1, \bar{s}\}$ be a multiplicative subset of R. Then R is a uniformly S-von Neumann regular ring, but R is not von Neumann regular. Thus there exists a u-S-absolutely pure module M which is not absolutely pure by Theorem 3.5.

Let R be a ring. An R-module M is said to be *u-S-divisible* if there exists $s \in S$ such that $sM = M$. Recall from [12] that a ring R is called a uniformly S-Noetherian ring provided that there exists an element $s \in S$ such that for any ideal J of R, $sJ \subseteq K$ for some finitely generated sub-ideal K of J. Following from Theorem [12, Theorem 4.10] that if S is a regular multiplicative subset of R (i.e., the multiplicative set S is composed of non-zero-divisors), then R is uniformly S-Noetherian if and only if any direct sum of injective modules is u-S-injective. Now we give a new characterization of uniformly S-Noetherian rings.

Theorem 3.7. *Let R be a ring, S a regular multiplicative subset of R. Then the following statements are equivalent*:

(1) *R is a uniformly S-Noetherian ring*;
(2) *any u-S-absolutely pure module is u-S-injective*;
(3) *any absolutely pure module is u-S-injective.*

Proof. (1) $\Rightarrow$ (2) Suppose R is a uniformly S-Noetherian ring. Let s be an element in S such that for any ideal J of R, $sJ \subseteq K$ for some finitely generated sub-ideal K of J. Let E be a u-S-absolutely pure module. Then there exists $s_2 \in S$ such that $s_2\mathrm{Ext}^1_R(N, E) = 0$ for any finitely presented R-module N. Let s_1 be an element in S. Consider the induced exact sequence $\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(Rs_1, E) \to \mathrm{Ext}^1_R(R/Rs_1, E) \to 0$. Since R/Rs_1 is finitely presented, $s_2\mathrm{Ext}^1_R(R/Rs_1, E) = s_2(E/s_1E) = 0$ since s_1 is a non-zero-divisor. Then $s_2E = s_1s_2E$, and thus s_2E is u-S-divisible. Since s_2E is u-S-isomorphic to E, s_2E is also u-S-absolutely pure by Proposition 3.3. Hence there exists $s_3 \in S$ such that $s_3\mathrm{Ext}^1_R(N, E) = 0$ for any finitely presented R-module N. Consider the induced u-S-exact sequence $\mathrm{Hom}_R(J/K, s_2E) \to \mathrm{Ext}^1_R(R/J, s_2E) \to \mathrm{Ext}^1_R(R/K, s_2E)$. Since R/K is finitely presented, we have $s_3\mathrm{Ext}^1_R(R/K, s_2E) = 0$. Note that $s\mathrm{Hom}_R(J/K, s_2E) = 0$. Then

$$ss_3\mathrm{Ext}^1_R(R/J, s_2E) = 0.$$

Since s_2E is u-S-divisible, we have s_2E is u-S-injective by [12, Proposition 4.9]. Since s_2E is u-S-isomorphic to E, it follows that E is also u-S-injective by [12, Proposition 4.7].

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a family of injective modules. Then $\bigoplus_{\lambda\in\Lambda} I_\lambda$ is absolutely pure, and thus is u-S-injective by assumption. Consequently, R is a uniformly S-Noetherian ring by [12, Theorem 4.10]. $\square$

It is well-known that any direct sum and any direct product of absolutely pure modules are also absolutely pure. However, it does not work for u-S-absolutely pure modules.

Example 3.8. Let $R = \mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$. Then an R-module M is a u-S-absolutely pure module if and only if it is u-S-injective by Theorem 3.7. Let $\mathbb{Z}/\langle p^k\rangle$ be a cyclic group of order p^k ($k \geq 1$). Then each $\mathbb{Z}/\langle p^k\rangle$ is u-S-torsion, and thus is u-S-absolutely pure. However, the product $M := \prod_{k=1}^{\infty} \mathbb{Z}/\langle p^k\rangle$ is not u-S-injective by [12, Remark 4.6], so it is also not u-S-absolutely pure.

We claim that the direct sum $N := \bigoplus_{k=1}^{\infty} \mathbb{Z}/\langle p^k \rangle$ is also not u-S-absolutely pure. Indeed, consider the following exact sequence induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$:

$$0 = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, N) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, N) \to 0.$$

Since the submodule $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N)$ is not u-S-torsion, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N)$ is also not u-S-torsion. Then N is not u-S-injective by [12, Theorem 4.3]. So the direct sum $N := \bigoplus_{k=1}^{\infty} \mathbb{Z}/\langle p^k \rangle$ is also not u-S-absolutely pure.

We also note that, in Theorem 3.2, the element $s \in S$ in the statement (6) (similar in the statement (7)) is uniform for all finitely presented R-modules N.

Example 3.9. Let $R = \mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$. Let J_p be the additive group of all p-adic integers (see [5] for example). Then $\mathrm{Ext}^1_R(N, J_p)$ is u-S-torsion for any finitely presented R-modules N. However, J_p is not u-S-absolutely pure.

Proof. Let N be a finitely presented R-module. Then, by [5, Chapter 3, Theorem 2.7], $N \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{m} (\mathbb{Z}^n/\langle p^i \rangle)^{n_i} \oplus T$, where T is a finitely generated torsion S-divisible torsion-module. Thus

$$\mathrm{Ext}^1_R(N, J_p) \cong \bigoplus_{i=1}^{m} \mathrm{Ext}^1_R(\mathbb{Z}^n/\langle p^i \rangle, J_p) \cong \bigoplus_{i=1}^{m} (J_p/p^i J_p) \cong \bigoplus_{i=1}^{m} \mathbb{Z}^n/\langle p^i \rangle$$

by [5, Chapter 9, Section 3(G)] and [5, Chapter 1, Exercise 3(10)]. So $\mathrm{Ext}^1_R(N, J_p)$ is obviously u-S-torsion. However, J_p is not u-S-injective by [12, Theorem 4.5]. So J_p is not u-S-absolutely pure by Theorem 3.7. □

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