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SCHUR CONVEXITY OF L-CONJUGATE MEANS AND ITS APPLICATIONS

Chun-Ru Fu, Huan-Nan Shi, and Dong-Sheng Wang

Abstract. In this paper, using the theory of majorization, we discuss the Schur m power convexity for L-conjugate means of n variables and the Schur convexity for weighted L-conjugate means of n variables. As applications, we get several inequalities of general mean satisfying Schur convexity, and a few comparative inequalities about n variables Gini mean are established.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vectors on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, \ldots, n\}.$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively.

$$A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i,\ G_n(\boldsymbol{x}) = \prod_{i=1}^n x_i^{\frac{1}{n}},\ H_n(\boldsymbol{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$$

are the arithmetic mean, geometric mean and harmonic mean of $\boldsymbol{x} \in \mathbb{R}^n_{++}$, respectively.

$$M_n^{[m]}(\boldsymbol{x}) = \left(\frac{\sum_{i=1}^n x_i^m}{n}\right)^{\frac{1}{m}} (m \neq 0)$$

is the m-order power mean of $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

Generally, let $\boldsymbol{x} \in I^n \subset \mathbb{R}^n_{++}$, $L(\boldsymbol{x})$: $I^n \to \mathbb{R}_{++}$ be a continuous function. We call $L(\boldsymbol{x})$ a mean if it has the following properties:

(i) $L(\boldsymbol{x})$ is symmetry with $x_1, \ldots, x_n$.

(ii) For any $\lambda > 0$, if $(\lambda x_1, \ldots, \lambda x_n) \in I^n$, then

$$L(\lambda x_1, \ldots, \lambda x_n) = \lambda L(x_1, \ldots, x_n).$$

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(iii) For any $a \in I$, $L(a, \dots, a) = a$.
(iv) If $0 < m \le x_i \le M$, $i = 1, \dots, n$, then $m \le L(x_1, \dots, x_n) \le M$.

The concept and method of mean value play a basic role in mathematical theory, and its mathematical theory research is mainly related to convex function and inequality theory. For a set of statistics, the mean value can be regarded as a representative quantity determined by certain criteria. Therefore, the theoretical study of the mean value is valuable.

The L-conjugate mean was originated from the study of the pseudo arithmetic mean. In the paper [7], the author studied the conjugate arithmetic mean:

Definition 1.1 ([7])**.** A function $M : I^2 \to I$ is called a conjugate arithmetic mean in I if there exists $\varphi \in CM(I)$ for which

$$M(x, y) = \varphi^{-1}\left(\varphi(x) + \varphi(y) - \varphi\left(\frac{x+y}{2}\right)\right) \tag{1.1}$$

for all $x, y \in I$, where $CM(I)$ is a set of all continuous and strictly monotonic real functions defined on I.

Daróczy and Dascăl [6] defined a weighted L-conjugate mean of two variables.

Definition 1.2 ([6])**.** Let $L : I^2 \to I$ be a fixed strict mean. A function $M : I^2 \to I$ is said to be an L-conjugate mean on I if there exist $p, q \in [0, 1]$ and $\varphi \in CM(I)$ such that

$$M(x, y) = \varphi^{-1}\left(p\varphi(x) + q\varphi(x) + (1 - p - q)L(x, y)\right), \ (x, y) \in I^2 \tag{1.2}$$

the numbers p,q are said to be the weights and the function is called the generating function of the mean M.

In paper [8], Daróczy and Páles introduced the notion of L-conjugate mean of $n > 2$ variables.

Definition 1.3 ([8, 21])**.** L-conjugate means of $n \ge 2$ variables defined by

$$L^*_\phi(x_1, \dots, x_n) = \phi^{-1}\left(\frac{\phi(x_1) + \cdots + \phi(x_n) - \phi(L(x_1, \dots, x_n))}{n-1}\right), \tag{1.3}$$

where $L : I^n \to I$ is a symmetric mean on the open real interval I and $\phi : I \to \mathbb{R}$ is a continuous and strictly monotonic function.

Let $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+$ and $\boldsymbol{w} = (w_1, \dots, w_m) \in \mathbb{R}^m_+$, when we say that a pair $(\boldsymbol{p}, \boldsymbol{w})$ is admissible, we mean that for all $i \in (1, \dots, n)$ inequality

$$p_i \ge \sum_{j=1}^{m} w_j$$

holds.

In 2007, Bakula et al. [11] defines weighted L-conjugate mean.

Definition 1.4 ([11])**.** Let $n \geq 2$, $m \geq 1$, $L = (L_1, \dots, L_m)$ be an m-tuple of fixed means of n variables on an open real interval I, and φ is a strictly monotonic and differentiable function. Let $\boldsymbol{x} \in I^n$ and $(\boldsymbol{p}, \boldsymbol{w})$ be an admissible pair, where $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$. The weighted L-conjugate mean L_φ^* of n-tuple $\boldsymbol{x}$ with weights $(\boldsymbol{p}, \boldsymbol{w})$ is defined as

$$L_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w}) = \varphi^{-1}\left(\frac{\sum_{i=1}^n p_i \varphi(x_i) - \sum_{j=1}^m w_j \varphi(L_j(\boldsymbol{x}))}{P_n - W_m}\right), \tag{1.4}$$

where

$$P_n = \sum_{i=1}^n p_i, \quad W_m = \sum_{j=1}^m w_j.$$

In recent years, the theory of majorization has been used as an important tool in studying the properties of the means (see [2–5, 9, 14–16, 18, 19, 22–26]).

In this paper, we discuss Schur convexity of weighted L-conjugate mean for n variables and Schur m power convexty of L-conjugate mean for n variables, as an application, some new inequalities about mean are obtained.

Our main result is as follows.

Theorem 1.5. *Let $\varphi(x)$ be a continuous function on $I \subset \mathbb{R}$, $D = \{\boldsymbol{x} : x_1 \geq \cdots \geq x_n\}$, and $L_j(\boldsymbol{x})$, $j = 1, \dots, m$, be m fixed means.*

(i) *If φ is strictly increasing and convex on I, L_j $(j = 1, \dots, m)$ is Schur concave and $p_1 \geq \cdots \geq p_n > 0$, then $L_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w})$ is Schur convex on $D \cap I$ with $\boldsymbol{x}$.*

If φ is strictly increasing and concave on I, L_j $(j = 1, \dots, m)$ is Schur convex and $0 < p_1 \leq \cdots \leq p_n$, then $L_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w})$ is Schur concave on $D \cap I$ with $\boldsymbol{x}$.

(ii) *If φ is strictly decreasing and concave on I, L_j $(j = 1, \dots, m)$ is Schur concave, and $p_1 \geq \cdots \geq p_n > 0$, then $L_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w})$ is Schur convex on $D \cap I$ with $\boldsymbol{x}$.*

If φ is strictly decreasing and convex on I, L_j $(j = 1, \dots, m)$ is Schur convex, and $0 < p_1 \leq \cdots \leq p_n$, then $L_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w})$ is Schur concave on $D \cap I$ with $\boldsymbol{x}$.

Theorem 1.6. *Let $\phi(x)$ be a strictly monotone continuous function on $I \subset \mathbb{R}$, $L(\boldsymbol{x})$ be a fixed mean, and $m \in \mathbb{R}$.*

(i) *For $m < 1$ and $m \neq 0$, if ϕ is strictly increasing and convex, or ϕ is strictly decreasing and concave, and $L(\boldsymbol{x})$ is Schur m power concave, then $L_\phi^*(\boldsymbol{x})$ is Schur m power convex with $\boldsymbol{x}$.*

(ii) *For $m = 1$,*

(1) *if ϕ is strictly increasing and convex, $L(\boldsymbol{x})$ is Schur concave, or ϕ is strictly decreasing and concave, and $L(\boldsymbol{x})$ is Schur concave, then $L_\phi^*(\boldsymbol{x})$ is Schur convex with $\boldsymbol{x}$;*

(2) *if ϕ is strictly increasing and concave, $L(\boldsymbol{x})$ is Schur convex, or ϕ is strictly decreasing and convex, $L(\boldsymbol{x})$ is Schur convex, then $L^*_\phi(\boldsymbol{x})$ is Schur concave with $\boldsymbol{x}$.*

(iii) *For $m > 1$, if ϕ is strictly increasing and concave, or ϕ is strictly decreasing and convex, $L(\boldsymbol{x})$ is Schur m power convex, then $L^*_\phi(\boldsymbol{x})$ is Schur m power concave with $\boldsymbol{x}$.*

(iv) *For $m = 0$,*

(1) *if ϕ is strictly increasing and convex, $L(\boldsymbol{x})$ is Schur geometrically concave, then $L^*_\phi(\boldsymbol{x})$ is Schur geometrically convex with $\boldsymbol{x}$;*

(2) *if ϕ is strictly decreasing and concave, $L(\boldsymbol{x})$ is Schur geometrically concave, then $L^*_\phi(\boldsymbol{x})$ is Schur geometrically concave with $\boldsymbol{x}$.*

2. Preliminaries

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 2.1 ([13]). Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) A set $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) Let $\Omega \subset \mathbb{R}^n$. A function φ: $\Omega \to \mathbb{R}$ is said to be a Schur convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is a Schur convex function.

Definition 2.2 ([27]). Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$.

(i) A set $\Omega \in \mathbb{R}^n_{++}$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_1^\beta) \in \Omega$ for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subset \mathbb{R}^n_+$. A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\log x_1, \ldots, \log x_n) \prec (\log y_1, \ldots, \log y_n)$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is a Schur geometrically convex function.

Definition 2.3 ([17]). Let $\Omega \subset \mathbb{R}^n_{++}$.

(i) A set Ω is said to be a harmonically convex set if $\frac{\mathbf{xy}}{\lambda\mathbf{x} + (1-\lambda)\mathbf{y}} \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\mathbf{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\mathbf{x}} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Definition 2.4 ([17])**.** Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x^m-1}{m}, & m \neq 0; \\ \log x, & m = 0. \end{cases} \tag{2.1}$$

Then a function $\varphi : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), \dots, f(x_n)) \prec (f(y_1), \dots, f(y_n))$$

for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \Omega$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

If $-\varphi$ is Schur m power convex, then we say that φ is Schur m power concave.

If putting $f(x) = x, \ln x, \frac{1}{x}$ in Definition 2.4, then the definitions of Schur convex, Schur geometrically convex, and Schur harmonically convex functions can be deduced, respectively.

Lemma 2.5 ([13, 20])**.** *Let $\Omega \subset \mathbb{R}^n$ be a convex set, and have a nonempty interior set Ω°. Let $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is the Schur convex (or Schur concave, respectively) function if and only if it is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\mathit{or}\ \leq 0,\ \mathit{respectively})$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^\circ$.

Remark 2.6. Lemma 2.5 equivalent to

$$\frac{\partial \varphi}{\partial x_i} \geq \frac{\partial \varphi}{\partial x_{i+1}} \quad (\text{or } \leq 0,\ \text{respectively}),\ i = 1, \dots, n-1,$$

for all $\mathbf{x} \in D \cap \Omega$, where $D = \{\mathbf{x} : x_1 \geq \cdots \geq x_n\}$.

Lemma 2.7 ([27])**.** *Let $\Omega \subset \mathbb{R}^n_{++}$ be a convex set with a nonempty interior set Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is the Schur geometrically convex (or Schur geometrically concave, respectively) function if and only if it is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\mathit{or}\ \leq 0,\ \mathit{respectively}) \tag{2.2}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^\circ$.

Lemma 2.8 ([17])**.** *Let $\Omega \subset \mathbb{R}_n$ be a symmetric harmonically convex set with a nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable on Ω. Then φ is the Schur harmonically convex (or Schur harmonically concave, respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\mathit{or}\ \leq 0,\ \mathit{respectively}) \tag{2.3}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^\circ$.

Lemma 2.9 ([17]). *Let $\Omega \subset \mathbb{R}_n$ be a symmetric set with a nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is the Schur m power convex (or Schur m power concave, respectively) function if and only if φ is symmetric on Ω and for $m \neq 0$,*

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial\varphi}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \quad (or\ \leq 0, \quad respectively) \tag{2.4}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^\circ$.

And for $m = 0$,

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (or\ \leq 0,\ respectively) \tag{2.5}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^\circ$.

Obviously, Lemma 2.9 contains Lemma 2.5, Lemma 2.7, Lemma 2.8.

Lemma 2.10 ([13, 17]). *Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$ is an arithmetic mean, $G_n(\boldsymbol{x}) = \prod_{i=1}^n x_i^{\frac{1}{n}}$ is a geometric mean, $M_n^{[m]}(\boldsymbol{x}) = \left(\frac{\sum_{i=1}^n x_i^m}{n}\right)^{\frac{1}{m}}$ $(m \neq 0)$ is an m-order power mean. Then*

(i)

$$(\underbrace{A_n(\boldsymbol{x}), \dots, A_n(\boldsymbol{x})}_{n}) \prec (x_1, \dots, x_n). \tag{2.6}$$

(ii)

$$\left(\underbrace{\frac{(M_n^{[m]}(\boldsymbol{x}))^m - 1}{m}, \dots, \frac{(M_n^{[m]}(\boldsymbol{x}))^m - 1}{m}}_{n}\right) \prec \left(\frac{x_1^m - 1}{m}, \dots, \frac{x_n^m - 1}{m}\right). \tag{2.7}$$

(iii)

$$\left(\underbrace{\log G_n(\boldsymbol{x}), \dots, \log G_n(\boldsymbol{x})}_{n}\right) \prec (\log x_1, \dots, \log x_n). \tag{2.8}$$

(iv) *If $0 < r \leq s$, then*

$$\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r}, \dots, \frac{x_n^r}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1^s}{\sum_{i=1}^n x_i^s}, \dots, \frac{x_n^s}{\sum_{i=1}^n x_i^s}\right). \tag{2.9}$$

(v) *Let $\sum_{i=1}^n x_i = s$. For any $c > 0$, then*

$$\left(\frac{x_1 + c}{s + nc}, \dots, \frac{x_n + c}{ns + c}\right) \prec \left(\frac{x_1}{s}, \dots, \frac{x_n}{s}\right). \tag{2.10}$$

(vi) *Let $\sum_{i=1}^n x_i = s$. For any $0 < c < \min\{x_i\}$, then*

$$\left(\frac{x_1 - c}{s - nc}, \dots, \frac{x_n - c}{s - nc}\right) \succ \left(\frac{x_1}{s}, \dots, \frac{x_n}{s}\right). \tag{2.11}$$

(vii) *Let* $\sum_{i=1}^{n} x_i = s$. *For any* $c \geq s$, *then*

$$\left(\frac{c-x_1}{nc-s}, \ldots, \frac{c-x_n}{nc-s}\right) \prec \left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}\right). \tag{2.12}$$

3. Proof of main results

3.1. Proof of Theorem 1.5

Write

$$h(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} p_i \varphi(x_i) - \sum_{j=1}^{m} w_j \varphi(L_j(\boldsymbol{x}))}{P_n - W_m},$$

where $\boldsymbol{x} \in D \cap I$, $D = \{\boldsymbol{x} : x_1 \geq \cdots \geq x_n\}$. By Definition 1.4, we have

$$\begin{aligned}\frac{\partial L_\varphi^*}{\partial x_i} &= \frac{\partial \varphi^{-1}}{\partial h} \cdot \frac{\partial h}{\partial x_i} \\ &= \frac{1}{P_n - W_m} \frac{\partial \varphi^{-1}}{\partial h} \left(p_i \frac{d\varphi(x_i)}{dx_i} - \sum_{j=1}^{m} w_j \frac{\partial \varphi}{\partial L_j} \frac{\partial L_j}{\partial x_i} \right),\end{aligned}$$

$$\begin{aligned}\frac{\partial L_\varphi^*}{\partial x_{i+1}} &= \frac{\partial \varphi^{-1}}{\partial h} \cdot \frac{\partial h}{\partial x_{i+1}} \\ &= \frac{1}{P_n - W_m} \frac{\partial \varphi^{-1}}{\partial h} \left(p_{i+1} \frac{d\varphi(x_{i+1})}{dx_{i+1}} - \sum_{j=1}^{m} w_j \frac{\partial \varphi}{\partial L_j} \frac{\partial L_j}{\partial x_{i+1}} \right),\end{aligned}$$

$$\begin{aligned}\frac{\partial L_\varphi^*}{\partial x_i} - \frac{\partial L_\varphi^*}{\partial x_{i+1}} &= \frac{1}{P_n - W_m} \frac{\partial \varphi^{-1}}{\partial h} \\ &\quad \times \left[\left(p_i \frac{d\varphi(x_i)}{dx_i} - p_{i+1} \frac{d\varphi(x_{i+1})}{dx_{i+1}} \right) + \sum_{j=1}^{m} w_j \frac{\partial \varphi}{\partial L_j} \left(\frac{\partial L_j}{\partial x_{i+1}} - \frac{\partial L_j}{\partial x_i} \right) \right].\end{aligned}$$

(i) If φ is strictly increasing and convex on I, then φ^{-1} is strictly increasing. If L_j is Schur concave and $p_1 \geq \cdots \geq p_n > 0$, then

$$\begin{aligned}&\frac{\partial \varphi^{-1}}{\partial h} > 0, \quad p_i \frac{d\varphi(x_i)}{dx_i} - p_{i+1} \frac{d\varphi(x_{i+1})}{dx_{i+1}} \geq 0, \\ &\frac{\partial \varphi}{\partial L_j} > 0, \quad \frac{\partial L_j}{\partial x_{i+1}} - \frac{\partial L_j}{\partial x_i} \geq 0.\end{aligned}$$

Therefore $\frac{\partial L_\varphi^*}{\partial x_i} - \frac{\partial L_\varphi^*}{\partial x_{i+1}} \geq 0$, by Lemma 2.5, L_φ^* is Schur convex with $\mathbf{x}$ on $D \cap I$.

If φ is strictly increasing and concave on I, then φ^{-1} is strictly increasing. If L_j is Schur convex, and $0 < p_1 \leq \cdots \leq p_n$, then

$$\frac{\partial \varphi^{-1}}{\partial h} > 0, \quad p_i \frac{d\varphi(x_i)}{dx_i} - p_{i+1} \frac{d\varphi(x_{i+1})}{dx_{i+1}} \leq 0,$$

$$\frac{\partial\varphi}{\partial L_j} > 0, \quad \frac{\partial L_j}{\partial x_{i+1}} - \frac{\partial L_j}{\partial x_i} \leq 0.$$

Therefore $\frac{\partial L_\varphi^*}{\partial x_i} - \frac{\partial L_\varphi^*}{\partial x_{i+1}} \leq 0$, by Lemma 2.5, L_φ^* is Schur concave with $\mathbf{x}$ on $D \cap I$.

Similar to prove (ii).

The proof of Theorem 1.5 is complete.

3.2. Proof of Theorem 1.6

Write

$$k(x_1, \ldots, x_n) = \frac{\phi(x_1) + \cdots + \phi(x_n) - \phi(L(x_1, \ldots, x_n))}{n-1},$$

by Definition 1.3, we have

$$x_1^{1-m}\frac{\partial L_\phi^*}{\partial x_1} = x_1^{1-m}\frac{d\phi^{-1}}{dk}\frac{\partial k}{\partial x_1} = \frac{x_1^{1-m}}{n-1}\frac{d\phi^{-1}}{dk}\left(\frac{d\phi(x_1)}{dx_1} - \frac{d\phi}{dL}\frac{\partial L}{\partial x_1}\right),$$

$$x_2^{1-m}\frac{\partial L_\phi^*}{\partial x_2} = x_2^{1-m}\frac{d\phi^{-1}}{dk}\frac{\partial k}{\partial x_2} = \frac{x_2^{1-m}}{n-1}\frac{d\phi^{-1}}{dk}\left(\frac{d\phi(x_2)}{dx_2} - \frac{d\phi}{dL}\frac{\partial L}{\partial x_2}\right),$$

for $m \neq 0$,

$$\begin{aligned}\triangle_m &:= \frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial L_\phi^*}{\partial x_1} - x_2^{1-m}\frac{\partial L_\phi^*}{\partial x_2}\right) \\ &= \frac{x_1^m - x_2^m}{(n-1)m}\frac{d\phi^{-1}}{dk}\left[\left(x_1^{1-m}\frac{d\phi(x_1)}{dx_1} - x_2^{1-m}\frac{d\phi(x_2)}{dx_2}\right) + \frac{d\phi}{dL}\left(x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1}\right)\right].\end{aligned}$$

It is easy to see that L_ϕ^* is symmetry with $x_1, \ldots, x_n$, without loss of generality, we might as well assume $x_1 \geq x_2 > 0$, and let $z = \frac{x_1}{x_2} \geq 1$, we have

$$\triangle_m = \frac{x_2(z^m - 1)}{(n-1)m}\frac{d\phi^{-1}}{dk}\left[\left(z^{1-m}\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2}\right) + \frac{d\phi}{dL}\left(x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1}\right)\right],$$

and note that for $m \neq 0$, $\frac{z^m-1}{m} \geq 0$, and ϕ, ϕ^{-1} have the same monotonicity.

(i) For $m < 1$ and $m \neq 0$,

(1) If ϕ is strictly increasing and convex, and $L(\mathbf{x})$ is Schur m power concave, then

$$\frac{d\phi^{-1}}{dk} \geq 0, \quad z^{1-m}\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \geq \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \geq 0,$$

$$\frac{d\phi}{dL} \geq 0, \quad x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1} \geq 0,$$

so that $\triangle_m \geq 0$.

(2) If ϕ is strictly decreasing and concave, and $L(\mathbf{x})$ is Schur m power concave, then

$$\frac{d\phi^{-1}}{dk} \leq 0, \quad z^{1-m}\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \leq \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \leq 0,$$

$$\frac{d\phi}{dL} \le 0, \quad x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1} \ge 0,$$

so that $\triangle_m \ge 0$.

By Lemma 2.9, $L_\phi^*(\mathbf{x})$ is Schur m power convex with $\mathbf{x}$.

(ii) For $m = 1$, in Theorem 1.5, taking $p_1 = \cdots = p_n = 1$, $w_1 = 1, w_2 = \cdots = w_m = 0$, we can see that (ii) is established.

(iii) For $m > 1$,

(1) If ϕ is strictly increasing and concave, and $L(\mathbf{x})$ is Schur m power convex, then

$$\frac{d\phi^{-1}}{dk} \ge 0, \quad z^{1-m}\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \le \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \le 0,$$

$$\frac{d\phi}{dL} \ge 0, \quad x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1} \le 0,$$

so that $\triangle_m \le 0$.

(2) If ϕ is strictly decreasing and convex, and $L(\mathbf{x})$ is Schur m power convex, then

$$\frac{d\phi^{-1}}{dk} \le 0, \quad z^{1-m}\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \ge \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \ge 0,$$

$$\frac{d\phi}{dL} \le 0, \quad x_2^{1-m}\frac{\partial L}{\partial x_2} - x_1^{1-m}\frac{\partial L}{\partial x_1} \le 0,$$

so that $\triangle_m \le 0$.

By Lemma 2.9, $L_\phi^*(\mathbf{x})$ is Schur m power concave with $\mathbf{x}$.

(iv) For $m = 0$,

$$\begin{aligned}\triangle_0 &:= (\log x_1 - \log x_2)\left(x_1\frac{\partial L_\phi^*}{\partial x_1} - \frac{\partial L_\phi^*}{\partial x_2}\right)\\ &= \frac{x_2(\log x_1 - \log x_2)}{n-1}\left[\left(z\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2}\right) + \frac{d\phi}{dL}\left(x_2\frac{\partial L}{\partial x_2} - x_1\frac{\partial L}{\partial x_1}\right)\right].\end{aligned}$$

(1) If ϕ is strictly increasing and convex, and $L(\mathbf{x})$ is Schur geometrically concave, then

$$z\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \ge \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \ge 0,$$

$$\frac{d\phi}{dL} \ge 0, \quad x_2\frac{\partial L}{\partial x_2} - x_1\frac{\partial L}{\partial x_1} \ge 0,$$

so that $\triangle_0 \ge 0$. By Lemma 2.7, $L_\phi^*(\mathbf{x})$ is Schur geometrically convex with $\mathbf{x}$.

(2) If ϕ is strictly decreasing and concave, and $L(\mathbf{x})$ is Schur geometrically concave, we have

$$z\frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \le \frac{d\phi(x_1)}{dx_1} - \frac{d\phi(x_2)}{dx_2} \le 0,$$

$$\frac{d\phi}{dL} \le 0, \quad x_2 \frac{\partial L}{\partial x_2} - x_1 \frac{\partial L}{\partial x_1} \ge 0,$$

so that $\triangle_0 \le 0$. By Lemma 2.7, $L^*_\phi(\mathbf{x})$ is Schur geometrically concave with $\mathbf{x}$.

The proof of Theorem 1.6 is complete.

4. Applications

As applications of Theorem 1.5 and Theorem 1.6, we establish the following new inequalities for the mean.

Theorem 4.1. *Let $\phi(x)$ be a strictly monotone continuous function on $I \subset \mathbb{R}$, $L(\boldsymbol{x})$ be a fixed mean with $\boldsymbol{x} = (x_1, \dots, x_n)$.*

(i) *For any real number $m < 1$ and $m \ne 0$, if ϕ is strictly increasing and convex, or ϕ is strictly decreasing and concave, $L(\boldsymbol{x})$ is Schur m power concave, then*

$$L^*_\phi(\boldsymbol{x}) \ge M_n^{[m]}(\boldsymbol{x}). \tag{4.1}$$

(ii) *For any real number $m > 1$, if ϕ is strictly increasing and concave, or ϕ is strictly decreasing and convex, $L(\boldsymbol{x})$ is Schur m power convex, then*

$$L^*_\phi(\boldsymbol{x}) \le M_n^{[m]}(\boldsymbol{x}). \tag{4.2}$$

Proof. (i) By Theorem 1.6(i), Lemma 2.10(ii), Definition 2.1 and the property of the mean, we have

$$\begin{aligned} L^*_\phi(\boldsymbol{x}) &\ge \phi^{-1}\left(\frac{\phi(M_n^{[m]}(\boldsymbol{x})) + \cdots + \phi(M_n^{[m]}(\boldsymbol{x})) - \phi(L(M_n^{[m]}(\boldsymbol{x})), \dots, M_n^{[m]}(\boldsymbol{x}))}{n-1}\right) \\ &= \phi^{-1}\phi(M_n^{[m]}(\boldsymbol{x})) = M_n^{[m]}(\boldsymbol{x}). \end{aligned}$$

Using similar methods, (ii) can be proved.

The proof of Theorem 4.1 is complete. □

By Theorem 1.6(iv), Lemma 2.10(iii), Definition 2.2 and the property of the mean, we have the following conclusion.

Theorem 4.2. *Let $\boldsymbol{x} \in \mathbb{R}^n_{++}$, $L(\boldsymbol{x})$ be a fixed mean.*

(i) *If ϕ is strictly increasing and convex, $L(\boldsymbol{x})$ is Schur geometrically concave, then*

$$L^*_\phi(\boldsymbol{x}) \ge G_n(\boldsymbol{x}). \tag{4.3}$$

(ii) *If ϕ is strictly decreasing and concave, $L(\boldsymbol{x})$ is Schur geometrically concave, then*

$$L^*_\phi(\boldsymbol{x}) \le G_n(\boldsymbol{x}). \tag{4.4}$$

Theorem 4.3. *Let* $\boldsymbol{x} \in \mathbb{R}^n_{++}$, $c > 0$ *and* $\sum_{i=1}^n x_i = s$. *If* $L(\boldsymbol{x})$ *is an arbitrary Schur convex mean, then*

$$\frac{L(x_1 + c, \ldots, x_n + c)}{L(x_1, \ldots, x_n)} \leq \left(\frac{s}{s + nc}\right)^{n-1} \prod_{i=1}^{n} \left(1 + \frac{c}{x_i}\right). \tag{4.5}$$

Proof. Let $\phi(x) = \log x$. By Theorem 1.6(ii), we know that $L^*_{\log}(\mathbf{x})$ is Schur concave, and according to majorizing inequality in Lemma 2.10(v):

$$\left(\frac{x_1 + c}{s + nc}, \ldots, \frac{x_n + c}{s + nc}\right) \prec \left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}\right),$$

by Definition 2.1 and notice the mean's property: $L(\lambda \mathbf{x}) = \lambda L(\mathbf{x})$, it is easy to prove the inequality (4.5) hold.

The proof of Theorem 4.3 is complete. □

For any Schur convex mean, we can obtain the following mean comparison theorem by combining Theorem 1.6 with majorizing inequality.

Theorem 4.4. *Let* $\boldsymbol{x} \in \mathbb{R}^n_{++}$, $L(\boldsymbol{x})$ *be an arbitrary Schur convex mean, and* $\sum_{i=1}^n x_i = s$.

(i) *If* $0 < c < \min\{x_i\}$, *then*

$$\frac{L(x_1 - c, \ldots, x_n - c)}{L(x_1, \ldots, x_n)} \geq \left(\frac{s}{s - nc}\right)^{n-1} \prod_{i=1}^{n} \left(1 - \frac{c}{x_i}\right). \tag{4.6}$$

(ii) *If* $c \geq s$, *then*

$$\frac{L(c - x_1, \ldots, c - x_n)}{L(x_1, \ldots, x_n)} \leq \left(\frac{s}{nc - s}\right)^{n-1} \prod_{i=1}^{n} \left(\frac{c}{x_i} - 1\right). \tag{4.7}$$

Proof. Let $\phi(x) = \log x$. By Theorem 1.6(ii), from majorizing inequality in Lemma 2.10(vi):

$$\left(\frac{x_1 - c}{s - nc}, \ldots, \frac{x_n - c}{s - nc}\right) \succ \left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}\right),$$

and majorizing inequality in Lemma 2.10(vii):

$$\left(\frac{c - x_1}{nc - s}, \ldots, \frac{c - x_n}{nc - s}\right) \prec \left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}\right),$$

it is easy to prove the inequality (4.6) and inequality (4.7) hold.

The proof of Theorem 4.4 is complete. □

Let

$$L(x_1, \ldots, x_n) = M_n^{[m]}(\mathbf{x}) = \left(\frac{\sum_{i=1}^n x_i^m}{n}\right)^{\frac{1}{m}}, \ (m \geq 1).$$

It is easy to see $M_n^{[m]}(\mathbf{x})$ is symmetry with $x_1, \ldots, x_n$, without loss of generality, we might as well assume $x_1 \geq x_2 > 0$. Write

$$k = \frac{1}{n}\sum_{i=1}^{n} x_i^m,$$

then

$$\frac{\partial M_n^{[m]}(\mathbf{x})}{\partial x_1} = \frac{k^{\frac{1}{m}-1}}{n} x_1^{m-1}, \quad \frac{\partial M_n^{[m]}(\mathbf{x})}{\partial x_2} = \frac{k^{\frac{1}{m}-1}}{n} x_2^{m-1},$$

$$\begin{aligned}\triangle_m &:= (x_1 - x_2)\left(\frac{\partial M_n^{[m]}(\mathbf{x})}{\partial x_1} - \frac{\partial M_n^{[m]}(\mathbf{x})}{\partial x_2}\right) \\ &= (x_1 - x_2)\frac{mk^{\frac{1}{m}-1}}{n}(x_1^{m-1} - x_2^{m-1}) \geq 0.\end{aligned}$$

So, when $m \geq 1$, $M_n^{[m]}(\mathbf{x})$ is Schur convex.

By Theorem 4.3 and Theorem 4.4(i), we get the following conclusion.

Corollary 4.5. *Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}_{++}^n$. If $m \geq 1$, $0 < c < \min\{x_i\}$, then*

$$\begin{aligned}\frac{A_n^{n-1}(\boldsymbol{x}+c)M_n^{[m]}(\boldsymbol{x}+c)}{G_n^n(\boldsymbol{x}+c)} &\leq \frac{A_n^{n-1}(\boldsymbol{x})M_n^{[m]}(\boldsymbol{x})}{G_n^n(\boldsymbol{x})} \\ &\leq \frac{A_n^{n-1}(\boldsymbol{x}-c)M_n^{[m]}(\boldsymbol{x}-c)}{G_n^n(\boldsymbol{x}-c)}.\end{aligned} \tag{4.8}$$

Let $m = 1$, by Corollary 4.5 we get

$$\frac{A_n(\mathbf{x}+c)}{G_n(\mathbf{x}+c)} \leq \frac{A_n(\mathbf{x})}{G_n(\mathbf{x})} \leq \frac{A_n(\mathbf{x}-c)}{G_n(\mathbf{x}-c)}. \tag{4.9}$$

Let $L(\mathbf{x}) = M_n^{[m]}(\mathbf{x})$, $(m \geq 1)$. When $\sum_{i=1}^n x_i \leq 1$, by Theorem 4.4(ii), we have

$$\left(\frac{M_n^{[m]}(1-\mathbf{x})(A_n(1-\mathbf{x}))^{n-1}}{M_n^{[m]}(\mathbf{x})(A_n(\mathbf{x}))^{n-1}}\right)^{\frac{1}{n}} \leq \frac{G_n(1-\mathbf{x})}{G_n(\mathbf{x})}. \tag{4.10}$$

Let $m = 1$, we get Ky Fan's inequality

$$\frac{G_n(\mathbf{x})}{G_n(1-\mathbf{x})} \leq \frac{A_n(\mathbf{x})}{A_n(1-\mathbf{x})}. \tag{4.11}$$

Theorem 4.6. *Let $D = \{\boldsymbol{x} : x_1 \geq \cdots \geq x_n\}$, $L_j(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}_{++}^n$ $(j = 1, \ldots, m)$ be m fixed means, $(\boldsymbol{p}, \boldsymbol{w})$ be an admissible pair. For $\forall \boldsymbol{x} \in D \cap \mathbb{R}_{++}^n$,*

(i) *If $p_1 \geq \cdots \geq p_n > 0$, $w_j > 0$, $j = 1, \ldots, m$ and L_j, $j = 1, \ldots, m$ are Schur concave, then for $r \geq 1$, we have*

$$\sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} p_i x_i^r - (P_n - W_m)A_n^r(\boldsymbol{x}). \tag{4.12}$$

(ii) *If* $0 < p_1 \leq \cdots \leq p_n$, $w_j > 0$, $j = 1, \ldots, m$ *and* L_j, $j = 1, \ldots, m$ *are Schur convex, then for* $0 < r \leq 1$, *we have*

$$\sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n) \geq \sum_{i=1}^{n} p_i x_i^r - (P_n - W_m) A_n^r(\boldsymbol{x}), \tag{4.13}$$

where $P_n = \sum_{i=1}^{n} p_i$, $W_m = \sum_{j=1}^{m} w_j$.

Proof. Let $\varphi(x) = x^r$ $(r \geq 1)$, $p_1 \geq \cdots \geq p_n > 0$, $w_j > 0$, $j = 1, \ldots, m$. Then

$$L_\varphi^*(x_1, \ldots, x_n; \boldsymbol{p}, \boldsymbol{w}) = \left(\frac{\sum_{i=1}^{n} p_i x_i^r - \sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n)}{P_n - W_m} \right)^{\frac{1}{r}}.$$

We know that φ is strictly increasing and convex on $\mathbb{R}_+$, L_j $(j = 1, \ldots, m)$ are Schur concave on $\mathbb{R}_{++}$, by Theorem 1.5(i), it follows that L_φ^* is Schur convex with $x_1, \ldots, x_n$. By Definition 2.1 and

$$(x_1, \ldots, x_n) \succ (\underbrace{A_n(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x})}_{n}),$$

we have

$$L_\varphi^*(x_1, \ldots, x_n; \boldsymbol{p}, \boldsymbol{w}) \geq L_\varphi^*(A_n(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x}); \boldsymbol{p}, \boldsymbol{w})$$

thus

$$\begin{aligned} & \left(\frac{\sum_{i=1}^{n} p_i x_i^r - \sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n)}{P_n - W_m} \right)^{\frac{1}{r}} \\ \geq & \left(\frac{\sum_{i=1}^{n} p_i A_n^r(\boldsymbol{x}) - \sum_{j=1}^{m} w_j L_j^r(A_n(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x}))}{P_n - W_m} \right)^{\frac{1}{r}} \\ \Rightarrow & \left(\frac{\sum_{i=1}^{n} p_i x_i^r - \sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n)}{P_n - W_m} \right)^{\frac{1}{r}} \\ \geq & \left(\frac{\sum_{i=1}^{n} p_i A_n^r(\boldsymbol{x}) - \sum_{j=1}^{m} w_j A_n^r(\boldsymbol{x})}{P_n - W_m} \right)^{\frac{1}{r}} \\ = & A_n(\boldsymbol{x}), \end{aligned}$$

that is,

$$\sum_{j=1}^{m} w_j L_j^r(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} p_i x_i^r - (P_n - W_m) A_n^r(\mathbf{x}).$$

By similar method, we can prove the inequality (4.13).

The proof of Theorem 4.6 is complete. □

As an example in Theorem 4.6, let $p_i = 1$, $w_i = \frac{1}{n}$, $i = 1, \ldots, n$, $L_j(\mathbf{x}) = A_n(\mathbf{x})$. For $r > 1$, we get the power mean inequality:

$$(A_n(\mathbf{x}))^r \leq \frac{1}{n}\sum_{i=1}^{n} x_i^r.$$

In 1938, Gini introduced a mean of two variables with double parameters.

Definition 4.7 ([10])**.** Let $(r, s) \in \mathbb{R}^2$, $(a, b) \in \mathbb{R}^2_{++}$. The Gini mean of two variables is defined as

$$G(r, s; a, b) = \left(\frac{a^s + b^s}{a^r + b^r}\right)^{\frac{1}{s-r}}, \ (s \neq r). \tag{4.14}$$

Gini mean of two variables contains many important mean, for example, $G(0, p; a, b)$, $p \neq 0$ is a p power mean of two variables, $G(p-1, p; a, b)$ is Lehmer mean of two variables.

Gini mean of two variables can naturally be extended to the form of n variables.

Definition 4.8 ([1])**.** Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $(r, s) \in \mathbb{R}^2$, $s \neq r$. The Gini mean of n is variables defined as

$$G(r, s; \mathbf{x}) = \left(\frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}} \quad (s \neq r). \tag{4.15}$$

We get the following conclusion for the comparison of arbitrary Schur convex mean with Gini mean.

Theorem 4.9. *Let $x_i \in \mathbb{R}_{++}$, $i = 1, \ldots, n$, $L(x_1, \ldots, x_n)$ be an arbitrary Schur convex mean. If $0 < r < s$, then*

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n-1}} \left(\frac{L(x_1^r, \ldots, x_n^r)}{L(x_1^s, \ldots, x_n^s)}\right)^{\frac{1}{(n-1)(s-r)}} \leq G(r, s; \boldsymbol{x}) \leq \left(\frac{L(x_1^s, \ldots, x_n^s)}{L(x_1^r, \ldots, x_n^r)}\right)^{\frac{1}{s-r}}. \tag{4.16}$$

Proof. Let $\phi(x) = \log x$. Then $\phi(x)$ is strictly increasing and concave, and $L(x_1, \ldots, x_n)$ is Schur convex, by Theorem 1.6(ii), it follows that

$$L^*_{\log}(x_1, \ldots, x_n) = \exp\left(\frac{\log x_1 + \cdots + \log x_n - \log L(x_1, \ldots, x_n)}{n-1}\right)$$

is Schur concave.

By the majorizing inequality in Lemma 2.10(iv):

$$\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r}, \ldots, \frac{x_n^r}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1^s}{\sum_{i=1}^n x_i^s}, \ldots, \frac{x_n^s}{\sum_{i=1}^n x_i^s}\right),$$

Definition 2.1, and notice that property of the mean: $L(\lambda\mathbf{x}) = \lambda L(\mathbf{x})$, we have

$$\begin{aligned}
&\exp\left(\frac{\log\frac{x_1^r}{\sum_{i=1}^n x_i^r}+\cdots+\log\frac{x_n^r}{\sum_{i=1}^n x_i^r}-\log L(\frac{x_1^r}{\sum_{i=1}^n x_i^r},\dots,\frac{x_n^r}{\sum_{i=1}^n x_i^r})}{n-1}\right)\\
&\geq \exp\left(\frac{\log\frac{x_1^s}{\sum_{i=1}^n x_i^s}+\cdots+\log\frac{x_n^s}{\sum_{i=1}^n x_i^s}-\log L(\frac{x_1^s}{\sum_{i=1}^n x_i^s},\dots,\frac{x_n^s}{\sum_{i=1}^n x_i^s})}{n-1}\right)
\end{aligned}$$

$$\Rightarrow \frac{\prod_{i=1}^n x_i^r}{\left(\sum_{i=1}^n x_i^r\right)^n L\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r},\dots,\frac{x_n^r}{\sum_{i=1}^n x_i^r}\right)} \geq \frac{\prod_{i=1}^n x_i^s}{\left(\sum_{i=1}^n x_i^s\right)^n L\left(\frac{x_1^s}{\sum_{i=1}^n x_i^s},\dots,\frac{x_n^s}{\sum_{i=1}^n x_i^s}\right)}$$

$$\Rightarrow \frac{\left(\sum_{i=1}^n x_i^s\right)^n}{\left(\sum_{i=1}^n x_i^r\right)^n} \geq \frac{\frac{\prod_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r} L(x_1^r,\dots,x_n^r)}{\frac{\prod_{i=1}^n x_i^r}{\sum_{i=1}^n x_i^s} L(x_1^s,\dots,x_n^s)}$$

$$\Rightarrow \left(\frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\right)^{n-1} \geq \prod_{i=1}^n x_i^{s-r}\frac{L(x_1^r,\dots,x_n^r)}{L(x_1^s,\dots,x_n^s)}$$

$$\Rightarrow \left[\left(\frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}}\right]^{n-1} \geq \prod_{i=1}^n x_i\left(\frac{L(x_1^r,\dots,x_n^r)}{L(x_1^s,\dots,x_n^s)}\right)^{\frac{1}{s-r}}$$

$$\Rightarrow G(r,s;\mathbf{x}) \geq \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n-1}}\left(\frac{L(x_1^r,\dots,x_n^r)}{L(x_1^s,\dots,x_n^s)}\right)^{\frac{1}{(n-1)(s-r)}}.$$

Because $L(x_1,\dots,x_n)$ is a Schur convex mean, by the majorizing inequality

$$\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r},\dots,\frac{x_n^r}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1^s}{\sum_{i=1}^n x_i^s},\dots,\frac{x_n^s}{\sum_{i=1}^n x_i^s}\right),$$

Definition 2.1, and notice that property of the mean: $L(\lambda\mathbf{x}) = \lambda L(\mathbf{x})$, we have

$$\begin{aligned}
&L\left(\frac{x_1^s}{\sum_{i=1}^n x_i^s},\dots,\frac{x_n^s}{\sum_{i=1}^n x_i^s}\right) \geq L\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r},\dots,\frac{x_n^r}{\sum_{i=1}^n x_i^r}\right)\\
&\Rightarrow \frac{L(x_1^s,\dots,x_n^s)}{\sum_{i=1}^n x_i^s} \geq \frac{L(x_1^r,\dots,x_n^r)}{\sum_{i=1}^n x_i^r}\\
&\Rightarrow \frac{L(x_1^s,\dots,x_n^s)}{L(x_1^r,\dots,x_n^r)} \geq \frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\\
&\Rightarrow G(r,s;\mathbf{x}) = \left(\frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}} \leq \left(\frac{L(x_1^s,\dots,x_n^s)}{L(x_1^r,\dots,x_n^r)}\right)^{\frac{1}{s-r}}.
\end{aligned}$$

The proof of Theorem 4.9 is complete. □

Let $L(x_1,\dots,x_n) = M_n^{[m]}(\mathbf{x})$ $(m \geq 1)$. By Theorem 4.9, we get the following conclusion.

Corollary 4.10. *Let* $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$. *If* $0 < r < s$, $m \geq 1$, *then*

$$G(r,s;\boldsymbol{x}) \geq [G_n(\boldsymbol{x})]^{1+\frac{1}{n-1}} \left[\frac{M_n^{[m]}(\boldsymbol{x}^r)}{M_n^{[m]}(\boldsymbol{x}^s)}\right]^{\frac{1}{(n-1)(s-r)}}. \tag{4.17}$$

Let $m = 1$, by Corollary 4.10, we have:

Corollary 4.11. *Let* $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$. *If* $0 < r < s$, *then*

$$G(r,s;\boldsymbol{x}) \geq G_n(\boldsymbol{x}). \tag{4.18}$$

Remark 4.12. The following inequalities were introduced in ([12], p. 215):
Let $\boldsymbol{a} = (a_1, \dots, a_n)$. If $a_k \geq 1$ $(1 \leq k \leq n)$, $p > 0$, then

$$H_n(\boldsymbol{a}) \sum_{k=1}^n a_k^p \leq \sum_k^n a_k^{p+1}. \tag{4.19}$$

By Corollary 4.11, we can improve the inequality (4.19) as follows.
Let $\boldsymbol{a} = (a_1, \dots, a_n)$. If $a_k > 0$ $(1 \leq k \leq n)$, $p > 0$, then

$$H_n(\boldsymbol{a}) \sum_{k=1}^n a_k^p \leq G_n(\boldsymbol{a}) \sum_{k=1}^n a_k^p \leq \sum_k^n a_k^{p+1}. \tag{4.20}$$

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Chun-Ru Fu
Applied college of science and technology
Beijing Union University
Beijing 102200, P. R. China
Email address: fuchunru2008@163.com

Huan-Nan Shi
Teacher's College
Beijing Union University
Beijing 100011, P. R. China
Email address: shihuannan2014@qq.com;sfthuannan@buu.edu.cn

Dong-Sheng Wang
Basic courses department
Beijing Polytechnic
Beijing 100176, P. R. China
Email address: wds000651225@sina.com