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SYMMETRY OF THE TWISTED GROMOV-WITTEN CLASSES OF PROJECTIVE LINE

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Abstract. We study the rationality and symmetry of the Gromov-Witten invariants of the projective line twisted by certain line bundles.

1. Introduction

1.1. Overview

Let X be a smooth algebraic variety and let S be a line bundle on X. Via some Gromov-Witten theories over X, we define certain classes in tautological ring $\mathcal{R}_{X,S}$ of X. See Section 4.3 for the definition of $\mathcal{R}_{X,S}$. Motivated from the rationality and symmetry of the Gromov-Witten invariants of total spaces of $\mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$, we study the rationality and symmetry of related Gromov-Witten classes in $\mathcal{R}_{X,S}$.

While the localization method works for both the Gromov-Witten and the stable quotient theories, in general calculations can be performed more efficiently on the stable quotient side. We study the stable quotient theory of $\mathcal{O}_{\mathbb{P}^1}(-2)$ and recover the Gromov-Witten theory via the wall-crossing formula in Section 2. Since the wall-crossing formula for $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ is trivial, we directly study Gromov-Witten theory of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ in Section 3.

The quasimap invariants of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$ were studied in [13, Theorem 4]. The Gromov-Witten invariants of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ were studied in [5,8] via localization and Hodge integrals over the moduli space of curves. The result for $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$ was studied in [11, Section 6.10] using symmetries on the symplectic invariants of STU model. For local toric Hirzebruch surfaces, another approach has been pursued by Buelles and Moreira via PT invariants [2].

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1.2. Gromov-Witten theory of $\mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-2)$

Let X be a smooth algebraic variety, and let S be a line bundle on X. Let π_i be the projection maps

$$\pi_1 : X \times \mathbb{P}^1 \to X, \quad \pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Denote by Y the total space of the line bundle

$$E := \pi_1^*(S^{-1}) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-2)$$

on $X \times \mathbb{P}^1$. For $\beta \in H_2(X, \mathbb{Z})$, $d \in \mathbb{Z}$, let π be the map

$$\pi : \overline{M}_{g,0}(X \times \mathbb{P}^1, (\beta, d)) \to \overline{M}_{g,0}(X, \beta)$$

induced by the projection map π_1.

For $g \geq 0$, $\beta \in H_2(X, \mathbb{Z})$, the Gromov-Witten series of Y is defined by

$$(1) \quad \mathcal{F}^Y_{g,\beta}(q) := \sum_{d \geq 0} q^d \pi_* \Big([\overline{M}_{g,0}(X \times \mathbb{P}^1, (\beta, d))]^{\mathrm{vir}} \cap e(-R^\bullet p_* f^* E) \Big) \in \mathcal{R}_{X,S}[[q]],$$

where $p : \mathcal{C} \to \overline{M}_{g,0}(X \times \mathbb{P}^1, (\beta, d))$ is the universal curve and $f : \mathcal{C} \to X \times \mathbb{P}^1$ is the universal map. The first result of the paper is the symmetric properties of the Gromov-Witten classes of Y.

Theorem 1. *For the Gromov-Witten classes of Y, we have*

(i) $\mathcal{F}^Y_{g,\beta}(q) \in \mathcal{R}_{X,S}[q, (1-q)^{-1}]$,

(ii) $\mathcal{F}^Y_{g,\beta}(1/q) = (-q)^{\int_\beta c_1(S)} \cdot \mathcal{F}^Y_{g,\beta}(q)$.

1.3. Gromov-Witten theory of $\mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

Let X be a smooth algebraic variety, and let S be a line bundle on X. Let π_i be the projections

$$\pi_1 : X \times \mathbb{P}^1 \to X, \quad \pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Denote by Z the total space of the line bundle

$$F := \Big(\pi_1^*(S^{-1}) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \Big)^{\oplus 2}$$

on $X \times \mathbb{P}^1$. Let π be the map

$$\pi : \overline{M}_{g,0}(X \times \mathbb{P}^1, (\beta, d)) \to \overline{M}_{g,0}(X, \beta)$$

induced by the projection map π_1. Note that π depends on the genus and number of markings, but we will use the same notation for π when the domain of π is clear from the context. Here we need to consider the moduli space with the markings in Section 2.1.

For $g \geq 0$, $\beta \in H_2(X, \mathbb{Z})$, the Gromov-Witten classes of Z is defined by

$$\mathcal{F}^Z_{g,\beta}(q) := \sum_{d \geq 0} q^d \, \pi_* \Big([\overline{M}_{g,0}(X \times \mathbb{P}^1, (\beta, d))]^{\mathrm{vir}} \cap e(-R^\bullet p_* f^* F) \Big) \in \mathcal{R}_{X,S}[[q]],$$

where $p : \mathcal{C} \to \overline{M}_{g,n}(X \times \mathbb{P}^1, (\beta, d))$ is the universal curve and $f : \mathcal{C} \to X \times \mathbb{P}^1$ is the universal map. The second result of the paper is the following symmetric properties of the Gromov-Witten classes of Z.

Theorem 2. *For the Gromov-Witten classes of Z, we have*

(i) $\mathcal{F}^Z_{g,\beta}(q) \in \mathcal{R}_{X,S}[q, (1-q)^{-1}]$,

(ii) $\mathcal{F}^Z_{g,\beta}(1/q) = (-q)^{\int_\beta c_1(S)} \cdot \mathcal{F}^Z_{g,\beta}(q)$.

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2. Gromov-Witten theory of $\mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-2)$

2.1. Stable quotient and wall crossing formula

We review here the stable quotient invariants and wall crossing formula [3, 16].

Let $(C, p_1, \ldots, p_n)$ be an n-pointed quasi-stable curve:

- C is a reduced, connected, complete scheme of dimension one with at worst nodal singularities,
- the markings p_i are distinct and lie in the non-singular locus of C.

Let q be a quotient of the rank 2 trivial bundle on C,

$$\mathbb{C}^2 \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0.$$

We say q is a *quasi-stable quotient* if the quotient sheaf Q is locally free at the nodes and markings of C. Quasi-stability of q implies the associated kernel,

$$0 \to T \to \mathbb{C}^2 \otimes \mathcal{C} \xrightarrow{q} Q \to 0,$$

is a locally free sheaf on C. We assume that the rank of T is one. Let $(C, p_1, \ldots, p_n)$ be an n-pointed quasi-stable curve equipped with a quasi-stable quotient q. The data $(C, p_1, \ldots, p_n, q)$ determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C(p_1 + \cdots + p_n) \otimes (T^*)^{\otimes \epsilon}$$

is ample on C for every positive $\epsilon \in \mathbb{Q}$.

Denote by $\overline{Q}^{\infty,0+}_{g,n}(X \times \mathbb{P}^1, (\beta, d))$ the moduli space parameterizing the data

$$(C, p_1, \ldots, p_n, 0 \to S \to \mathbb{C}^2 \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,\ f : C \to X),$$

where q is a quasi-stable quotient with $\deg(T) = -d$ and f is a quasi-stable map with $\deg(f) = \beta \in H_2(X, \mathbb{Z})$ such that either q is a stable quotient or f is a stable map.

Combining the usual argument in the moduli space of stable maps and the argument in [16], we get the following results.

Theorem 3. *$\overline{Q}^{\infty,0+}_{g,n}(X\times\mathbb{P}^1,(\beta,d))$ is a separated and proper Delinge-Mumford stack of finite type over $\mathbb{C}$. Moreover it admits a perfect obstruction theory.*

Over the moduli space $\overline{Q}^{\infty,0+}_{g,n}(X\times\mathbb{P}^1,(\beta,d))$, there is a universal n-pointed curve

$$p:\mathcal{C}\to\overline{Q}^{\infty,0+}_{g,n}(X\times\mathbb{P}^1,(\beta,d))$$

with a universal quotient

$$0\to\mathcal{T}\to\mathbb{C}^2\otimes\mathcal{O}_{\mathcal{C}}\to\mathcal{Q}\to 0.$$

The subsheaf $\mathcal{T}$ is locally free on $\mathcal{C}$ because of the stability condition. We have the natural map

$$\pi:\overline{Q}^{\infty,0+}_{g,0}(X\times\mathbb{P}^1,(\beta,d))\to\overline{M}_{g,0}(X,\beta).$$

We define the stable quotient series by

$$\mathcal{F}^{\mathsf{SQ}}_{g,\beta}(q):=\sum_{d\geq 0}q^d\pi_*\Big([\overline{Q}^{\infty,0+}_{g,0}(X\times\mathbb{P}^1,(\beta,d))]^{\mathrm{vir}}\cap e(-R^\bullet p_*(f^*(S^{-1})\otimes\mathcal{T}^{\otimes 2}))\Big),$$

where $p:\mathcal{C}\to\overline{M}_{g,n}(X\times\mathbb{P}^1,(\beta,d))$ is the universal curve and $f:\mathcal{C}\to X\times\mathbb{P}^1$ is the universal map.

Recall the Gromov-Witten series $\mathcal{F}^Y_{g,\beta}$ of Y defined by (1). More generally, we define the Gromov-Witten series of Y with insertion,

$$\begin{aligned}&\mathcal{F}^Y_{g,n,\beta}[\gamma_1,\gamma_2,\dots,\gamma_n](q)\\ :=&\sum_{d\geq 0}q^d\pi_*\Big([\overline{M}_{g,n}(X\times\mathbb{P}^1,(\beta,d))]^{\mathrm{vir}}\cap e(-R^\bullet\pi_*f^*E)\cup\prod_{k=1}^{n}\mathrm{ev}^*(\gamma_k)\Big),\end{aligned}$$

where $\gamma_k\in H^*(X\times\mathbb{P}^1)$. Here $\pi:\overline{M}_{g,n}(X\times\mathbb{P}^1,(\beta,d))\to\overline{M}_{g,n}(X,\beta)$ and note that $\mathcal{F}^Y_{g,0,\beta}=\mathcal{F}^Y_{g,\beta}$. Let $H\in H^2(\mathbb{P}^1)$ be the hyperplane class of $\mathbb{P}^1$ and $B=c_1(S)\in H^2(X)$. The relationship between the Gromov-Witten and stable quotient series can be proved using the argument in the proof of Theorem 1.3.2 in [3]:

$$\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{F}^Y_{g,\beta}[I_1(q)(H+\frac{1}{2}B),\dots,I_1(q)(H+\frac{1}{2}B)](q)=\mathcal{F}^{\mathsf{SQ}}_{g,\beta}(q),\tag{2}$$

where $I_1(q)$ is defined by

$$I_1(q)=-2\log\left(1+\sqrt{1-4q}\right)+2\log 2.$$

2.2. Localizations

We fix a torus action $\mathsf{T}=(\mathbb{C}^*)^2$ on $\mathbb{P}^1$ with weights λ_0, λ_1 on the vector space $\mathbb{C}^2$. The T-weight on the fiber over p_i of the canonical bundle $\mathcal{O}_{\mathbb{P}^1}(-2)\to\mathbb{P}^1$ is $-2\lambda_i$. We use the specialization

$$\lambda_0=1,\ \ \lambda_1=-1.$$

Proposition 4. *For the quasimap invariants of $\mathcal{O}_{\mathbb{P}^1}(-2)$, we have*

$$\mathcal{F}^{\mathsf{SQ}}_{g,\beta}(q) \in \mathcal{R}_{X,S}[(1-4q)^{-1}].$$

Proof. Define the I-function

$$\mathbb{I} := \sum_{d=0}^{\infty} q^d \frac{\prod_{k=0}^{2d-1}(-2H-B-kz)}{\prod_{i=0}^{1}\prod_{k=1}^{d}(H-\lambda_i+kz)}.$$

Define

$$\mathbb{S}(1) = \mathbb{I},$$
$$\mathbb{S}(H) = \frac{\mathsf{M}\,\mathbb{S}(1)}{L_0} - \Big(\frac{1}{2} - \frac{1}{L_0}\Big)\mathbb{S}(1), \tag{3}$$

where $\mathsf{M} := H + z\frac{qd}{dq}$ and $L_0(q) = (1-4q)^{-1/2}$.

The series

$$\mathbb{S}_i(1) := \mathbb{S}|_{H=\lambda_i},\ \mathbb{S}_i(H) := \mathbb{S}(H)|_{H=\lambda_i}$$

have the following asymptotic expansions:

$$\mathbb{S}_i(1) = e^{\frac{\sum_{k=0}^{\infty}\mu_{k,i}B^k}{z}}\Big(\sum_{j\geq 0,k\geq 0} R_{0jk,i}\, B^k z^j\Big),$$
$$\mathbb{S}_i(H) = e^{\frac{\sum_{k=0}^{\infty}\mu_{k,i}B^k}{z}}\Big(\sum_{j\geq 0,k\geq 0} R_{1jk,i}\, B^k z^j\Big), \tag{4}$$

with series $\mu_{k,i}, R_{ljk,i} \in \mathbb{Q}[[q]]$. The first equality can be obtained by directly analyzing the I-function ([17, Lemma 1]). See [12, Lemma 41] for a geometric proof. The second equality can be obtained from (3).

Define the series $L_{k,i}$ for $k \in \mathbb{Z}_{\geq 0}$ by

$$L_{0,i} = \mathsf{D}\mu_{0,i} + \lambda_i,$$
$$L_{k,i} = \mathsf{D}\mu_{k,i} \quad \text{for } k \geq 1, \tag{5}$$

where $\mathsf{D} := \frac{qd}{dq}$. We have the following result for the series $L_{k,i}$, $R_{ljk,i}$.

Lemma 5. *For k, l, $j \geq 0$ and $i = 0, 1$, we have*

$$L_{k,i}, R_{ljk,i} \in \mathbb{Q}[L_0].$$

Proof. The function $\mathbb{I}$ satisfies the following Picard-Fuchs equation,

$$\Big((\mathsf{M}-\lambda_0)(\mathsf{M}-\lambda_1) - q(-2\mathsf{M}-B)(-2\mathsf{M}-B-z)\Big)\mathbb{I} = 0. \tag{6}$$

The lemma follows by applying the asymptotic forms (4) to above equation. The coefficient of z^0 in (6) is calculated as

$$(1-4q)\mathcal{L}_{B,i}^2 - 8qB\mathcal{L}_{B,i} - (1+4qB^2) = 0, \tag{7}$$

where we used the notation

$$\mathcal{L}_{B,i} := L_{0,i} + L_{1,i}B + L_{2,i}B^2 + \cdots .$$

The coefficient of B^0 in (7) gives

$$(1-4q)L_{0,i}^2 - 1 = 0. \tag{8}$$

Therefore we obtain

$$L_{0,i} = (-1)^i \Big(\frac{1}{1-4q}\Big)^{1/2} := (-1)^i L_0.$$

Note that the choice of two roots of the equation (8) corresponds to the choice of two fixed points in $\mathbb{P}^1$. The coefficient of B in (7) gives

$$2L_{0,i}L_{1,i}(1-4q) - 8qL_{0,i} = 0. \tag{9}$$

The coefficient of B^2 in (7) gives

$$(L_{1,i}^2 + 2L_{2,i}L_{0,i})(1-4q) - 4q - 8qL_{1,i} = 0.$$

Therefore we obtain the result of Lemma 5 for $L_{1,i}$ and $L_{2,i}$ from above two equations. For $k \geq 3$, the coefficient of B^k in (7) gives

$$\Big(\sum_{j=0}^{k} L_{j,i}L_{k-j,i}\Big)(1-4q) - 8qL_{k-1,i} = 0.$$

Therefore we obtain the result of Lemma 5 for $L_{k,i}$ inductively on k. Similarly we can calculate the coefficient of z^j in the Picard-Fuchs equation (6) for $j \geq 1$ to obtain the result for $R_{0jk,i}$. Similar calculations were performed explicitly in [17, Theorem 3]. The result of Lemma 5 for $R_{1jk,i}$ follows easily from the previous results for $L_{k,i}, R_{0jk,i}$, the definition of the series $R_{1jk,i}$ in (4) and the definition of $\mathbb{S}(H)$ in (3). □

Define the series $Q_{ljk,i}$ by the equations

$$\sum_{j\geq 0, k\geq 0} Q_{ljk,i}B^k z^j = \Big[\Big(2\lambda_i(-2\lambda_i - B)\Big)^{-\frac{1}{2}} \exp\Big(\Big(\sum_{k=2}^{\infty} \mu_{k,i}B^k - B + \big(B+2\lambda_i\big)\log\Big(1+\frac{B}{2\lambda_i}\Big)\Big)/z\Big)\exp\Big(\sum_{k=1}^{\infty} -\frac{N_{k,i}\,B_{k+1}}{k(k+1)}z^k\Big) \sum_{j\geq 0, k\geq 0} R_{ljk,i}B^k z^j\Big]_+,$$

where $N_{k,i} = \Big(\frac{1}{\lambda_i - \lambda_{i+1}}\Big)^k + \Big(\frac{1}{-2\lambda_i - B}\Big)^k$ and B_k are the Bernoulli numbers. For a Laurent series F in z, $[F]_+$ is the non-negative part of F.

Using the localization formula [7, 12, 14], we have

$$\mathcal{F}_g^{\mathsf{SQ}} = \sum_{\Gamma \in \mathsf{G}_{g,0,\beta}^{\mathsf{Loc}}(X)} \frac{1}{\mathrm{Aut}(\Gamma)} [\Gamma, \prod_{v\in\mathsf{V}} \kappa_v \prod_{e\in\mathsf{E}} \Delta_e] \in \mathcal{R}_{X,S}[[q]], \tag{10}$$

where

- for $v \in \mathsf{V}$ let

$$\kappa_v = \mathrm{Vert}_v \cdot \kappa\Big(T - T \sum_{k\geq 0, j\geq 0} Q_{0jk,\mathsf{p}(v)} B^k (-T)^j\Big),$$

with

$$\mathrm{Vert}_v = \Big[\exp\big(\mu_{1,\mathsf{p}(v)} + \log(-2\lambda_{\mathsf{p}(v)})\big)\Big]^{\int_{\mathsf{d}(v)} B},$$

- for $e \in \mathsf{E}$, let

$$\begin{aligned}\Delta_e = \frac{1}{\psi' + \psi''}\Big[&- 2 \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_1)} B^k (-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_2)} B^k (-\psi'')^j \\ &- B \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_1)} B^k (-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_2)} B^k (-\psi'')^j \\ &- B \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_1)} B^k (-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_2)} B^k (-\psi'')^j \\ &- 2 \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_1)} B^k (-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_2)} B^k (-\psi'')^j \Big],\end{aligned}$$

where ψ', ψ'' are the ψ-classes corresponding to the half-edges.

See the appendix for the definition of $\mathsf{G}^{\mathsf{Loc}}_{g,0,\beta}(X)$. For a power series with vanishing constant and linear terms in X,

$$f(T,B) \in (T^2, TB)\mathbb{Q}[B][[T]]$$

we define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!} p_{m*}\Big(f(\psi_{n+1}, \mathrm{ev}^*_{n+1}(B)) \cdots f(\psi_{n+m}, \mathrm{ev}^*_{n+m}(B))\Big) \in R^*(\overline{M}_{g,n}(X,\beta)).$$

From the formula (10) and Lemma 5, we conclude that

$$\mathcal{F}^{\mathsf{SQ}}_g \in \mathcal{R}_{X,S}[L_0].$$

Moreover it is easy to check that only L_0^{2k} terms are non-zero for $k \in \mathbb{Z}_{\geq 0}$ in the formula (10). This is due to the fact that $R_{ljk,i}$ for $i = 0, 1$ in the proof of Lemma 5 satisfy the same differential equation with the choice of two initial conditions $L_{0,i} = (-1)^i L_0$ and the fact that the localization formula for $\mathcal{F}^{\mathsf{SQ}}_g$ in (10) is symmetric with respect to the two fixed points in $\mathbb{P}^1$. The proof of the proposition follows from $L_0(q)^2 = (1-4q)^{-1}$. ☐

2.3. Proof of Theorem 1

Recall that

$$I_1(q) = 2\log 2 - 2\log\big(1 + \sqrt{1-4q}\big).$$

If we define x by

$$x = q \cdot \exp(2\log 2 - 2\log(1 + \sqrt{1-4q})),$$

we have

$$q = \frac{x}{(1+x)^2}.$$

Therefore (2) yields the following equality:

$$\mathcal{F}^{Y}_{g,\beta}(x) = \mathcal{F}^{\mathsf{SQ}}_{g,\beta}(x/(1+x)^2) \cdot (1/(1+x))^{\int_\beta c_1(S)}. \tag{11}$$

Since we have

$$\frac{1}{1-4q} = \Big(\frac{1+x}{1-x}\Big)^2,$$

the proof of Theorem 1 follows from the above equation and Proposition 4. Note that the factor $(1/(1+x))^{\int_\beta c_1(S)}$ in (11) is canceled with $\mathrm{Vert}_v = \Big[\exp\big(\mu_{1,\mathsf{p}(v)} + \log(-2\lambda_{\mathsf{p}(v)})\big)\Big]^{\int_{\mathsf{d}(v)} B}$ in the formula (10), since we can easily calculate

$$\exp(\mu_{1,\mathsf{p}(v)}) = \frac{1}{1-4q}$$

from the equation (9).

3. Gromov-Witten theory of $\mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

3.1. Multiple cover formula

Let $\pi : U \to \overline{M}_{g,0}(\mathbb{P}^1, d)$ be the universal family over the moduli space. Let $f : U \to \mathbb{P}^1$ be the universal evaluation map. For $N := \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $R^1\pi_* f^* N$ is a vector bundle on $\overline{M}_{g,0}(\mathbb{P}^1, d)$. The following result was obtained using torus localization and Hodge integrals over the moduli space of curves [5, 8].

$$\int_{[\overline{M}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(R^1\pi_* f^* N) = \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!}. \tag{12}$$

Define the Gromov-Witten series of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by

$$\mathcal{F}_g(q) := \sum_{d=0}^{\infty} q^d \int_{[\overline{M}_{g,0}(\mathbb{P}^1,d)]} e(R^1\pi_* f^* N).$$

From the equation (12) and the following equations

$$\mathsf{D}^m\Big(\frac{1}{1-q}\Big) = \sum_{k=1}^{\infty} k^m q^k,$$

we can easily prove

$$\mathcal{F}_g(1/q) = \mathcal{F}_g(q).$$

For the generalization of the result, we give another proof of the above equation.

Proposition 6. *The Gromov-Witten series of* $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ *satisfies*

$$\mathcal{F}_g(1/q) = \mathcal{F}_g(q).$$

Proof. We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with weights λ_0, λ_1, γ_0, γ_1, so that the associated I-function is

$$\mathbb{I} := \sum_{d=0}^{\infty} q^d \frac{\prod_{i=0}^{1} \prod_{k=0}^{d-1} \Big(-H - kz - \gamma_i \Big)}{\prod_{i=0}^{1} \prod_{k=1}^{d} \Big(H + kz - \lambda_i \Big)}, \tag{13}$$

where $H \in H^2(\mathbb{P}^1)$ is the hyperplane class. Here the first $(\mathbb{C}^*)^2$ in T acts coordinate-wisely on $\mathbb{P}^1$ and the second $(\mathbb{C}^*)^2$ acts coordinate-wisely on the fiber $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. With the induced T-action on $\overline{M}_{g,0}(\mathbb{P}^1, d)$, define the T-equivariant Gromov-Witten series of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by

$$\mathcal{F}_g^{\mathsf{T}}(q) := \sum_{d=0}^{\infty} q^d \int_{[\overline{M}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir},\mathsf{T}}} e^{\mathsf{T}}(R^1\pi_* f^* N).$$

Here, $[\overline{M}_{g,0}(\mathbb{P}^1, d)]^{\mathrm{vir}}$ is the corresponding equivariant virtual class and e^{T} is the equivariant Euler class. Note that $\mathcal{F}^{\mathsf{T}}(q)$ does not depend on s and t, since the corresponding virtual dimension is zero. Therefore we have the following equality:

$$\mathcal{F}_g(q) = \mathcal{F}_g^{\mathsf{T}}(q).$$

We use the specialization

$$\lambda_i = (-1)^i s, \ \ \gamma_i = (-1)^i t. \tag{14}$$

Define

$$\mathbb{S}(1) = \mathbb{I}, \ \ \mathbb{S}(H) = \mathsf{M}\,\mathbb{S}(1),$$

where $\mathsf{M} := H + z\frac{qd}{dq}$. The series

$$\mathbb{S}_i(1) := \mathbb{S}|_{H=\lambda_i}, \ \ \mathbb{S}_i(H) := \mathbb{S}(H)|_{H=\lambda_i}$$

have the following asymptotic expansions:

$$\begin{aligned} \mathbb{S}_i(1) &= e^{\frac{\mu_i}{z}} \Big(R_{00,i} + R_{01,i} z + R_{02,i} z^2 + \cdots \Big), \\ \mathbb{S}_i(H) &= e^{\frac{\mu_i}{z}} \Big(R_{10,i} + R_{11,i} z + R_{12,i} z^2 + \cdots \Big), \end{aligned} \tag{15}$$

with series $\mu_i, R_{lj,i} \in \mathbb{Q}(s,t)[[q]]$. We have the following result for the series μ_i, $R_{lj,i}$.

Lemma 7. *For $k \geq 0$, $l = 0, 1$ and $i = 0, 1$, we have*

$$(-1)^i s + \mathsf{D}\mu_i = (-1)^i L, \ \ R_{lj,i} \in \mathbb{Q}(s,t)[L^{1/2}, L^{-1/2}],$$

where $L(q) = \sqrt{\frac{s^2 - q\,t^2}{1-q}}$.

Proof. The function $\mathbb{I}$ satisfies the following Picard-Fuchs equation

$$\Big((\mathsf{M} - \lambda_0)(\mathsf{M} - \lambda_1) - q(-\mathsf{M} - \gamma_0)(-\mathsf{M} - \gamma_1) \Big) \mathbb{I} = 0,$$

or equivalently,

$$\Big[\Big((\mathsf{M}-\lambda_0)(\mathsf{M}-\lambda_1)-q(-\mathsf{M}-\gamma_0)(-\mathsf{M}-\gamma_1)\Big)\mathbb{I}\Big]\mathbb{I}^{-1}=0. \tag{16}$$

The lemma will follow by applying the asymptotic forms (15) to above equation. Instead of the asymptotic expansion (15) of $\mathbb{I}$, we use the following exponential form:

$$\mathbb{I}|_{H=\lambda_i}=\exp\Big(\frac{\mu_i+a_{0,i}z+a_{1,i}z^2+\cdots}{z}\Big). \tag{17}$$

The evaluations of $R_{0k,i}$ can be obtained from those of $a_{k,i}$ by the equation

$$e^{\frac{\mu_i}{z}}\Big(R_{00,i}+R_{01,i}z+R_{02,i}z^2+\cdots\Big)=\exp\Big(\frac{\mu_i+a_{0,i}z+a_{1,i}z^2+\cdots}{z}\Big). \tag{18}$$

If we apply (17) to the Picard-Fuchs equation (16), the coefficient of z^0 in the equation is given by

$$(1-q)(\lambda_i+\mathsf{D}\mu_i)^2-s^2+q\,t^2=0.$$

Therefore μ_i satisfies

$$\lambda_i+\mathsf{D}\mu_i=(-1)^iL,$$

where L is the root of the polynomial

$$(1-q)\mathcal{L}^2-s^2+q\,t^2=0$$

with $L|_{q=0}=s$. From the above equation we obtain

$$q=\frac{L^2-s^2}{L^2-t^2},$$

$$\mathsf{D}L=\frac{(L^2-s^2)(L^2-t^2)}{2L(s^2-t^2)}.$$

The coefficient of z in the equation (16) is given by

$$(1-q)\Big(2L\mathsf{D}a_{0,i}+\mathsf{D}((-1)^iL)\Big)=0.$$

Therefore we calculate

$$\mathsf{D}a_{0,i}=-\frac{(L^2-s^2)(L^2-t^2)}{4L^2(s^2-t^2)}.$$

By solving above differential equation with the initial condition $a_{0,i}|_{q=0}=0$, we obtain

$$a_{0,i}=-\frac{\log\Big((-1)^iL/s\Big)}{2}.$$

From the equation (18) we obtain

$$R_{00,i}=\Big(\frac{s}{(-1)^iL}\Big)^{1/2}.$$

For $k \geq 2$, the coefficient of z^k in the equation (16) is given by

$$(1-q)\Big(2L\,\mathsf{D}a_{k-1,i} + \mathsf{D}^2 a_{k-2,i} + \sum_{j=0}^{k-2} \mathsf{D}a_{j,i}\mathsf{D}a_{k-2-j,i}\Big) = 0.$$

We can inductively solve the differential equation

$$2L\,\mathsf{D}a_{k-1,i} + \mathsf{D}^2 a_{k-2,i} + \sum_{j=0}^{k-2} \mathsf{D}a_{j,i}\mathsf{D}a_{k-2-j,i} = 0 \tag{19}$$

with the initial conditions $a_{k,i} = 0$ for $k \geq 1$ to obtain each $a_{k,i}$ for $k \geq 1$ as a Laurent polynomial in L up to possible extra factor $\log L$. This argument yields weaker result,

$$a_{k,i} \in \mathbb{Q}(s,t)[L, L^{-1}, \log L],$$

and hence

$$R_{lj,i} \in \mathbb{Q}(s,t)[L^{1/2}, L^{-1/2}, \log L],$$

by the equation (18).

To prove the result

$$R_{lj,i} \in \mathbb{Q}(s,t)[L^{1/2}, L^{-1/2}],$$

we use the *saddle point method* for finding asymptotic behaviour of the oscillatory integral occurring in Givental's equivariant mirror [6]. This argument was explained by Iritani. See [15, Appendix] for the introduction to this method. Similar argument was also used in [9, Appendix A.6]. Here we follow the notation of [15, Appendix]. The equivariant mirror for local $\mathbb{P}^1$ was introduced by Givental as the Landau-Ginzburg potential

$$W(w_0, w_1, w_2, w_3) = w_0 + w_1 + w_2 + w_3 - t\log(w_1/w_2) + s\log(q\,w_3/w_0),$$

defined on the family of affine varieties

$$M_q = \{(w_0, w_1, w_2, w_3) \in \mathbb{C}^4 : w_0 w_3 = q\,w_1 w_2\}.$$

The associated oscillatory integral is defined by

$$\mathcal{I} = \int_{\Gamma \subset M_q} e^{W/z}\omega, \tag{20}$$

where ω is the (meromorphic) volume form on M_q:

$$\omega = \frac{d\log w_0 \wedge d\log w_1 \wedge d\log w_2 \wedge d\log w_3}{d\log q}.$$

The integral in (20) is along 3-cycles Γ through a specific critical point of the Landau-Ginzburg potential W which can be constructed via Morse theory of the real part of W/z. A relationship between the formal asymptotic expansion of the mirror oscillatory integral (20) and the equivariant I-function (13) was proven in [4, Proposition 6.9]. Denote by

$$\mathrm{Asym}_{\mathrm{cr}_i}(e^{W/z}\omega)$$

be the formal asymptotic expansion of (20) at the critical point cr_i. Applying [4, Proposition 6.9] to our settings, we obtain

$$e^{W(\mathrm{cr}_i)/z} \cdot \mathrm{Asym}_{\mathrm{cr}_i}(e^{W/z}\omega) = e^{\frac{\mu_i}{z}} \cdot \left(1 + \frac{R_{01,i}}{R_{00,i}}\, z + \frac{R_{02,i}}{R_{00,i}}\, z^2 + \cdots\right).$$

Now the asymptotic behaviour of the oscillatory integral (20) can be explicitly calculated via the saddle point method as follows:

$$\mathrm{Asym}_{\mathrm{cr}_i}(e^{W/z}\omega) = \frac{1}{\sqrt{\det(h_{j,k})}} \left[e^{-\frac{z}{2}\sum_{j,k} h^{j,k}\partial_j \partial_k} e^{W_{\geq 3}/z}\right]_{t=\mathrm{cr}_i}, \tag{21}$$

where $h_{j,k} = \partial_j \partial_k W(\mathrm{cr}_i)$ is the Hessian matrix, $(h^{j,k})$ are the coefficients of the matrix inverse to $(h_{j,k})$, $\partial_j = \frac{\partial}{\partial w_j}$, and

$$W_{\geq 3}(w) = W(w) - W(\mathrm{cr}_i) - \frac{1}{2}\sum_{j,k} h_{j,k}(w_j - \mathrm{cr}_i^j)(w_k - \mathrm{cr}_i^k).$$

See [15, Appendix A.1] for more explanations. The coordinates (a, b, u) are more convenient for the calculations,

$$w_0 = u,\ w_1 = au,\ w_2 = bu,\ w_3 = qabu.$$

Then we can rewrite

$$W(a, b, u) = u(1 + a + b + q\,ab) - t\log(a/b) + s\log(q\,ab),$$

and its critical points are given by

$$\begin{aligned} u &= s - (\pm L),\\ ua &= t + (\pm L),\\ ub &= -t + (\pm L),\\ uab\,q &= -s - (\pm L). \end{aligned}$$

Note that the choice of the root $(\pm L)$ corresponds to the choice of two critical points. The Hessian of W with respect to the logarithmic coordinates $(\log a, \log b, \log u)$ at this critical point is given by

$$\mathrm{Hess}(W) = 2L(t^2 - s^2).$$

Therefore the formal asymptotic expansion (21) of the oscillatory integral (20) have the coefficients (which correspond to the normalized forms $R_{0j,i}/R_{00,i}$) lying in the ring $\mathbb{Q}(s, t)[L, L^{-1}]$. This concludes the proof of the statement for $R_{0k,i}$. The statement for $R_{1k,i}$ follows easily from the definition of $\mathbb{S}(H) := \mathsf{M}\,\mathbb{S}(1)$. □

Define the series $Q_{lj,i}$ by the equations

$$\sum_{j=0}^{\infty} Q_{lj,i}\, z^j = \frac{1}{R_{00,i}} \exp\Big(\sum_{k=0}^{\infty} -\frac{N_{j,i}\, B_{j+1}}{j(j+1)} z^j\Big) \sum_{j=0} R_{lj,i}\, z^j, \tag{22}$$

where $N_{j,i} = \big(\frac{1}{\lambda_i-\lambda_{i+1}}\big)^j + \big(\frac{1}{-\lambda_i-\gamma_0}\big)^j + \big(\frac{1}{-\lambda_i-\gamma_1}\big)^j$ and B_j are the Bernoulli numbers.

Lemma 8. *For $j \geq 0$, $l = 0, 1$ and $i = 0, 1$, we have*

$$Q_{lj,i} = \sum_{m=-3j}^{j} \frac{q_{ljm}}{(s^2-t^2)^j}\Big((-1)^i L\Big)^{m+l},$$

where $q_{ljm} \in \mathbb{Q}[s,t]$ satisfies $q_{ljm}(s,t) = q_{ljm}(t,s)$.

Proof. Recall we use the specialization (14). In the proof of Lemma 7, the differential equation (19) do not depend on s and t. Since

$$\mathsf{D}L = \frac{(L^2-s^2)(L^2-t^2)}{2L(s^2-t^2)},$$

we conclude that $a_{k,i}$ have same forms as in the statement of Lemma 8. Then the statement of Lemma 8 for $Q_{0j,i}$ follows easily, since $Q_{0j,i}$ and $a_{j,i}$ are related by the equations (18) and (22). The statement for $Q_{1j,i}$ also follows easily from the definition of $R_{1j,i}$ and $Q_{1j,i}$. Note that the factor $(-1)^i$ in the equation of Lemma 8 is due to the fact that $a_{j,i}$ for $i = 0, 1$ are the solutions of same differential equation (19) with different initial conditions. The choice of initial conditions corresponds to the choice of two roots L or $-L$ of the defining polynomial

$$(1-q)\mathcal{L}^2 - s^2 + q\,t^2 = 0.$$

□

Using the localization formula [7,12,14], we have

$$\mathcal{F}_g^{\mathsf{T}} = \sum_{\Gamma \in \mathsf{G}_{g,0}^{\mathsf{Loc}}} \frac{1}{\mathrm{Aut}(\Gamma)}[\Gamma, \prod_{v\in\mathsf{V}} \kappa_v \prod_{e\in\mathsf{E}} \Delta_e], \tag{23}$$

where

- for $v \in \mathsf{V}$ let
$$\kappa_v = \kappa\Big(T - T\sum_{j=0}^{\infty} Q_{0j,\mathsf{p}(v)}(-T)^j\Big),$$
- for $e \in \mathsf{E}$, let
$$\Delta = \frac{s^2-t^2}{\psi'+\psi''}\Big[\sum_{j=0}^{\infty} Q_{0j,\mathsf{p}(e_1)}(-\psi')^j \sum_{j=0}^{\infty} Q_{1j,\mathsf{p}(e_2)}(-\psi'')^j + \sum_{j=0}^{\infty} Q_{0j,\mathsf{p}(e_1)}(-\psi')^j \sum_{j=0}^{\infty} Q_{1j,\mathsf{p}(e_2)}(-\psi'')^j\Big],$$

  where ψ', ψ'' are the ψ-classes corresponding to the half-edges.

Lemma 9. *We have*

$$\mathcal{F}_g^{\mathsf{T}} \in \mathbb{Q}(s,t)[L^2, L^{-2}],$$

where $L(q) = \sqrt{\frac{s^2 - q\,t^2}{1-q}}$. *Moreover, each coefficients of* L^k *for* $k \in \mathbb{Z}$ *in* $\mathcal{F}_g^{\mathsf{T}}$ *are symmetric with respect to* s *and* t.

Proof. We get the result by applying Lemma 8 to the formula (23). Note that the odd powers of L in the formula (23) disappear by the factor $(-1)^i$ in the equation of Lemma 8 and the symmetry of the formula (23) with respect to two fixed points p_0 and p_1 in $\mathbb{P}^1$. □

Now we have the following equations which complete the proof of the proposition,

$$\begin{aligned}\mathcal{F}_g(q) &= \mathcal{F}_q^{\mathsf{T}}(q)|_{s=1,t=0}\\ &= \mathcal{F}_g^{\mathsf{T}}(q)|_{s=0,t=1}\\ &= \mathcal{F}_g^{\mathsf{T}}(1/q)|_{s=1,t=0}\\ &= \mathcal{F}_g(1/q).\end{aligned}$$

The second equality above holds since $\mathcal{F}_g^{\mathsf{T}}(q)$ do not depend on s and t by the dimension argument. The third equality follows from Lemma 9 and the following equality,

$$L(q)^2|_{s=1,t=0} = L(1/q)^2|_{s=0,t=1}.$$ □

3.2. Proof of Theorem 2

Let $\mathsf{T} = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ act on Z. We choose a torus action $\mathsf{T} = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ on Z with weights λ_0, λ_1, γ_0, γ_1, so that the associated I-function is

$$\mathbb{I} := \sum_{d=0}^{\infty} q^d \frac{\prod_{i=0}^{1}\prod_{k=0}^{d-1}\Big(-H - B - kz - \gamma_i\Big)}{\prod_{i=0}^{1}\prod_{k=1}^{d}\Big(H + kz - \lambda_i\Big)},$$

where $H \in H^2(\mathbb{P}^1)$ is the hyperplane class of $\mathbb{P}^1$ and $B = c_1(S) \in H^2(X)$. We use the specialization

$$\lambda_i = (-1)^i s, \ \ \gamma_i = (-1)^i t.$$

We define the equivariant Gromov-Witten class of Z by

$$\mathcal{F}_{g,\beta}^{Z,\mathsf{T}}(q) := \sum_{d\geq 0} q^d\, \pi_*\Big([\overline{M}_{g,0}(X\times\mathbb{P}^1,(\beta,d))]^{\mathrm{vir},\mathsf{T}} \cap e^{\mathsf{T}}(-R^{\bullet}\pi_*F)\Big) \in \mathcal{R}_{X,S}(s,t)[[q]].$$

Define

$$\mathbb{S}(1) := \mathbb{I},\ \ \mathbb{S}(H) := \mathsf{M}\mathbb{S}(1), \tag{24}$$

where $\mathsf{M} = H + z\frac{qd}{dq}$. We can show the series

$$\mathbb{S}_i(1) := \mathbb{S}|_{H=\lambda_i},\ \mathbb{S}_i(H) := \mathbb{S}_i(H)|_{H=\lambda_i}$$

have the following asymptotic expansions:

$$\begin{aligned} \mathbb{S}_i(1) &= e^{\frac{\sum_{k=0}^{\infty} \mu_{k,i} B^k}{z}} \Big(\sum_{j\geq 0, k\geq 0} R_{0jk,i}\, B^k z^j \Big), \\ \mathbb{S}_i(H) &= e^{\frac{\sum_{k=0}^{\infty} \mu_{k,i} B^k}{z}} \Big(\sum_{j\geq 0, k\geq 0} R_{1jk,i}\, B^k z^j \Big), \end{aligned} \tag{25}$$

with series $\mu_{k,i}$, $R_{ljk,i} \in \mathbb{Q}(s,t)[[q]]$.

Lemma 10. *For $k \geq 0$, $l = 0, 1$ and $i = 0, 1$, we have*

(i) *for $k \geq 0$,*

$$R_{ljk,i} \in \mathbb{Q}(s,t)[L_0^{1/2}, L_0^{-1/2}],$$

(ii) $\lambda_i + q\frac{d}{dq}\mu_{0,i} = (-1)^i L_0$, $q\frac{d}{dq}\mu_{1,i} = \frac{L_0^2 - s^2}{s^2 - t^2}$, *and for $k \geq 2$,*

$$\mathsf{D}\mu_{k,i} \in \mathbb{Q}(s,t)[L_0, L_0^{-1}],$$

where $L_0(q) = \sqrt{\frac{s^2 - q\,t^2}{1-q}}$.

Proof. We will use the notations

$$\begin{aligned} L_{0,i} &:= \lambda_i + \mathsf{D}\mu_{0,i}, \\ L_{k,i} &:= \mathsf{D}\mu_{k,i} \ \text{ for } k \geq 1. \end{aligned}$$

The function $\mathbb{I}$ satisfies the following Picard-Fuchs equation

$$\Big((\mathsf{M} - \lambda_0)(\mathsf{M} - \lambda_1) - q(-\mathsf{M} - B - \gamma_0)(-\mathsf{M} - B - \gamma_1) \Big) \mathbb{I} = 0,$$

or equivalently,

$$\Big[\Big((\mathsf{M} - \lambda_0)(\mathsf{M} - \lambda_1) - q(-\mathsf{M} - B - \gamma_0)(-\mathsf{M} - B - \gamma_1) \Big) \mathbb{I} \Big] \mathbb{I}^{-1} = 0. \tag{26}$$

The lemma follows by applying the asymptotic forms (25) to above equation. Note that the statement of Lemma 10 for $k = 0$ recover Lemma 7. The coefficient of z^0 in (26) is given by

$$(1 - q)\mathcal{L}_{B,i} - 2qB\mathcal{L}_{B,i} - s^2 + q(t^2 - B^2) = 0, \tag{27}$$

where we used the notation

$$\mathcal{L}_{B,i} = L_{0,i} + L_{1,i}B + L_{2,i}B^2 + \cdots .$$

The coefficient of B^0 in (27) is given by

$$(1 - q)L_{0,i}^2 - s^2 + qt^2 = 0. \tag{28}$$

Therefore we obtain

$$L_{0,i} = (-1)^i \Big(\frac{s^2 - qt^2}{1 - q} \Big)^{1/2} := (-1)^i L_0.$$

Note that the choice of two roots of the equation (28) corresponds to the choice of two fixed points in $\mathbb{P}^1$. We also obtain the following equations from (28),

$$q = \frac{L_0^2 - s^2}{L_0^2 - t^2},$$
$$\mathsf{D}L_0 = \frac{(L_0^2 - s^2)(L_0^2 - t^2)}{2L_0(s^2 - t^2)}.$$

The coefficient of B in (27) gives

$$(1-q)2L_{0,i}L_{1,i} - 2qL_{0,i} = 0.$$

The coefficient of B^2 in (27) gives

$$(1-q)(L_{1,i}^2 + 2L_{0,i}L_{2,i}) - q - 2qL_{1,i} = 0.$$

We obtain the result of Lemma 10 for $L_{1,i}$ and $L_{2,i}$ from above two equations. For $k \geq 3$, the coefficient of B^k in (27) gives

$$(1-4q)\Big(\sum_{j=0}^{k} L_{j,i}L_{k-j,i}\Big) - 2qL_{k-1,i} = 0.$$

Therefore we obtain the result of Lemma 10 for $L_{k,i}$ inductively on k. Similarly we can calculate the coefficient of z^j in the Picard-Fuchs equation (26) for $j \geq 1$ to obtain the result for $R_{0jk,i}$. Similar calculations were performed explicitly in [17, Theorem 3]. The result of Lemma 10 for $R_{1jk,i}$ follows easily from the previous results for $L_{k,i}, R_{0jk,i}$, the definition of the series $R_{1jk,i}$ in (25) and the definition of $\mathbb{S}(H)$ in (24). □

Define the series $Q_{ljk,i}$ by the equations

$$\begin{aligned}
&\sum_{j\geq 0,\, k\geq 0} Q_{ljk,i}B^k z^j \\
&= \Big[\Big(2\lambda_i(-\lambda_i - B - \gamma_0)(-\lambda_i - B - \gamma_1)\Big)^{-\frac{1}{2}} \\
&\quad \cdot \exp\Big(\Big(\sum_{k=2}^{\infty}\mu_{k,i}B^k + \sum_{i=0}^{1}\big(-B + (B+s+t_i)\log(1+\frac{B}{s+t_i})\big)\Big)/z\Big) \qquad (29)\\
&\quad \cdot \exp\Big(\sum_{k=1}^{\infty} -\frac{N_{k,i}B_{k+1}}{k(k+1)}z^k\Big)\sum_{j\geq 0,\, k\geq 0} R_{ljk,i}B^k z^j\Big]_+,
\end{aligned}$$

where $N_{k,i} = \left(\frac{1}{\lambda_i - \lambda_{i+1}}\right)^k + \left(\frac{1}{-\lambda_i - B - \gamma_0}\right)^k + \left(\frac{1}{-\lambda_i - B - \gamma_1}\right)^k$ and B_k are the Bernoulli numbers. For a Laurent series F in z, $[F]_+$ is the non-negative part of F.

Lemma 11. *We have*

$$Q_{ljk,i} = \sum_{r=-3j-k}^{k} \frac{q_{ljkr}}{(s^2-t^2)^{j+k}}((-1)^i L_0)^{r+l},$$

where $q_{ljkr} \in \mathbb{Q}[s,t]$ *are some polynomials in* s *and* t *such that* $q_{ljkr}(s,t) = q_{ljkr}(t,s)$.

Proof. It is easy to check that in the proof of Lemma 10, the differential equation for $R_{0jk,i}$ obtained from the coefficient of z^j in (26) do not depend on s and t. These calculations are parallel to the calculations given in the proof of Lemma 7. Since

$$\mathsf{D}L_0 = \frac{(L_0^2-s^2)(L_0^2-t^2)}{2L_0(s^2-t^2)},$$

we conclude that $R_{0jk,i}$ have same forms as in the statement of Lemma 11. Then the statement of Lemma 11 for $Q_{0jk,i}$ follows easily, since $Q_{0jk,i}$ and $R_{0jk,i}$ are related by the equation (29). The statement for $Q_{1jk,i}$ also follows easily from the previous result for $Q_{0jk,i}$ and the definitions of $R_{1jk,i}$, $Q_{1jk,i}$. Note that the factor $(-1)^i$ in the equation of Lemma 8 is due to the fact that $R_{0jk,i}$ for $i = 0,1$ are the solutions of same differential equation (26) with different initial conditions. The choice of initial conditions corresponds to the choice of two roots L_0 or $-L_0$ of the defining polynomial

$$(1-q)\mathcal{L}^2 - s^2 + q\,t^2 = 0.$$

□

For a power series with vanishing constant and linear terms in X,

$$f(X,Y) \in (X^2, XY)\mathbb{Q}[Y][[X]]$$

we define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!} p_{m*}\Big(f(\psi_{n+1}, \mathrm{ev}_{n+1}^*(B)) \cdots f(\psi_{n+m}, \mathrm{ev}_{n+m}^*(B))\Big) \in R^*(\overline{M}_{g,n}(X,\beta)).$$

Using the localization formula [7, 12, 14], we have

$$\mathcal{F}_{g,\beta}^{Z,\mathsf{T}} = \sum_{\Gamma\in \mathsf{G}_{g,0}^{\mathsf{Loc}}(X)} \frac{1}{\mathrm{Aut}(\Gamma)}[\Gamma, \prod_{v\in\mathsf{V}} \kappa_v \prod_{e\in\mathsf{E}} \Delta_e] \in \mathcal{R}_{X,S}[[q]], \tag{30}$$

where

- for $v \in \mathsf{V}$ let

$$\kappa_v = \mathrm{Vert}_v \cdot \kappa\Big(T - T\sum_{k\geq 0, j\geq 0} Q_{0jk,\mathsf{p}(v)} B^k (-T)^j\Big),$$

 with

$$\mathrm{Vert}_v = \Big[\exp\Big(\mu_{1,\mathsf{p}(v)} + \log\Big((-\lambda_{\mathsf{p}(v)} - \gamma_0)(-\lambda_{\mathsf{p}(v)} - \gamma_1)\Big)\Big)\Big]^{\int_{\mathsf{d}(v)} B},$$

- for $e \in \mathsf{E}$, let

$$\Delta_e = \frac{1}{\psi' + \psi''}\Big[2Bs^2 \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_1)} B^k(-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_2)} B^k(-\psi'')^j$$
$$+ (s^2 - t^2 + B^2) \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_1)} B^k(-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_2)} B^k(-\psi'')^j$$
$$+ (s^2 - t^2 + B^2) \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_1)} B^k(-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{0jk,\mathsf{p}(e_2)} B^k(-\psi'')^j$$
$$+ 2 \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_1)} B^k(-\psi')^j \sum_{j\geq 0, k\geq 0} Q_{1jk,\mathsf{p}(e_2)} B^k(-\psi'')^j\Big],$$

where ψ', ψ'' are the ψ-classes corresponding to the half-edges.

Lemma 12. *We have*

$$\mathcal{F}^{Z,\mathsf{T}}_{g,\beta} \in \Big(1/(1-q)\Big)^{\int_\beta c_1(S)} \cdot \mathcal{R}_{X,S}(s,t)[L_0^2, L_0^{-2}],$$

where $L_0(q) = \sqrt{\frac{s^2 - q\,t^2}{1-q}}$. *Moreover, each coefficients of* L_0^k *for* $k \in \mathbb{Z}$ *in* $\mathcal{F}^{Z,\mathsf{T}}_{g,\beta}$ *are symmetric with respect to* s *and* t.

Proof. First we explain the factor $\Big(1/(1-q)\Big)^{\int_\beta c_1(S)}$. In the formula (30), for a fixed Γ, all vertex factors of Vert_v, for $v \in \mathsf{V}$ contribute to $\mathrm{Vert}_v^{\int_\beta c_1(S)}$. Since $e^{\mu_{1,i}} = 1/(1-q)$ from Lemma 10, we get the factor $\Big(1/(1-q)\Big)^{\int_\beta c_1(S)}$.

From equation (30), we can consider $\mathcal{F}^{Z,\mathsf{T}}_{g,\beta}$ as a formal power series in B. Now using Lemma 11 and the following equation

$$\mathbb{S}_i(H) = \mathsf{M} \cdot \mathbb{S}(1)$$

we can prove the result of Lemma 12 from the formula (30). The odd powers of L_0 in $\mathcal{F}^{Z,\mathcal{T}}_{g,\beta}$ vanish due to the fact that $R_{ljk,i}$ for $i = 0, 1$ in the proof of Lemma 10 satisfy the same differential equation (26) with the choice of two initial conditions $L_{0,i} = (-1)^i L_0$ and the fact that the localization formula for $\mathcal{F}^{\mathsf{SQ}}_g$ in (10) is symmetric with respect to the two fixed points in $\mathbb{P}^1$. $\square$

We finally have the following equations which complete the proof of the theorem:

$$\begin{aligned}\mathcal{F}^{Z}_{g,\beta}(q) &= \mathcal{F}^{Z,\mathsf{T}}_{g,\beta}(q)|_{s=1,t=0}\\ &= \mathcal{F}^{Z,\mathsf{T}}_{g,\beta}(q)|_{s=0,t=1}\\ &= (-q)^{\int_\beta c_1(S)} \mathcal{F}^{Z,\mathsf{T}}_{g,\beta}(1/q)|_{s=1,t=0}\\ &= (-q)^{\int_\beta c_1(S)} \mathcal{F}^{Z}_{g,\beta}(1/q).\end{aligned}$$

The second equality above holds since $\mathcal{F}^{Z,\mathsf{T}}_{g,\beta}(q)$ do not depend on s and t by the dimension argument. The third equality follows from Lemma 12 and the following equation

$$L_0(q)^2|_{s=1,t=0} = L_0(1/q)^2|_{s=0,t=1}.$$

The factor $(-q)^{\int_\beta c_1(S)}$ in the third equality comes from the vertex factor Vert_v in the formula (30) and the following equation

$$e^{\mu_{1,i}} = \frac{L_0^2 - t^2}{s^2 - t^2} = \frac{1}{1-q},$$

which can be obtained from (ii) in Lemma 10 and $\mu_{1,i}|_{q=0} = 0$.

4. Appendix

4.1. Graphs

In the localization formula, the T-fixed loci are represented in terms of dual graphs. Let the genus g and the number of markings n for the moduli space be in the stable range

$$2g - 2 + n > 0.$$

A *localization graph* $\Gamma \in \mathsf{G}^{\mathsf{Loc}}_{g,n}$ consists of the data $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g}, \mathsf{p})$, where

(i) V is the vertex set,
(ii) E is the edge set (allowing possible self-edges),
(iii) $\mathsf{N} : \{1, 2, \ldots, n\} \to \mathsf{V}$ is the marking assignment,
(iv) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment with

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma)$$

and for which $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g})$ is a stable graph,
(v) $\mathsf{p} : \mathsf{V} \to \{0, 1\}$ is an extra assignment.

4.2. *X*-valued stable graphs

Let X be a nonsingular projective variety over $\mathbb{C}$ and let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class. We review the X-valued stable graphs introduced in [1]. Boundary strata of the moduli space of stable maps to X correspond to *X-valued stable graphs*

$$\Gamma = (\mathsf{V}, \mathsf{H}, \mathsf{g} : V \to \mathbb{Z}_{\geq 0}, \mathsf{d} : V \to H_2(X, \mathbb{Z}), \mathsf{v} : H \to V, \mathsf{i} : H \to H)$$

satisfying the following properties:

(i) V is a vertex set with a genus function $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ and a degree function $\mathsf{d} : \mathsf{V} \to H_2(X, \mathbb{Z})$,
(ii) H is a half-edge set equipped with a vertex assignment $\mathsf{v} : \mathsf{H} \to \mathsf{V}$ and an involution i,
(iii) E, the edge set, is defined by the 2-cycle of i in H (self-edges at vertices are allowed),

(iv) L, the set of legs, is defined by the fixed points of i and endowed with a bijective correspondence with a set of markings,
(v) the pair (V, E) defines a *connected* graph,
(vi) for each vertex $v \in \mathsf{V}$, the stability condition holds:

$$2\mathsf{g}(v) - 2 + \mathsf{n}(v) > 0 \ \text{ if } \mathsf{d}(v) = 0,$$

where (v) is the valence of Γ at v including both edges and legs,
(vii) the degree condition holds

$$\sum_{v \in \mathsf{V}} \mathsf{d}(v) = \beta.$$

An automorphism of Γ consist of automorphisms of the sets V and H which leave invariant the structures $\mathsf{g}, \mathsf{d}, \mathsf{i}$, and v (and hence respect E). Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by

$$g(\Gamma) = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma).$$

A boundary stratum of the moduli space $\overline{M}_{g,n}(X, \beta)$ of stable maps naturally determines a stable graph of genus g, degree d with n legs by considering the dual graph of a generic pointed domain curve parameterized by the stratum. Let $\mathsf{G}_{g,n,\beta}(X)$ be the set of isomorphism classes of X-valued stable graphs of genus g and degree β with n legs. We also define $\mathsf{G}^{\mathsf{Loc}}_{g,n,\beta}(X)$ to be the set of isomorphism classes of X-valued stable graphs of genus g, degree β, n legs and extra assignment

$$\mathsf{p} : \mathsf{V} \to \{0,\ 1\}.$$

The set $\{0, 1\}$ in the assignment p will correspond to two fixed points of the action $(\mathbb{C}^*)^2$ on $\mathbb{P}^1$ in the localization formula (10) and (30).

To each stable graph Γ, we associate the moduli space $\overline{M}_\Gamma$ which is the substack of the product

$$\prod_{v \in V} \overline{M}_{g(v),n(v)}(X, \beta(v))$$

cut out by the inverse image of the diagonal $\Delta_X \subset X \times X$ under the evaluation maps associated to all edges $e = (h, h') \in E$,

$$\prod_{v \in V} \overline{M}_{g(v),n(v)}(X, \beta(v)) \xrightarrow{\mathrm{ev}_e} X \times X.$$

Let π_v be the projection from $\overline{M}_\Gamma$ to $\overline{M}_{g(v),n(v)}(X, \beta(v))$ associated to the vertex v. There is a canonical morphism

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}(X, \beta) \tag{31}$$

with the image equal to the boundary stratum associated to the graph Γ. To construct ξ_Γ, a family of stable maps over $\overline{M}_\Gamma$ is required. Such a family is easily obtained by gluing pull-backs of the universal families over each of the $\overline{M}_{g(v),n(v)}(X,\beta(v))$ along the sections corresponding to half-edges. The moduli space $\overline{M}_\Gamma$ carries a natural virtual fundamental class $[\overline{M}_\Gamma]^{\mathrm{vir}}$ induced by the Gysin pull-back along diagonals

$$[\overline{M}_\Gamma]^{\mathrm{vir}} = \prod_{e\in E} \mathrm{ev}_e^{-1}(\Delta) \cap \prod_{v\in V} [\overline{M}_{g(v),n(v)}(X,\beta(v))]^{\mathrm{vir}}.$$

4.3. Strata algebra

For any target X, we can associate a $\mathbb{Q}$-algebra, called the X-valued strata algebra [1], which represents tautological classes on $\overline{M}_{g,n}(X,\beta)$. In this paper, we will restrict to the subalgebra of X-valued strata algebra associated to a fixed line bundle on X. Let S be a line bundle over X. There are two canonical line bundles on the universal curve

$$\pi : \mathcal{C}_{g,n,\beta}(X) \to \overline{M}_{g,n}(X,\beta).$$

The first one is the relative dualizing sheaf ω_π and the second one is the pull-back f^*S of the line bundle S via the universal map,

$$f : \mathcal{C}_{g,n,\beta}(X) \to X.$$

Let s_i be the i-th section of π, and let

$$D_i \subset \mathcal{C}_{g,n,\beta}(X)$$

be the corresponding divisor. Denote by $\omega_{\log}$ the relative logarithmic line bundle

$$\omega_\pi\Big(\sum_i^n D_i\Big)$$

with the first Chern class $c_1(\omega_{\log})$. Let $\xi = c_1(f^*S)$ be the first Chern class of the pull-back of S. Tautological classes ψ, ξ, and η classes on $\overline{M}_{g,n}(X,\beta)$ are defined as follows:

$$\psi_i := c_1(s_i^*\omega_\pi), \;\; \xi_i := s_i^*\xi, \;\; \eta_{a,b} = \pi_*(c_1(\omega_{\log})^a\xi^b).$$

Definition 13. A *decorated X-valued stable graph* $[\Gamma,\gamma]$ is an X-valued stable graph $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ together with the following decoration data γ:

(i) each leg $i \in \mathsf{L}$ is decorated with a monomial $\psi_i^a\xi_i^b$,
(ii) each half-edge $h \in \mathsf{H}\setminus\mathsf{L}$ is decorated with a monomial ψ_h^a,
(iii) each edge $e \in \mathsf{E}$ is decorated with a monomial ξ_e^a,
(iv) each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b\geq 2}$.

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n,\beta}(X,S)$ whose basis consists of the isomorphism classes of a decorated X-valued stable graph $[\Gamma,\gamma]$.

There is a product structure on $\mathcal{S}_{g,n,\beta}(X,S)$ which generalizes the intersection product on the strata algebra $\mathcal{S}_{g,n}$ of $\overline{M}_{g,n}$ ([1]). If we assign a grading

$$\deg[\Gamma,\gamma] = |\mathsf{E}| + \deg_{\mathbb{C}}(\gamma),$$

to each basis element $[\Gamma, \gamma]$, $\mathcal{S}_{g,n,\beta}(X)$ is a graded $\mathbb{Q}$-algebra

$$\mathcal{S}_{g,n,\beta}(X,S) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k_{g,n,\beta}(X,S).$$

Via this intersection product, $\mathcal{S}_{g,n,\beta}(X,S)$ is a $\mathbb{Q}$-algebra which we call the *strata algebra* (associated to S) following [1, 10].

To each element $[\Gamma, \gamma] \in \mathcal{S}_{g,n,\beta}(X,S)$, we assign a cycle class $\xi_{\Gamma *}[\gamma]$ obtained by the push-forward via

$$\overline{M}_\Gamma \to \overline{M}_{g,n}(X,\beta)$$

of the action of the product of the ψ, ξ and η decorations on $[\overline{M}_\Gamma]^{\mathrm{vir}}$

$$\xi_{\Gamma *}[\gamma] := \xi_{\Gamma *}\Big(\gamma \cap [\overline{M}_\Gamma]^{\mathrm{vir}}\Big) \in A_*(\overline{M}_{g,n}(X,\beta))_{\mathbb{Q}}.$$

Then ξ_Γ defines a $\mathbb{Q}$-linear map

$$\mathsf{q} : \mathcal{S}_{g,n,\beta}(X,S) \to A_*(\overline{M}_{g,n}(X,\beta)), \;\; \mathsf{q}([\Gamma,\gamma]) = \xi_{\Gamma *}[\gamma]$$

and it is known that the kernel of q is an ideal. We denote by $R^*_S(\overline{M}_{g,n}(X,\beta))$ the image of q. We write

$$\mathcal{R}_{X,S} := \bigoplus_{n \in \mathbb{Z},\, \beta \in H_2(X,\mathbb{Z})} R^*_S(\overline{M}_{g,n}(X,\beta)).$$

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