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HESITANT FUZZY MINIMAL AND MAXIMAL OPEN SETS

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Abstract. The aim of this article is to extend hesitant fuzzy minimal open and hesitant fuzzy maximal open sets in hesitant fuzzy topological space. Further, we investigate some properties with these new sets.

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1. Introduction

Zadeh[8] established fuzzy set in 1965 and Chang[1] introduced fuzzy topology in 1968. Ittanagi and Wali[4] instigated the notions of fuzzy maximal and minimal open sets. The idea of hesitant fuzzy set introduced by Torra[7] in 2010 which is an addendum to fuzzy sets. In 2019 Deepak et. al. [2] introduced hesitant fuzzy topological space and extended the study to hesitant connectedness and compactness in hesitant fuzzy topological space. In section 2, we study few known results. In section 3, we introduce hesitant fuzzy minimal open sets and some of their related results. In section 4, we study hesitant fuzzy maximal open sets and some of their proerties.

Throughout this paper the terms HFTS, HFMAO, HFMIO and HFS are respectively denoted as "hesitant fuzzy topological space, hesitant fuzzy maximal open, hesitant fuzzy minimal open sets and hesitant fuzzy sets".

2. Prelimiaries

In this section basic ideas of hesitant fuzzy topological space are studied.

Definition 2.1. [5] A HFS h in X is a function $h : X \to P[0,1]$, where $P[0,1]$ represents the power set of [0,1].

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We define the hesitant fuzzy empty set h^0(resp. whole set h^1) is a HFS in X as follows: $h^0(x) = \phi$ (resp. $h^1(x) = [0, 1]), \forall x \in X$. $HS(X)$ stands for collection of HFS in X.

Definition 2.2. [3] Two HFS $h_1, h_2 \in HS(X)$ such that $h_1(x) \subset h_2(x), \forall x \in X$, then h_1 is contained in h_2.

Definition 2.3. [3] Two HFS h_1 and h_2 of X are said to be equal if $h_1 \subset h_2$ and $h_2 \subset h_1$.

Definition 2.4. [5] Let $h \in HS(X)$ for any nonempty set X. Then h^c is the complement of h which is HFS in X such that $h^c(x) = [h(x)]^c = [0, 1] \backslash h(x)$.

Definition 2.5. [6] Let (X, τ) be a HFTS. Let $x_\lambda \in H_p(X)$ and $N \in HS(G)$. Then the hesitant fuzzy neighbourhood N of x_λ is defined as if for an hestitant fuzzy set $U \in \tau$ such that $x_\lambda \in U \subset N$.

3. HESITANT FUZZY MINIMAL OPEN SETS

Definition 3.1. A proper nonzero HFO set h in HFTS (X, τ) is said to be HFMIO iff HFO set contained in h is h^0 or h.

Lemma 3.2. *Let (X, τ) be a HFTS.*
(i) If h_1 is HFMIO and h_2 is HFO in X, then $h_1 \cap h_2 = h^0$ or $h_1 \subset h_2$.
(ii) If h_1 and h_2 are HFMIO, then $h_1 \cap h_2 = h^0$ or $h_1 = h_2$.

Proof. (i) Let us assume that h_2 is HFO in X such that $h_1 \cap h_2 \neq h^0$. Since h_1 is HFMIO, and $h_1 \cap h_2 \subset h_1$, then $h_1 \cap h_2 = h_1$ implies that $h_1 \subset h_2$.

(ii) Suppose that $h_1 \cap h_2 \neq h^0$, then clearly from(ii), $h_1 \subset h_2$ and $h_2 \subset h_1$ as h_1 and h_2 are HFMIO. Hence $h_1 = h_2$. □

Theorem 3.3. *Let h and h_i are HFMIO sets for any $i \in M$. If $h \subseteq \bigcup_{i \in M} h_i$, then $h = h_j$ for any $j \in M$.*

Proof. Suppose $h \subseteq \bigcup_{i \in M} h_i$, then $h = h \cap \left(\bigcup_{i \in M} h_i \right) = \bigcup_{i \in M} (h \cap h_i)$. By deploying lemma 3.2(ii), $h \cap h_i = h^0$ or $h = h_i$ as h and h_i are HFMIO sets. If $h \cap h_i = h^0$, then $h = h^0$ which contradicts that h is a HFMIO set. Hence if $h \cap h_i \neq h^0$ then $h = h_j$ for any $j \in M$. □

Theorem 3.4. *If h and h_i are HFMIO sets for any $i \in M$ and $h \neq h_i$, then $h \cap \left(\bigcup_{i \in M} h_i \right) = h^0$ for any $i \in M$.*

Proof. Let $h \cap \left(\bigcup_{i \in M} h_i \right) \neq h^0$, then $h \cap h_i \neq h^0$ for any $i \in M$. By deploying lemma 3.2(ii), $h = h_i$ contradictory to $h \neq h_i$. Hence $h \cap \left(\bigcup_{i \in M} h_i \right) = h^0$. □

Theorem 3.5. *If h_i is a HFMIO for any $i \in M$ ($|M| \geq 2$) and $h_i \neq h_j$ for any distinct $i, j \in M$. Then* $\left(\bigcup_{i \in M \setminus \{j\}} h_i\right) \cap h_j = h^0$ *for any $j \in M$.*

Proof. Let $\left(\bigcup_{i \in M \setminus \{j\}} h_i\right) \cap h_j \neq h^0$. Then $\bigcup_{i \in M \setminus \{j\}} (h_i \cap h_j) \neq h^0 \Rightarrow (h_i \cap h_j) \neq h^0$. By lemma 3.2(ii), $h_i = h_j$, a contradiction. Hence $\left(\bigcup_{i \in M \setminus \{j\}} h_i\right) \cap h_j = h^0$ for any $j \in M$. □

Theorem 3.6. *If h_i is a HFMIO for any $i \in M$, ($|M| \geq 2$) and $h_i \neq h_j$ for any distinct $i, j \in M$. If K is a proper HFS of M, then* $\left(\bigcup_{i \in M \setminus K} h_i\right) \bigcap \left(\bigcup_{s \in K} h_s\right) = h^0$.

Proof. Let $\left(\bigcup_{i \in M \setminus K} h_i\right) \bigcap \left(\bigcup_{s \in K} h_s\right) \neq h^0$. It implies that $\bigcup (h_i \cap h_s) \neq h^0$ for $i \in M \setminus K$ and $s \in K$ implies that $h_i \cap h_s \neq h^0$ for some $i \in M$ and $s \in K$. By lemma3.2(ii), $h_i = h_s$, which is a contradiction. Hence $\left(\bigcup_{i \in M \setminus K} h_i\right) \bigcap \left(\bigcup_{s \in K} h_s\right) = h^0$. □

Theorem 3.7. *If h_i is a HFMIO for any $i \in M$ such that $h_i \neq h_j$ for any distinct $i, j \in M$. If S is a proper nonzero HFS of M, then* $\left[\bigcup_{i \in M \setminus k} h_i\right] \cap \left[\bigcup_{k \in S} h_k\right] = h^0$.

Proof. Assume that $\cup [h_i \cap h_k] \neq h^0$ for $i \in M \setminus k, k \in S$. Clearly, for some $i \in M, k \in S$ we have $[h_i \cap h_k] \neq h^0$. By deploying lemma 3.2(ii) $h_i = h_k$, a contradiction. □

Theorem 3.8. *If h_i and h_k are HFMIO sets for any $i \in M$ and $k \in S$ and if $\exists$ an $n \in S$ such that $h_i \neq h_n$ for any $i \in M$, then* $\left[\bigcup_{n \in K} h_n\right] \not\subset \left[\bigcup_{i \in M} h_i\right]$.

Proof. Assume that $\exists$ an $n \in S$ such that $h_i \neq h_n$ for any $i \in M$, then $\left[\bigcup_{n \in K} h_n\right] \subset \left[\bigcup_{i \in M} h_i\right]$.
$\Rightarrow h_n \subset \left[\bigcup_{i \in M} h_i\right]$ for some $n \in K$.
$\Rightarrow h_i \neq h_n$ for any $i \in M$, by theorem 3.3, which is a contradiction. Hence $\left[\bigcup_{n \in K} h_n\right] \not\subset \left[\bigcup_{i \in M} h_i\right]$. □

Theorem 3.9. *If h_i is a HFMIO for any $i \in M$ such that $h_i \neq h_j$ for any distinct $i, j \in M$, then* $\left[\bigcup_{k\in K} h_k\right] \subsetneqq \left[\bigcup_{i\in M} h_i\right]$ *for any proper nonzero subset K of M.*

Proof. Let $m \in M \backslash K$, then h_m is a HFMIO set of the family $\{h_m | m \in M\backslash K\}$ of HFMIO sets. Clearly $h_m \cap \left[\bigcup_{k\in K} h_k\right] = \bigcup_{k\in K} [h_m \cap h_k] = h^0$. Also $h_m \cap \left[\bigcup_{i\in M} h_i\right] = \bigcup_{i\in M} [h_m \cap h_i] = h_m$.

If $\left[\bigcup_{k\in K} h_k\right] = \left[\bigcup_{i\in M} h_i\right]$, then $h_m = h^0$ which is a contradiction that h_m is a HFMIO set. Hence $\left[\bigcup_{k\in K} h_k\right] \subsetneqq \left[\bigcup_{i\in M} h_i\right]$. □

Theorem 3.10. *If h_i is a HFMIO set for any $i \in M$ such that $h_i \neq h_j$ for any distinct $i, j \in M$, then*
(i) $h_j \subset \left[\bigcup_{i\in M\backslash\{j\}} h_i\right]^c$ *for some $j \in M$.*
(ii) $\bigcup_{i\in M\backslash\{j\}} h_i \neq h^1$ *for any $j \in M$.*

Proof. (i) By hypothesis, $h_i \neq h_j$ for any distinct $i, j \in M$.

By theorem 3.4, $\left[\bigcup_{i\in M} h_i\right] \cap h_j = h^0$ which is true for any $j \in M$.
$\Rightarrow \bigcup_{i\in M} [h_i \cap h_j] = h^0$
$\Rightarrow h_i \cap h_j = h^0$ (By Lemma 3.2(ii))
$\Rightarrow h_i \subset {h_j}^c$
$\Rightarrow \bigcup_{i\in M\backslash\{j\}} h_i \subset {h_j}^c$. Hence proved.

(ii) Let $j \in M$ such that $\bigcup_{i\in M\backslash\{j\}} h_i = h^1$
$\Rightarrow h_i = h^0$
$\Rightarrow h_i$ is not a HFMIO set, a contradiction. Hence $\bigcup_{i\in M\backslash\{j\}} h_i \neq h^1$ for any $j \in M$. □

Corollary 3.11. *If h_i is a HFMIO set for any $i \in M$ such that $h_i \neq h_j$ for any distinct $i, j \in M$. Then $h_i \cup h_j \neq h^1$ for any distinct $i, j \in M$.*

Proof. Similar to that of "Theorem 3.10(ii)." □

Theorem 3.12. *If h_i is a HFMIO sets for any $i \in M$ such that $h_i \neq h_j$ for any distinct $i, j \in M$, then* $h_j = \left[\bigcup_{i\in M} h_i\right] \cap \left[\bigcup_{i\in M\backslash\{j\}} h_i\right]^c$ *for any $j \in M$.*

Proof. For any $j \in M \Rightarrow \left[\bigcup_{i\in M} h_i\right] \bigcap \left[\bigcup_{i\in M\setminus\{j\}} h_i\right]^c = \left[\bigcup_{i\in M\setminus\{j\}} h_i \bigcup h_j\right] \cap \left[\bigcup_{i\in M\setminus\{j\}} h_i\right]^c$
$= \left[\left(\bigcup_{i\in M\setminus\{j\}} h_i\right) \cap \left(\bigcup_{i\in M\setminus\{j\}} h_i\right)^c\right] \bigcup \left[h_j \cap \left(\bigcup_{i\in M\setminus\{j\}} h_i\right)^c\right]$
$= h^0 \cup h_j$
$= h_j$ for any $j \in M$. □

Proposition 3.13. *Let G be a HFMIO set. If $x_\lambda \in G$, then $G \subset G_1$ for any hesitant fuzzy open neighbourhood G_1 of x_λ.*

Proof. Let G_1 be an hesitant fuzzy open neighbourhood of x_λ such that $G \not\subset G_1$. Clearly $G \cap G_1$ is an HFO such that $G \cap G_1 \subsetneq G$ and $G \cap G_1 \neq h^0$. This implies that G is a HFMIO set which a contradiction. □

Proposition 3.14. *Let G be a HFMIO set. Then*
$G = \bigcap\{G_1 |\ G_1 \quad is \quad an \quad hesitant fuzzy open neighbourhood \quad of \quad x_\lambda\}$ *for any* $x_\lambda \in G$.

Proof. By deploying Proposition 3.13 and as G is an hesitant fuzzy open neighbourhood of x_λ, we have
$G \subset \bigcap\{G_1 |\ G_1 \quad is \quad an \quad hesitant\ fuzzy\ open\ neighbourhood \quad of \quad x_\lambda\} \subset G$. This completes the proof. □

Theorem 3.15. *Let G be a HFMIO set. Then the following conditions are equivalent.*
(i) G is HFMIO set.
(ii)$G \subset Cl_H(K)$ for any nonzero subset K of G.
(iii)$Cl_H(G) = Cl_H(K)$ for any nonzero subset K of G.

Proof. $(i) \Rightarrow (ii)$: By deploying "proposition 3.13"for any $x_\lambda \in G$ and hesitant fuzzy open neighbourhood M of x_λ, we have $K = (G \cap K) \subset (M \cap K)$ for any proper nonzero hesitant fuzzy subset $K \subset G$. Therefore, we have $(M \cap K) \neq h^0$ and $x_\lambda \in Cl_H(K)$. It follows that $G \subset Cl_H(K)$.

$(ii) \Rightarrow (iii)$: For any proper hesitant fuzzy subset K of G, $Cl_H(G) \subset Cl_H(K)$. Also by(ii) $Cl_H(G) \subset Cl_H\left(Cl_H(K)\right) = Cl_H(K)$. Hence proved.

$(iii) \Rightarrow (i)$: Let us assume that G is not HFMIO. Then $\exists$ a proper HFO D such that $D \subset G$.Then $\exists\ y_\alpha \in G$ such that $y_\alpha \notin D$. Then $Cl_H(\{y_\alpha\}) \in D^c$ implies that $Cl_H(\{y_\alpha\}) \neq Cl_H(G)$, a contradiction. This completes our proof. □

4. HESITANT FUZZY MAXIMAL OPEN SETS AND ITS PROPERTIES

Definition 4.1. A proper nonzero HFO set h of a HFTS (X, τ) is said to HFMAO if any HFO set which contains h is either h or h^1.

Example 4.2. Let $X = \{a, b\}$ and $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4\}$ with

$$h_1(a) = [0.3, 1],\ h_1(b) = \{0.2, 0.6.0.9\}$$

$$h_2(a) = [0.3, 1),\ h_2(b) = \{0.2, 0.6.0.8\}$$
$$h_3(a) = [0.3, 1),\ h_3(b) = \{0.2, 0.6\}$$
$$h_4(a) = [0.3, 1],\ h_4(b) = \{0.2, 0.6, 0.8, 0.9\}$$

is a HFTS X. Here h_3 is an HFMIO set and h_4 is an HFMAO set.

Lemma 4.3. *Let (X, τ) be a HFTS. Then*
(i) If h_1 is a HFMAO and h_2 is HFO in X, then $h_1 \cup h_2 = h^1$ or $h_2 \subset h_1$.
(ii) If h_1 and h_3 are HFMAO sets, then either $h_1 \cup h_3 = h^1$ or $h_1 = h_3$.

Proof. (i) Assume that $h_2 \not\subset h_1$. Clearly, $h_1 \subset (h_1 \cup h_2)$ a contrary to h_1 is a HFMAO set if $h_1 \cup h_2 \neq h^1$. Hence, $h_1 \cup h_2 = h^1$.
(ii) Let h_1 and h_3 are HFMAO sets. Then from(i) $h_3 \subset h_1$ and $h_1 \subset h_3$ implies that $h_1 = h_3$. □

Theorem 4.4. *If h_1, h_2 and h_3 are HFMAO sets such that $h_1 \neq h_2$ and $(h_1 \cap h_2) \subset h_3$, then either $h_1 = h_3$ or $h_2 = h_3$.*

Proof. Suppose that h_1, h_2 and h_3 are HFMAO sets with $h_1 \neq h_2$, $(h_1 \cap h_2) \subset h_3$ and if $h_1 \neq h_3$, then
$(h_2 \cap h_3) \ = h_2 \cap (h_3 \cap h^1)$
$= h_2 \cap [h_3 \cap (h_1 \cup h_2)]$, by lemma 4.3(ii)
$= h_2 \cap [(h_3 \cap h_1) \cup (h_3 \cap h_2)]$
$= [h_2 \cap h_3 \cap h_1] \cup [h_2 \cap h_3 \cap h_2]$
$= [h_2 \cap h_1] \cup [h_2 \cap h_3]$
$= h_2 \cap [h_1 \cup h_3]$
$= h_2 \cap h^1$
$= h_2$
$(h_2 \cap h_3) = h_2 \Rightarrow h_2 \subset h_3$. As h_2 and h_3 are HFMAO sets, $h_3 \subset h_2$. Hence $h_2 = h_3$.

□

Theorem 4.5. *For any distinct HFMAO sets h_1, h_2, h_3*
$[h_1 \cap h_2] \not\subset [h_1 \cap h_3]$.

Proof. Consider $[h_1 \cap h_2] \subset [h_1 \cap h_3]$ for any distinct HFMAO sets h_1, h_2 and h_3. Then
$[h_1 \cap h_2] \cup [h_2 \cap h_3] \subset [h_1 \cap h_3] \cup [h_2 \cap h_3]$
$= [h_1 \cup h_3] \cap h_2 \subset [h_1 \cup h_2] \cap h_3$
$= h^1 \cap h_2 \subset h^1 \cap h_3$
$= h_2$ is contained in h_3
a contradiction to h_1, h_2 and h_3 are distinct . Hence $[h_1 \cap h_2] \not\subset [h_1 \cap h_3]$. □

Remark 4.1. Proofs of "Theorem 4.6, Corollary 4.7, Theorem 4.8 and Theorem 4.9" are similar to proofs of "Theorem 3.10, Corollary 3.11, Theorem 3.12 and Theorem 3.9" respectively. Hence the proofs are omitted.

Theorem 4.6. *If h_i is a HFMAO sets for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$, then*

(i) $\left[\bigcap\limits_{i \in M \setminus \{j\}} h_i\right]^c \subset h_j$ *for any* $j \in M$

(ii) $\bigcap\limits_{i \in M \setminus \{j\}} h_i \neq h^0$ *for any* $j \in M$.

Corollary 4.7. *If h_i is a HFMAO sets for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$ then,$h_i \cap h_j \neq h^0$ for any distinct $i, j \in M$.*

Theorem 4.8. *If h_i is a HFMAO sets for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$, then $h_j = \left[\bigcap\limits_{i \in M} h_i\right] \cup \left[\bigcap\limits_{i \in M \setminus \{j\}} h_i\right]^c$ for any $j \in M$.*

Theorem 4.9. *If h_i is a HFMAO sets for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$ and if K is a proper nonzero subset of M, then $\bigcap\limits_{i \in M} h_i \subsetneqq \bigcap\limits_{k \in K} h_k$.*

Theorem 4.10. *If h_i is a HFMAO sets for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$ and if $\bigcap\limits_{i \in M} h_i$ is a HFC set, then h_j is a HFC set for any $j \in M$.*

Proof. By "Theorem 4.8", we have $h_j = \left[\bigcap\limits_{i \in M} h_i\right] \cup \left[\bigcap\limits_{i \in M \setminus \{j\}} h_i\right]^c$ for any $j \in M$.

$h_j = \left[\bigcap\limits_{i \in M} h_i\right] \cup \left[\bigcup\limits_{i \in M \setminus \{j\}} h_i^c\right]$.

Since M is finite, $\bigcup\limits_{i \in M \setminus \{j\}} h_i^c$ is HFC. Hence h_j is HFC for any $j \in M$. □

Theorem 4.11. *If h_i is a HFMAO set for any $i \in M, M$ is a finite set and $h_i \neq h_j$ for any distinct $i, j \in M$. If $\bigcap\limits_{i \in M} h_i = h^0$,then $\{h_i / i \in M\}$ is the set of all HFMAO sets of HFTS X.*

Proof. Suppose that $\exists$ another HFMAO h_k of a HFTS X such that $h_k \neq h_i, \forall i \in M$. Clearly, $h^0 = \bigcap\limits_{i \in M} h_i = \bigcap\limits_{i \in (M \cup k) \setminus \{k\}} h_i \neq h^0$, by Theorem 4.6(ii), a contradiction.

Hence $\{h_i / i \in M\}$ is the family of all HFMAO sets of HFTS X. □

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