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REFINEMENTS OF FRACTIONAL VERSIONS OF HADAMARD INEQUALITY FOR LIOUVILLE-CAPUTO FRACTIONAL DERIVATIVES

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Abstract. The Hadamard type inequalities for fractional integral operators of convex functions are studied at very large scale. This paper provides the Hadamard type inequalities for refined $(\alpha,h\text{-}m)$-convex functions by utilizing Liouville-Caputo fractional (L-CF) derivatives. These inequalities give refinements of already existing (L-CF) inequalities of Hadamard type for many well known classes of functions provided the function h is bounded above by $\frac{1}{\sqrt{2}}$.

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1. Introduction

Convex functions are very important and useful in the study of integral inequalities. In recent years, integral inequalities for various kinds of convex functions have been published. Due to generalizations and extensions of convex functions, it becomes possible to get generalizations and extensions of classical inequalities. For example, Bombardelli and Varosanec [5] gave Hermite-Hadamard-Fejér inequalities for h-convex functions. Chen and Wu [9] proved Hermite-Hadamard type inequalities for harmonically convex functions. Dragomir [12] established Ostrowski like inequalities for convex functions. İşcan [15] proved Hermite-Hadamard type inequalities for harmonically (α, m)-convex functions. İşcan [16] proved Ostrowski type inequalities for p-convex functions. Kunt and İşcan [19] proved Hermite-Hadamard-Fejér type inequalities for p-convex functions. Mehreen and Anwar [21] proved Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions. Özdemir

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et al. [22] have obtained Hadamard inequality by (h-m)-convexity. Özdemir et al. [23] have established Ostrowski's type inequalities for (α, m)-convex functions. Obeidat et al. [24] have proved Fejér and Hermite-Hadamard type inequalities involving h-convex functions. Sezer [26] gave the Hermite-Hadamard inequality for s-convex function in the third sense.

Motivated by recent research articles, we aim to present the (L-CF) derivative inequalities of Hadamard type for refined (α,h-m)-convex functions. In the following we give the definitions of convex function, (α,h-m)-convex function, refined (α,h-m)-convex function, (L-CF) derivative operators and beta function respectively.

Definition 1.1. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called convex function, if the undermentioned inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \ \forall t \in [0,1], \ a, b \in I. \tag{1}$$

The classical Hadamard inequality is an interpretation of convex function. It is stated as follows:

Theorem 1.2. *A convex function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ satisfies the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

where $a, b \in I$ and $a < b$. If order in (2) is reversed, then it holds for concave function.

Many authors have studied the Hadamard inequality for different fractional integral operators. For example, Sarikaya et al. gave the Hadamard inequality for classical Riemann-Liouville fractional integral operators in [25]. Kang et al. gave the Hadamard inequality for (L-CF) derivatives in [17]. Mehmood et al. proved the Hadamard inequality for generalized fractional integral operators containing Mittag-Leffler functions in [20]. Anastassiou have obtained generalized fractional Hermite-Hadamard inequalities involving m-convexity and (s,m)-convexity in [1]. Agarwal et al. presented certain Hermite-Hadamard type inequalities via generalized k-fractional integrals in [3]. Chen has established Hermite-Hadamard type inequalities for R-L fractional integrals via two kinds of convexity in [8]. Chen and Katugampola have introduced Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals in [7]. Farid et al. presented fractional integral inequalities of Hadamard type for m-convex function via Caputo k-fractional derivatives in [13]. Farid et al. have introduced k-fractional integral inequalities of Hadamard type for (h-m)-convex functions in [14].

There are many types of convex functions which have been formulated from the analytical definition of convex functions, one of them is (α,h-m)-convex function given in the following definition:

Definition 1.3. [14] Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A function $f : I \to \mathbb{R}$ is called $(\alpha$,h-$m)$-convex functions, if the undermentioned inequality holds:

$$f(ta+m(1-t)b) \leq h(t^{\alpha})f(a)+mh(1-t^{\alpha})f(b),\ \forall t \in [0,1],\ a,b \in I,\ (\alpha,m) \in [0,1]^2. \tag{3}$$

Recently, in [28], Wu et al. introduced a new class of convex functions, namely refined $(\alpha$,h-$m)$-convex functions defined as follows:

Definition 1.4. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A function $f : I \to \mathbb{R}$ is called refined $(\alpha$,h-$m)$-convex functions, if the undermentioned inequality holds:

$$f(ta+m(1-t)b) \leq h(t^{\alpha})h(1-t^{\alpha})(f(a)+mf(b)),\ \forall t \in [0,1],\ a,b \in I, \tag{4}$$
$$(\alpha,m) \in [0,1]^2.$$

Remark 1.1. Several definitions of convex functions are reproduced from refined $(\alpha$,h-$m)$-convex functions. For example, for $\alpha = 1$, $h(t) = t^s$ and $m = 1$ in (4) the definition of $s - tgs$-convex functions is reproduced given in [4], for $\alpha = 1$, $h(t) = t^{-s}$ and $m = 1$ in (4) the definition of Godunova-Levin-Dragomir tgs-convex functions is reproduced given in [4], for $\alpha = 1$, $h(t) = t$ and $m = 1$ in (4) the definition of tgs-convex functions is reproduced given in [27], for $\alpha = 1$, $h(t) = 1$ and $m = 1$ in (4) the definition of p-functions is reproduced given in [11].

Remark 1.2. A lot of new definitions of convex functions can be deduced from refined $(\alpha$,h-$m)$-convex functions for different choices of h, m and α, we leave it for interested readers.

Fractional calculus is the extension of concepts of classical calculus related to derivatives and integrals. In [6], Caputo made the most significant contribution to fractional calculus by giving improved formulas of fractional derivatives. The (L-CF) derivatives are defined as follows:

Definition 1.5. ([6, 18]) Let $f \in AC^n[a,b]$ and $n = [\Re(\beta)] + 1$. Then (L-CF) derivatives of order $\beta \in \mathbb{C}$, $\Re(\beta) > 0$ of f are defined as follows:

$${}^{C}D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\beta-n+1}}dt, \quad x > a \tag{5}$$

and

$${}^{C}D^{\beta}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\beta-n+1}}dt, \quad x < b. \tag{6}$$

If $\beta = n \in \{1,2,3,...\}$ and usual derivative of order n exists, then (L-CF) derivative $({}^{C}D^{\beta}_{a+}f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D^{\beta}_{b-}f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular, we have

$$({}^C D^0_{a+} f)(x) = ({}^C D^0_{b-} f)(x) = f(x) \tag{7}$$

where $n = 1$ and $\beta = 0$.

We also use the well-known beta function defined as follows:

Definition 1.6. [10] The beta function of two variables x and y is defined as:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Re(x), \Re(y) > 0.$$

In the upcoming section, we give two versions of the Hadamard inequalities for refined (α,h-m)-convex functions. To prove these inequalities (L-CF) derivatives are utilized. Further, the Hadamard inequalities for refined (h-m)-convex functions, refined (α-m)-convex functions, refined (s-m)-convex functions, refined h-convex functions and refined m-convex functions are given. In whole paper, we assume f and g be real valued and non-negative functions defined on I. Also, I and J are the intervals in $\mathbb{R}$ and $(0,1) \subseteq J$.

2. Main Results

First, we give the Hadamard inequality for refined (α,h-m)-convex functions via (L-CF) derivatives. Also from now to onward we use the notation $RC^m_h(\alpha)$ for refined (α,h-m)-convex.

Theorem 2.1. *Let f be a positive, integrable and $RC^m_h(\alpha)$ functions. Then the following inequality for (L-CF) derivatives holds:*

$$\frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})} \leq \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D^\beta_{v-} f\right)\left(\frac{u}{m}\right) + \left({}^C D^\beta_{u+} f\right)(mv)\right]$$
$$\leq (n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(n-\beta-1)}dt. \tag{8}$$

Proof. Since $f^{(n)}$ is $RC^m_h(\alpha)$ function for $x, y \in [u,v]$, $t \in [0,1]$. Then, we have

$$f^{(n)}\left(\frac{mx+y}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)(mf^{(n)}(x) + f^{(n)}(y)). \tag{9}$$

Let $x = (1-t)\frac{u}{m} + tv \leq v$ and $y = m(1-t)v + tu \geq u$ in (9), we have

$$f^{(n)}\left(\frac{u+mv}{2}\right)$$
$$\leq h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mf^{(n)}\left((1-t)\frac{u}{m}+tv\right) + f^{(n)}(m(1-t)v+tu)\right).$$

By multiplying above inequality with $t^{n-\beta-1}$ and then doing integration on $[0,1]$, the following inequality is yielded

$$f^{(n)}\left(\frac{u+mv}{2}\right)\int_0^1 t^{n-\beta-1}dt \le h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right) \tag{10}$$
$$\times\left[m\int_0^1 f^{(n)}\left((1-t)\frac{u}{m}+tv\right)t^{n-\beta-1}dt+\int_0^1 f^{(n)}(m(1-t)v+tu)t^{n-\beta-1}dt\right].$$

The above inequality takes the following form by considering change of variables

$$\frac{1}{n-\beta}f^{(n)}\left(\frac{u+mv}{2}\right)$$
$$\le h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(\frac{m}{(mv-u)^{n-\beta}}\int_{\frac{u}{m}}^{v}\left(x-\frac{u}{m}\right)^{n-\beta-1}f^{(n)}(x)dx\right. \tag{11}$$
$$\left.+\frac{1}{(mv-u)^{n-\beta}}\int_u^{mv}(mv-y)^{n-\beta-1}f^{(n)}(y)dy\right).$$

Multiplying by $(n-\beta)$ and using Definition 1.1, the first inequality of (8) can be obtained. Again by using $RC_h^m(\alpha)$ity of $f^{(n)}$, we have

$$f^{(n)}(tu+m(1-t)v)+mf^{(n)}\left((1-t)\frac{u}{m}+tv\right)$$
$$\le h(t^\alpha)h(1-t^\alpha)\left(f^{(n)}(u)+2mf^{(n)}(v)+m^2f^{(n)}\left(\frac{u}{m^2}\right)\right).$$

Multiplying the above inequality by $t^{n-\beta-1}$ and integrating over $[0,1]$, then by using change of variables and Definition 1.1, the second inequality of (8) is obtained. □

The extension of inequality (8) is given in the following result:

Theorem 2.2. *Let $h(t)\le\frac{1}{\sqrt{2}}$ along with same assumptions as stated in Theorem 2.1. Then the following inequality is valid:*

$$2f^{(n)}\left(\frac{u+mv}{2}\right)\le\frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})}\le\frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D_{v^-}^{\beta}f\right)\left(\frac{u}{m}\right)\right.$$
$$\left.+\left({}^C D_{u^+}^{\beta}f\right)(mv)\right]\le(n-\beta)\left(f^{(n)}(u)+2mf^{(n)}(v)+m^2f^{(n)}\left(\frac{u}{m^2}\right)\right)$$
$$\times\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(n-\beta-1)}dt\le\frac{1}{2}\left(f^{(n)}(u)+2mf^{(n)}(v)+m^2f^{(n)}\left(\frac{u}{m^2}\right)\right). \tag{12}$$

Proof. It is given that $h(t)\le\frac{1}{\sqrt{2}}$, so we can write

$$\int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(n-\beta-1)}dt\le\frac{1}{2(n-\beta)}$$

and

$$\frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})}\ge 2f^{(n)}\left(\frac{u+mv}{2}\right).$$

By using these inequalities along with inequality (8), we will get inequality (12). □

Corollary 2.3. *By using $\alpha = 1$ in (8), the inequality for (L-CF) derivatives of $RC_h^m(1)$ functions is obtained:*

$$\frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{(h(\frac{1}{2}))^2} \le \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D_{v^-}^{\beta} f\right)\left(\frac{u}{m}\right) + \left({}^C D_{u^+}^{\beta} f\right)(mv)\right]$$
$$\le (n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)\int_0^1 h(t)h(1-t)t^{(n-\beta-1)}dt. \tag{13}$$

Corollary 2.4. *By using $\alpha = 1$ in (12), the inequality for (L-CF) derivatives of $RC_h^m(1)$ functions is obtained:*

$$2f^{(n)}\left(\frac{u+mv}{2}\right) \le \frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{(h(\frac{1}{2}))^2} \le \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D_{v^-}^{\beta} f\right)\left(\frac{u}{m}\right)\right.$$
$$\left. + \left({}^C D_{u^+}^{\beta} f\right)(mv)\right] \le (n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)$$
$$\times \int_0^1 h(t)h(1-t)t^{(n-\beta-1)}dt \le \frac{1}{2}\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right). \tag{14}$$

Corollary 2.5. *By using $h(t) = t$ in (8), the inequality for (L-CF) derivatives of $RC_{I_d}^m(\alpha)$ functions is obtained:*

$$\frac{2^{2\alpha} f^{(n)}\left(\frac{u+mv}{2}\right)}{2^\alpha - 1} \le \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D_{v^-}^{\beta} f\right)\left(\frac{u}{m}\right) + \left({}^C D_{u^+}^{\beta} f\right)(mv)\right]$$
$$\le \frac{\alpha(n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)}{(\alpha+n-\beta)(2\alpha+n-\beta)}. \tag{15}$$

Corollary 2.6. *By using $h(t) = t$ in (12), the inequality for (L-CF) derivatives of $RC_{I_d}^m(\alpha)$ function is obtained:*

$$2f^{(n)}\left(\frac{u+mv}{2}\right) \le \frac{2^{2\alpha} f^{(n)}\left(\frac{u+mv}{2}\right)}{2^\alpha - 1} \le \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D_{v^-}^{\beta} f\right)\left(\frac{u}{m}\right)\right.$$
$$\left. + \left({}^C D_{u^+}^{\beta} f\right)(mv)\right] \le \frac{\alpha(n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)}{(\alpha+n-\beta)(2\alpha+n-\beta)}$$
$$\le \frac{1}{2}\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right). \tag{16}$$

Corollary 2.7. *By using $\alpha = 1$ and $h(t) = t^s$ in (8), the inequality for (L-CF) derivatives of $RC^m_{t^s}(1)$ functions is obtained:*

$$2^{2s} f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D^{\beta}_{v^-} f\right)\left(\frac{u}{m}\right) + \left({}^C D^{\beta}_{u^+} f\right)(mv)\right]$$
$$\leq (n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)\beta(s+1, n-\beta+s). \tag{17}$$

Corollary 2.8. *By using $\alpha = 1$ and $h(t) = t^s$ in (12), the inequality for (L-CF) derivatives of $RC^m_{t^s}(1)$ functions is obtained:*

$$2f^{(n)}\left(\frac{u+mv}{2}\right) \leq 2^{2s} f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D^{\beta}_{v^-} f\right)\left(\frac{u}{m}\right)\right.$$
$$\left. + \left({}^C D^{\beta}_{u^+} f\right)(mv)\right] \leq (n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)\beta(s+1, n-\beta+s) \tag{18}$$
$$\leq \frac{1}{2}\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right).$$

Corollary 2.9. *By using $\alpha = 1$ and $m = 1$ in (8), the inequality for (L-CF) derivatives of $RC^1_h(1)$ functions is obtained:*

$$\frac{f^{(n)}\left(\frac{u+v}{2}\right)}{(h(\frac{1}{2}))^2} \leq \frac{\Gamma(n-\beta+1)}{(v-u)^{n-\beta}}\left[(-1)^n\left({}^C D^{\beta}_{v^-} f\right)(u) + \left({}^C D^{\beta}_{u^+} f\right)(v)\right]$$
$$\leq 2(n-\beta)\left(f^{(n)}(u) + f^{(n)}(v) +\right)\int_0^1 h(t)h(1-t)t^{(n-\beta-1)}dt. \tag{19}$$

Corollary 2.10. *By using $h(t) = t$ and $\alpha = 1$in (8), the inequality for (L-CF) derivatives of $RC^m_{I_d}(1)$ functions is obtained:*

$$4f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[(-1)^n m^{n-\beta+1}\left({}^C D^{\beta}_{v^-} f\right)\left(\frac{u}{m}\right) + \left({}^C D^{\beta}_{u^+} f\right)(mv)\right]$$
$$\leq \frac{(n-\beta)\left(f^{(n)}(u) + 2mf^{(n)}(v) + m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right)}{(1+n-\beta)(2+n-\beta)}. \tag{20}$$

Theorem 2.11. *Let f be a positive, integrable and $RC^m_h(\alpha)$ function. Then the following inequality for (L-CF) derivatives holds:*

$$\frac{1}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})} f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[\left({}^C D^{\beta}_{\frac{u+mv}{2}^+} f\right)(mv)\right.$$
$$\left. + m^{n-\beta+1}(-1)^n\left({}^C D^{\beta}_{\frac{u+mv}{2m}^-} f\right)\left(\frac{u}{m}\right)\right] \leq (n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)\right.$$
$$\left. + f^{(n)}(u) + 2mf^{(n)}(v)\right)\int_0^1 h\left(\frac{t}{2}\right)^{\alpha} h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right) t^{n-\beta-1}dt. \tag{21}$$

Proof. From $RC^m_h(\alpha)$ity of $f^{(n)}$, we have

$$f^{(n)}\left(\frac{mx+y}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) h\left(1-\frac{1}{2^\alpha}\right)(mf^{(n)}(x) + f^{(n)}(y)). \tag{22}$$

Let $x=\frac{u}{m}\left(\frac{2-t}{2}\right)+\frac{vt}{2}$ and $y=\frac{ut}{2}+m\frac{(2-t)}{2}v$ in (9), $t\in[0,1]$. Then we have

$$f^{(n)}\left(\frac{u+mv}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mf^{(n)}\left(\frac{u}{m}\left(\frac{2-t}{2}\right)+\frac{vt}{2}\right)+f^{(n)}\left(\frac{ut}{2}+m\frac{(2-t)}{2}v\right)\right).$$

By multiplying above inequality with $t^{n-\beta-1}$ and then doing integration on $[0,1]$, the following inequality is yielded

$$f^{(n)}\left(\frac{u+mv}{2}\right)\int_0^1 t^{n-\beta-1}dt \leq h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right) \tag{23}$$
$$\left[\int_0^1 mf^{(n)}\left(\frac{u}{m}\left(\frac{2-t}{2}\right)+\frac{vt}{2}\right)t^{n-\beta-1}dt+\int_0^1 f^{(n)}\left(\frac{ut}{2}+m\frac{(2-t)}{2}v\right)t^{n-\beta-1}dt\right].$$

The above inequality takes the following form by considering change of variables

$$\frac{1}{n-\beta}f^{(n)}\left(\frac{u+mv}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2^{n-\beta}}{(mv-u)^{n-\beta}}\int_{\frac{u}{m}}^{\frac{u+mv}{2m}} m^{n-\beta+1}\left(x-\frac{u}{m}\right)^{n-\beta-1}f^{(n)}(x)dx + \frac{2^{n-\beta}}{(mv-u)^{n-\beta}}\int_{\frac{u+mv}{2}}^{mv}(mv-y)^{n-\beta-1}f^{(n)}(y)dy\right). \tag{24}$$

Which after using Definition 1.1 and multiplying the resulting inequality with $(n-\beta)$, the first inequality of (21) is obtained.
Again by using $RC_h^m(\alpha)$ity of $f^{(n)}$, we have

$$mf^{(n)}\left(\frac{u}{m}\left(\frac{2-t}{2}\right)+\frac{vt}{2}\right)+f^{(n)}\left(\frac{ut}{2}+m\frac{(2-t)}{2}v\right) \leq h\left(\frac{t}{2}\right)^\alpha h\left(1-\left(\frac{t}{2}\right)^\alpha\right)\left(m^2f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right).$$

Multiplying the above inequality by $t^{n-\beta-1}$ and integrating over $[0,1]$, then by using change of variables and Definition 1.1, the second inequality of (21) is obtained. □

The extension of inequality (21) is given in the following result.

Theorem 2.12. *Let $h(t)\leq\frac{1}{\sqrt{2}}$ along with same assumptions as stated in Theorem 2.11. Then the following inequality is valid:*

$$2f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{1}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})}f^{(n)}\left(\frac{u+mv}{2}\right)$$
$$\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[\left({}^CD^\beta_{\frac{u+mv}{2}+}f\right)(mv)+m^{n-\beta+1}(-1)^n\left({}^CD^\beta_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)\right]$$
$$\leq (n-\beta)\left(m^2f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\int_0^1 h\left(\frac{t}{2}\right)^\alpha h\left(1-\left(\frac{t}{2}\right)^\alpha\right)t^{n-\beta-1}dt$$
$$\leq \frac{1}{2}\left(m^2f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right). \tag{25}$$

Proof. It is given $h(t) \leq \frac{1}{\sqrt{2}}$, so we have

$$\int_0^1 h\left(\frac{t}{2}\right)^\alpha h\left(1-\left(\frac{t}{2}\right)^\alpha\right) t^{n-\beta-1}dt \leq \frac{1}{2(n-\beta)}$$

and

$$\frac{1}{h(\frac{1}{2^\alpha})h(1-\frac{1}{2^\alpha})} f^{(n)}\left(\frac{u+mv}{2}\right) \geq 2f^{(n)}\left(\frac{u+mv}{2}\right).$$

By using these inequalities along with inequality (21), we will get inequality (25). □

Corollary 2.13. *By using $\alpha = 1$ in (21), the inequality for (L-CF) derivatives of $RC_h^m(1)$ functions is obtained:*

$$\begin{aligned}
&\frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{(h(\frac{1}{2}))^2} \leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\Big[m^{n-\beta+1}(-1)^n\left({}^C D^\beta_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)\\
&+\left({}^C D^\beta_{\frac{u+mv}{2}+}f\right)(mv)\Big] \leq (n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\\
&\times \int_0^1 h\left(\frac{t}{2}\right)h\left(1-\left(\frac{t}{2}\right)\right)t^{n-\beta-1}dt.
\end{aligned} \tag{26}$$

Corollary 2.14. *By using $\alpha = 1$ in (25), the inequality for (L-CF) derivatives of $RC_h^m(1)$ functions is obtained:*

$$\begin{aligned}
&2f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{f^{(n)}\left(\frac{u+mv}{2}\right)}{(h(\frac{1}{2}))^2}\\
&\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[\left({}^C D^\beta_{\frac{u+mv}{2}+}f\right)(mv)+m^{n-\beta+1}(-1)^n\left({}^C D^\beta_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)\right]\\
&\leq (n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\int_0^1 h\left(\frac{t}{2}\right)h\left(1-\left(\frac{t}{2}\right)\right)t^{n-\beta-1}dt\\
&\leq \frac{1}{2}\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right).
\end{aligned} \tag{27}$$

Corollary 2.15. *By using $h(t) = t$ in (21), the inequality for (L-CF) derivatives of $RC_{I_d}^m(\alpha)$ functions is obtained:*

$$\begin{aligned}
&\frac{2^{2\alpha}f^{(n)}\left(\frac{u+mv}{2}\right)}{2^\alpha-1}\\
&\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^C D^\beta_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)+\left({}^C D^\beta_{\frac{u+mv}{2}+}f\right)(mv)\right]\\
&\leq \frac{(n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\left[2^\alpha(2\alpha+n-\beta)-(\alpha+n-\beta)\right]}{2^{2\alpha}(2\alpha+n-\beta)(\alpha+n-\beta)}.
\end{aligned} \tag{28}$$

Corollary 2.16. *By using $h(t) = t$ in (25), the inequality for (L-CF) derivatives of $RC^m_{I_d}(\alpha)$ functions is obtained:*

$$
\begin{aligned}
&2f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{2^{2\alpha} f^{(n)}\left(\frac{u+mv}{2}\right)}{2^{\alpha}-1} \\
&\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^{C}D^{\beta}_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)+\left({}^{C}D^{\beta}_{\frac{u+mv}{2}+}f\right)(mv)\right] \\
&\leq \frac{(n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\left[2^{\alpha}(2\alpha+n-\beta)-(\alpha+n-\beta)\right]}{2^{2\alpha}(2\alpha+n-\beta)(\alpha+n-\beta)} \\
&\leq \frac{1}{2}\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right).
\end{aligned} \tag{29}
$$

Corollary 2.17. *By using $\alpha = 1$ and $h(t) = t^s$ in (21), the inequality for (L-CF) derivatives of $RC^m_{t^s}(1)$ functions is obtained:*

$$
\begin{aligned}
&2^{2s} f^{(n)}\left(\frac{u+mv}{2}\right) \\
&\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^{C}D^{\beta}_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)+\left({}^{C}D^{\beta}_{\frac{u+mv}{2}+}f\right)(mv)\right] \\
&\leq \frac{(n-\beta)}{2^{2s}}\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\int_0^1 t^{s+n-\beta-1}(2-t)^s dt.
\end{aligned} \tag{30}
$$

Corollary 2.18. *By using $\alpha = 1$ and $h(t) = t^s$ in (25), the inequality for (L-CF) derivatives of $RC^m_{t^s}(1)$ functions is obtained:*

$$
\begin{aligned}
&2f^{(n)}\left(\frac{u+mv}{2}\right) \leq 2^{2s} f^{(n)}\left(\frac{u+mv}{2}\right) \\
&\leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[\left({}^{C}D^{\beta}_{\frac{u+mv}{2}+}f\right)(mv)+m^{n-\beta+1}(-1)^n\left({}^{C}D^{\beta}_{\frac{u+mv}{2m}-}f\right)\left(\frac{u}{m}\right)\right] \\
&\leq \frac{(n-\beta)}{2^{2s}}\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right)\int_0^1 t^{s+n-\beta-1}(2-t)^s dt \\
&\leq \frac{1}{2}\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right)+f^{(n)}(u)+2mf^{(n)}(v)\right).
\end{aligned} \tag{31}
$$

Corollary 2.19. *By using $\alpha = 1$ and $m = 1$ in (21), the inequality for (L-CF) derivatives of $RC_h^1(1)$ functions is obtained:*

$$\frac{f^{(n)}\left(\frac{u+v}{2}\right)}{(h(\frac{1}{2}))^2} \leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(v-u)^{n-\beta}}\left[(-1)^n\left({}^C D^{\beta}_{\frac{u+v}{2}-} f\right)(u) + \left({}^C D^{\beta}_{\frac{u+v}{2}+} f\right)(v)\right] \leq 2(n-\beta)\left(f^{(n)}(u) + f^{(n)}(v)\right) \times \int_0^1 h\left(\frac{t}{2}\right) h\left(1-\left(\frac{t}{2}\right)\right) t^{n-\beta-1} dt. \tag{32}$$

Corollary 2.20. *By using $h(t) = t$ and $\alpha = 1$in (21), the inequality for (L-CF) derivatives of $RC_{I_d}^m(1)$ functions is obtained:*

$$4f^{(n)}\left(\frac{u+mv}{2}\right) \leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(mv-u)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^C D^{\beta}_{\frac{u+mv}{2m}-} f\right)\left(\frac{u}{m}\right) + \left({}^C D^{\beta}_{\frac{u+mv}{2}+} f\right)(mv)\right] \leq \frac{(n-\beta)(3+n-\beta)\left(m^2 f^{(n)}\left(\frac{u}{m^2}\right) + f^{(n)}(u) + 2mf^{(n)}(v)\right)}{4(2+n-\beta)(1+n-\beta)}. \tag{33}$$

3. Conclusions

This paper investigates the refinements of inequalities for Liouville-Caputo fractional derivatives. Hadamard type inequalities are established for refined convex functions utilizing Liouville-Caputo fractional derivatives. Many known inequalities and their refinements are special cases of results of this paper. Also, some new inequalities are deduced from main results.

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