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CONTROLLABILITY OF IMPULSIVE NEUTRAL STOCHASTIC FUNCTIONAL INTEGRODIFFERENTIAL SYSTEM VIA RESOLVENT OPERATOR

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Abstract. This paper is concerned by the controllability results of impulsive neutral stochastic functional integrodifferential equations (INSFIDEs) driven by fractional Brownian motion with infinite delay in a real separable Hilbert space. The controllability results are obtained using stochastic analysis, Krasnoselkii fixed point method and the theory of resolvent operator in the sense of Grimmer. A practical example is provided to illustrate the viability of the abstract result of this work.

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1. Introduction

Over the past few decades, controllability is one of the fundamental concept in mathematical control theory and plays an important role in both deterministic and stochastic control theories. Controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [9, 10, 19, 1, 20] and references therein.

Fractional Brownian motion (fBm) $\left\{\mathtt{B}^{\mathrm{H}} : \mathrm{t} \in \mathbb{R}\right\}$ is a Gaussian stochastic process, which depends on a parameter $\mathrm{H} \in (0,1)$ called the Hurst index. This stochastic process has self-similarity, stationary increments, and long-range dependence properties. It is known that fBm is generalization of Brownian motion

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and it reduces to a standard Brownian motion when $\mathrm{H} = \frac{1}{2}$. FBm have attracted significant interest [9, 12, 13] and for potential applications in finance market, telecommunications networks, biology and other fields see [22, 15]. For additional details on fBm, we fefer the monographs [16] and the articles therein [1, 19, 6].

The asymptotic behaviours along with the existence and uniqueness for class of INSFIDEs driven by fBm with delays and infinite delays are established in [2, 7] with the mild solutions. Moreover, several upcoming researchers are keen interest to study the solvation of control problems in the field of stochastic systems. A through survey of literature revals that a very little work has been done for the fBm in stochastic control problems. Chen [6] concerned the approximate controllability for semilinear stochastic equations with fBm. Several researchers reported the use of fBm in stochastic integrodifferential equations (see refer to [12, 13, 19] and references therein). Moreover, the controllability of INSFIDEs systems with infinite delay driven by fBm is an untreated topic in the literature so far. Thus, we will make the first attempt to study such problem in this paper.

The goal of present research work is focus to study the controllability of INSFIDEs of the form:

$$
\begin{aligned}
d\left[\mathtt{x}(\mathrm{t})-\Gamma(\mathrm{t},\mathtt{x}_{\mathrm{t}})\right] &= \mathfrak{A}\left[\mathtt{x}(\mathrm{t})-\Gamma(\mathrm{t},\mathtt{x}_{\mathrm{t}})+\mathfrak{B}\mathbf{u}(\mathrm{t})\right]dt \\
&+ \int_0^{\mathrm{t}} O(\mathrm{t}-s)\left[\mathtt{x}(s)-\Gamma(s,\mathtt{x}_s)\right]ds d\mathrm{t}+\Lambda(\mathrm{t},\mathtt{x}_{\mathrm{t}})dt \\
&+ \Xi(\mathrm{t})d\mathtt{B}^{\mathrm{H}}(\mathrm{t}),\ \mathrm{t}\in[0,\mathbb{T}], \qquad (1)\\
\Delta\mathtt{x}\big|_{\mathrm{t}-\mathrm{t}_{\mathrm{k}}} &= \mathtt{x}(\mathrm{t}_{\mathrm{k}}^{+})-\mathtt{x}(\mathrm{t}_{\mathrm{k}}^{-})=\mathrm{I}_{\mathrm{k}}(\mathtt{x}(\mathrm{t}_{\mathrm{k}}^{-})),\ \mathrm{k}=1,...,m, \qquad (2)\\
\mathtt{x}(\mathrm{t}) &= \varphi(\mathrm{t})\in\mathcal{L}_2^0(\Omega,\mathscr{B}_{\mathfrak{H}}),\ \text{for } a.e.\ \mathrm{t}\in(-\infty,0]. \qquad (3)
\end{aligned}
$$

Where, $\mathfrak{A}$ is the infinitesimal generator of a strongly continuous bounded linear operator $(\mathbb{T}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on a Hilbert space $\mathbb{X}$ with domain $\mathfrak{D}(\mathfrak{A})$; $O(\mathrm{t})$ is a closed linear operator on $\mathbb{X}$ with domain $\mathfrak{D}(O)\supset\mathfrak{D}(\mathfrak{A})$; $\mathtt{B}^{\mathrm{H}}$ is a fBm with Hurst parameter $\mathrm{H} > \frac{1}{2}$ on a real and separable Hilbert space $\mathbb{Y}$; and $\mathbf{u}(\cdot)$ denots the control function takes values $\mathcal{L}^2([0,\mathbb{T}],\mathbb{U})$, the Hilbert space of admissionble control functions for a separable Hilbert space $\mathbb{U}$; and $\mathfrak{B}$ is a bounded linear operator from $\mathbb{U}$ into $\mathbb{X}$. The history $\mathtt{x}_{\mathrm{t}} : (-\infty,0]\to\mathbb{X}$, $\mathtt{x}_{\mathrm{t}}(\theta)=\mathtt{x}(\mathrm{t}+\theta)$, belongs to an abstract phase space $\mathscr{B}_{\mathfrak{H}}$ defined axiomatically, and $\Gamma,\Lambda : [0,\mathbb{T}]\times\mathscr{B}_{\mathfrak{H}}\to\mathbb{X}$, $\Xi : [0,\mathbb{T}]\to\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$, are appropriate functions, where $\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$ denotes the space of all Q-Hilbert-Schmit operators from $\mathbb{Y}$ into $\mathbb{X}$. Moreover, the fixed moments of time t_{k} satisfy $0<\mathrm{t}_1<\mathrm{t}_2<...<\mathrm{t}_m<\mathbb{T}$, $\mathtt{x}(\mathrm{t}_{\mathrm{k}}^{-})$ and $\mathtt{x}(\mathrm{t}_{\mathrm{k}}^{+})$ represent the left and right limits of $\mathtt{x}(\mathrm{t})$ at time t_{k} respectively. $\Delta\mathtt{x}(\mathrm{t}_{\mathrm{k}})$ denotes the jump in the state $\mathtt{x}$ at time t_{k} with $\mathrm{I} : \mathbb{X}\to\mathbb{X}$ determining the size of the jump.

2. Preliminaries

Fix a time interval $[0,\mathbb{T}]$ and let $(\Omega,\Im,\mathbb{P})$ be a complete probability space and $\left\{\beta^{\mathrm{H}}(\mathrm{t}) : \mathrm{t}\in[0,\mathbb{T}]\right\}$ be a one-dimensional fBm with Hurst parameter $\mathrm{H}\in(\frac{1}{2},1)$. By definition, β^{H} is a centered Gaussian process with covariance function

$$R_{\mathrm{H}(\mathrm{t},s)} \;=\; \mathbf{E}\left[\beta^{\mathrm{H}}(\mathrm{t})\beta^{\mathrm{H}}(s)\right]=\frac{1}{2}\left(|\mathrm{t}|^{2\mathrm{H}}+|s|^{2\mathrm{H}}-|\mathrm{t}-s|^{2\mathrm{H}}\right).$$

Moreover, β^{H} has the following Wiener integral representation

$$\beta^{\mathrm{H}}(\mathrm{t}) \;=\; \int_0^{\mathrm{t}}\mathrm{K}_{\mathrm{H}}(\mathrm{t},s)d\beta(s),$$

where $\beta=\{\beta(\mathrm{t});\mathrm{t}\in[0,\mathbb{T}]\}$ is a Wiener process and kernel $\mathrm{K}_{\mathrm{H}}(\mathrm{t},s)$ is given by

$$\mathrm{K}_{\mathrm{H}}(\mathrm{t},s) \;=\; c_{\mathrm{H}}S^{\frac{1}{2}=\mathrm{H}}\int_0^{\mathrm{t}}(u-s)^{\mathrm{H}-\frac{3}{2}}u^{\mathrm{H}-\frac{1}{2}}du,$$

for $\mathrm{t}>s$, where $c_{\mathrm{H}}=\sqrt{\frac{\mathrm{H}(2\mathrm{H}-1)}{g(2-2\mathrm{H},\mathrm{H}-\frac{1}{2})}}$ and $g(\cdot,\cdot)$ denotes the Beta function. We take $\mathrm{K}_{\mathrm{H}}(\mathrm{t},s)=0$ if $\mathrm{t}\le s$. We will denote by $\mathscr{H}$ the repreducing kernal Hilbert space of the fBm. Precisely, $\mathscr{H}$ is the closure of set of indicator functions $\left\{1_{[0,\mathrm{t}]} : \mathrm{t}\in[0,\mathbb{T}]\right\}$ with respect to the scalar product $\left\langle 1_{[0,\mathrm{t}]},1_{[0,s]}\right\rangle_{\mathscr{H}}=R_{\mathrm{H}}(\mathrm{t},s)$. The mapping $1_{[0,\mathrm{t}]}\to\beta^{\mathrm{H}}(\mathrm{t})$ can be extended to an isometry between $\mathscr{H}$ and the first Wiener chaos and we will denote by $\beta^{\mathrm{H}}(\varphi)$ the image of φ by the previous isometry.

Let $\mathbb{X}$ and $\mathbb{Y}$ be two real separable Hilbert spaces and let $\mathcal{L}(\mathbb{Y},\mathbb{X})$ be the space of bounded linear operator from $\mathbb{Y}$ to $\mathbb{X}$. Let $Q\in\mathcal{L}(\mathbb{X},\mathbb{Y})$ be an operator defined by $Qe_n=\lambda_ne_n$ with fiinite trace

$$\mathrm{tr}Q \;=\; \sum_{n=1}^{\infty}\lambda_n<\infty,$$

where $\lambda_n\ge 0$ $(n=1,2,...)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2,...)$ is a complete orthonormal basis in $\mathbb{Y}$. We define the infinte dimensional fBm on $\mathbb{Y}$ with covariance Q as

$$\mathtt{B}^{\mathrm{H}}(\mathrm{t})=\mathtt{B}^{\mathrm{H}}_Q(\mathrm{t})=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^{\mathrm{H}}(\mathrm{t}).$$

where β_n^{H} are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance

$$\mathbf{E}\left\langle\mathtt{B}^{\mathrm{H}}(\mathrm{t}),\mathtt{x}\right\rangle\left\langle\mathtt{B}^{\mathrm{H}}(s),\mathtt{y}\right\rangle \;=\; R(s,\mathrm{t})\left\langle Q(\mathtt{x}),\mathtt{y}\right\rangle \text{ for } \mathtt{x},\mathtt{y}\in\mathbb{Y} \text{ and } \mathrm{t},s\in[0,\mathbb{T}].$$

Now, define the Weiner integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0=\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$ of all Q-Hilbert-Schmidt operators $\zeta:\mathbb{Y}\to\mathbb{X}$. We recall that $\zeta\in\mathcal{L}(\mathbb{Y},\mathbb{X})$ is called a Q-Hilbert-Schmidt operator, if

$$\|\zeta\|^2_{\mathcal{L}_2^0} \;=\; \sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\zeta e_n\right\|^2<\infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $< \varphi, \zeta >_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} < \varphi e_n, \zeta e_n >$ is a separable Hilbert space. Let $\phi(s) : s \in [0, \mathbb{T}]$ be a function with values in $\mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ such that

$$\sum_{n=1}^{\infty} \left\| \mathrm{K}^* \phi Q^{1/2} e_n \right\|_{\mathcal{L}_2^0}^2 < \infty.$$

The Weiner integral of ϕ with respect to $\mathtt{B}^{\mathrm{H}}$ is defined by

$$\int_0^{\mathrm{t}} \phi(s) d\mathtt{B}^{\mathrm{H}} = \sum_{n=1}^{\infty} \int_0^{\mathrm{t}} \sqrt{\lambda_n} \phi(s) e_n d\beta_n^{\mathrm{H}}(s). \tag{4}$$

Lemma 2.1. *If* $\zeta : [0, \mathbb{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ *satisfies* $\int_0^{\mathrm{t}} \|\zeta(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$, *then (4) is well defined as an* $\mathbb{X}$*-valued random variable and*

$$\mathbf{E} \left\| \int_0^{\mathrm{t}} \zeta(s) d\mathtt{B}^{\mathrm{H}}(s) \right\|^2 \leq 2\mathrm{H}\mathrm{t}^{2\mathrm{H}-1} \int_0^{\mathrm{t}} \|\zeta\|_{\mathcal{L}_2^0}^2 ds.$$

Let the phase space $\mathscr{B}_{\mathfrak{H}}$ is a linear space of functions from $(-\infty, 0]$ into $\mathbb{X}$, endowed with a norm $\|\cdot\|_{\mathscr{B}_{\mathfrak{H}}}$. First, we present the abstract phase space $\mathscr{B}_{\mathfrak{H}}$. Suppose that the continuous function $\mathfrak{H} : (-\infty, 0] \to [0, \infty)$ endowed with

$$l = \int_{-\infty}^{0} \mathfrak{H}(s) ds < \infty.$$

Define the abstract phase space $\mathscr{B}_{\mathfrak{H}}$ by $\mathscr{B}_{\mathfrak{H}} = \Big\{ \zeta : (-\infty, 0] \to \mathbb{X}$ for any $\tau > 0$, $(\mathbf{E}\|\zeta\|^2)^{1/2}$ is bounded and measurable function $[\tau, 0] \int_{-\infty}^{0} \mathfrak{H}(\mathrm{t}) \sup_{\mathrm{t} \leq \tau \leq 0} (\mathbf{E}\|\zeta(s)\|^2)^{1/2} d\mathrm{t} < \infty \Big\}$. Let us define the norm

$$\|\zeta\|_{\mathscr{B}_{\mathfrak{H}}} = \int_{-\infty}^{0} \mathfrak{H}(\mathrm{t}) \sup_{\mathrm{t} \leq s \leq 0} (\mathbf{E}\|\zeta\|^2)^{1/2} d\mathrm{t},$$

then it is clear that $(\mathscr{B}_{\mathfrak{H}}, \|\cdot\|_{\mathscr{B}_{\mathfrak{H}}})$ is a Banach space.

Consider the space $\mathscr{B}_{\mathfrak{D}\mathrm{I}}$ [$\mathfrak{D}$ and I stand for delay and impulse, respectively] given by $\mathscr{B}_{\mathfrak{D}\mathrm{I}} = \Big\{ \mathtt{x} : (-\infty, \mathbb{T}] \to \mathbb{X} : \mathtt{x}|\mathrm{I}_\mathrm{k} \in \mathfrak{C}(\mathrm{I}_\mathrm{k}, \mathbb{X})$ and $\mathtt{x}(\mathrm{t}_\mathrm{k}^+), \mathtt{x}(\mathrm{t}_\mathrm{k}^-)$ exist with $\mathtt{x}(\mathrm{t}_\mathrm{k}^+) - \mathtt{x}(\mathrm{t}_\mathrm{k}^-), \mathrm{k} = 1, 2, ..., m$ $\mathtt{x}_0 - \varphi \in \mathscr{B}_{\mathfrak{H}}$ and $\sup_{0 \leq \mathrm{t} \leq \mathbb{T}} \mathbf{E}(\|\mathtt{x}(\mathrm{t})\|^2) < \infty \Big\}$, where $\mathtt{x}|\mathrm{I}_\mathrm{k}$ is the restiction of $\mathtt{x}$ to the interval $\mathrm{I}_\mathrm{k} = (\mathrm{t}_\mathrm{k}, \mathrm{t}_{\mathrm{k}+1}]$, $\mathrm{k} = 1, 2, ..., m$. Then the function $\|\cdot\|_{\mathscr{B}_{\mathfrak{H}}}$ to be a semi-norm in $\mathscr{B}_{\mathfrak{D}\mathrm{I}}$, it is defined by

$$\|\mathtt{x}\|_{\mathscr{B}_{\mathfrak{D}\mathrm{I}}} = \|\mathtt{x}_0\|_{\mathscr{B}_{\mathfrak{H}}} + \sup_{0 < \mathrm{t} < \mathbb{T}} (\mathbf{E}(\|\mathtt{x}(\mathrm{t})\|^2))^{1/2}.$$

Lemma 2.2. [14] *Suppose* $\mathtt{x} \in \mathscr{B}_{\mathfrak{D}\mathrm{I}}$, *then for all* $\mathrm{t} \in [0, \mathbb{T}]$, $\mathtt{x}_\mathrm{t} \in \mathscr{B}_{\mathfrak{H}}$ *and*

$$l(\mathbf{E}(\|\mathtt{x}(\mathrm{t})\|^2))^{\frac{1}{2}} \leq l \sup_{0 \leq s \leq t} (\mathbf{E}\|\mathtt{x}(s)\|^2)^{\frac{1}{2}} + \|\mathtt{x}_0\|_{\mathscr{B}_{\mathfrak{H}}},$$

where $l = \int_{-\infty}^{0} \mathfrak{H}(s) ds < \infty$.

2.1. Partial integrodifferential equations in Banach spaces. Further, we recollect some basic results related to resolvent operators. Regarding the theory of resolvent operators, we refer the reader to [8]. Let $\mathfrak{A}$ and $O(\mathrm{t})$ are closed linear operator on $\mathbb{X}$ and $\mathbb{Y}$ represents the Banach space $\mathfrak{D}(\mathfrak{A})$ equipped with the graph norm

$$|\mathtt{y}|_{\mathbb{Y}} := |\mathfrak{A}\mathtt{y}| + |\mathtt{y}| \text{ for } \mathtt{y} \in \mathbb{Y}.$$

The notation $\mathfrak{C}([0,\infty);\mathbb{Y})$ stands for the space of all continuous functions from $[0,\infty)$ into $\mathbb{Y}$. We consider the following Cauchy problem

$$\begin{cases} \mathtt{x}'(\mathrm{t}) = \mathfrak{A}\mathtt{x}(\mathrm{t}) + \int_0^{\mathrm{t}} O(\mathrm{t}-s)\mathtt{x}(s)ds \text{ for } \mathrm{t} \geq 0, \\ \mathtt{x}(0) = \mathtt{x}_0 \in \mathbb{X}. \end{cases} \tag{5}$$

Definition 2.3. [8] A resolvent operator for Equation (5) is a bounded linear operator valued function $\mathfrak{R}(\mathrm{t}) \in \mathcal{L}(\mathbb{X})$ for $\mathrm{t} \geq 0$, satisfying the following properties:

(i) $\mathfrak{R}(0) = \mathrm{I}$ and $\|\mathfrak{R}(\mathrm{t})\| \leq Me^{\lambda \mathrm{t}}$ for some constants M and λ.

(ii) For each $\mathtt{x} \in \mathbb{X}$, $\mathfrak{R}(\mathrm{t})\mathtt{x}$ is strongly continuous for $\mathrm{t} \geq 0$.

(iii) For $\mathtt{x} \in \mathbb{Y}$, $\mathfrak{R}(\cdot)\mathtt{x} \in \mathfrak{C}^1([0,\infty);\mathbb{X}) \bigcap \mathfrak{C}([0,\infty);\mathbb{Y})$ and

$$\begin{aligned} \mathfrak{R}'(\mathrm{t})\mathtt{x} &= \mathfrak{A}\mathfrak{R}(\mathrm{t})\mathtt{x} + \int_0^{\mathrm{t}} O(\mathrm{t}-s)\mathfrak{R}(s)\mathtt{x}ds \\ &= \mathfrak{R}(\mathrm{t})\mathfrak{A}\mathtt{x} + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)O(s)\mathtt{x}ds \text{ for } \mathrm{t} \geq 0. \end{aligned} \tag{6}$$

For additional details on resolvent operators, we refer the reader to [8]. In what follows we suppose the following assumptions:

(H1) The operator $\mathfrak{A}$ generator a $\mathfrak{C}_0$-semigroup $(\mathbb{T}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on $\mathbb{X}$.

(H2) For all $\mathrm{t} \geq 0$, $O(\mathrm{t})$ is a continuous linear operator from $(\mathbb{Y}, |\cdot|_{\mathbb{Y}})$ into $(\mathbb{X}, |\cdot|_{\mathbb{X}})$. Moreover, there exists an integrable function $\mathfrak{C} : [0,\infty) \to \mathbb{R}^+$ such that for any $\mathtt{y} \in \mathbb{Y}$, $\mathtt{y} \to \Theta(t)\mathtt{y}$ belongs to $\mathrm{W}^{1,1}([0,\infty);\mathbb{X})$ and

$$\left| \frac{d}{d\mathrm{t}} O(\mathrm{t})(\mathrm{t})\mathtt{y} \right|_{\mathbb{X}} \leq \mathfrak{C}(\mathrm{t}) \, |\mathtt{y}|_{\mathbb{Y}} \text{ for } \mathtt{y} \in \mathbb{Y} \text{ and } \mathrm{t} \geq 0.$$

Theorem 2.4. *Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then there exists a unique resolvent operator for the Cauchy problem (5).*

Lemma 2.5. [11] *There exists a constant $\mathcal{L} = \mathcal{L}(\mathbb{T})$ such that*

$$||\mathfrak{R}(\mathrm{t}+\epsilon) - \mathfrak{R}(\epsilon)\mathfrak{R}(\mathrm{t})||_{\mathcal{L}(\mathbb{X})} \leq \mathcal{L}\epsilon \text{ for } 0 \leq \epsilon \leq \mathrm{t} \leq \mathbb{T}.$$

In the sequel, we recall some results on existence of solutions for the following integrodifferential equation

$$\begin{cases} \mathtt{x}'(\mathrm{t}) = \mathfrak{A}\mathtt{x}(\mathrm{t}) + \int_0^{\mathrm{t}} O(\mathrm{t}-s)\mathtt{x}(s)ds + q(\mathrm{t}) \text{ for } \mathrm{t} \geq 0, \\ \mathtt{x}(0) = \mathtt{x}_0 \in \mathbb{X}; \end{cases} \tag{7}$$

where $q : [0, \infty) \to \mathbb{X}$ is a continuous function.

Definition 2.6. A continuous function $\mathtt{x} : [0, \infty) \to \mathbb{X}$ is said to be a strict solution of equation (7) if
(i) $\mathtt{x} \in \mathfrak{C}^1([0, \infty); \mathbb{X}) \bigcap \mathfrak{C}([0, \infty); \mathbb{Y})$,
(ii) $\mathtt{x}$ satisfies equation (7) for $\mathrm{t} \geq 0$.

Theorem 2.7. *Assume that* **(H1)**-**(H2)** *hold. If* $\mathtt{x}$ *is a strict solution of Equation (7), then the following variation of constants formula holds*

$$\mathtt{x}(\mathrm{t}) \quad = \quad \mathfrak{R}(\mathrm{t})\mathtt{x}_0 + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)q(s)ds \textit{ for } \mathrm{t} \geq 0. \tag{8}$$

Now, we have the following definition for mild solution of (7):

Definition 2.8. An $\mathbb{X}$-valued process $\{\mathtt{x}(\mathrm{t}) : \mathrm{t} \in (-\infty, \mathbb{T}]\}$ is called a mild solution of (1)-(3),
1. $\mathtt{x}(\mathrm{t})$ is measurable for each $\mathrm{t} > 0$, $\mathtt{x}(\mathrm{t}) = \varphi(\mathrm{t})$ on $(\infty, 0]$,

$$\Delta \mathtt{x}\big|_{\mathrm{t}-\mathrm{t}_\mathrm{k}} = \mathrm{I}_\mathrm{k}(\mathtt{x}(\mathrm{t}_\mathrm{k}^-)), \ \mathrm{k} = 1, 2, ...m;$$

the restriction of $\mathtt{x}(\cdot)$ to $[0, \mathbb{T}] - \{\mathrm{t}_1, \mathrm{t}_2, ...\mathrm{t}_m\}$ is continuous.
2. For every $0 \leq s \leq \mathrm{t}$, the process $\mathtt{x}$ satisfies the following integral equation:

$$\begin{aligned} \mathtt{x}(\mathrm{t}) \quad &= \quad \mathfrak{R}(\mathrm{t})\left[\varphi(0) - \Gamma(0, \varphi)\right] + \Gamma(\mathrm{t}, \mathtt{x}_\mathrm{t}) + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\Lambda(s, \mathtt{x}_s)ds \\ &+ \quad \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\mathfrak{B}\mathbf{u}(s)ds + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\Xi(s)d\mathtt{B}^{\mathrm{H}}(s) \\ &+ \quad \sum_{0<\mathrm{t}_\mathrm{k}<\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\mathrm{I}_\mathrm{k}(\mathtt{x}(\mathrm{t}_\mathrm{k}^-)), \ \mathtt{P} - a.s. \end{aligned} \tag{9}$$

Definition 2.9. A function $\mathtt{x} : [0, \infty) \to \mathbb{X}$ is called a mild solution of (7) if $\mathtt{x}$ satisfies the variation of constants formula (8) for $\mathtt{x}_0 \in \mathbb{X}$.

Definition 2.10. A bounded closed and convex subset $\mathbb{V}$ of a Banach space $\mathbb{X}$ and let Π_1, Π_2 be two operators of $\mathbb{V}$ into $\mathbb{X}$ satisfying
(i) $\Pi_1(\mathtt{x}) + \Pi_2(\mathtt{x}) \in \mathbb{V}$ whenever $\mathtt{x} \in \mathbb{V}$,
(ii) Π_1 is a contraction mapping, and
(iii) Π_2 is completely continuous.
Then, $\exists \ \ az \in \mathbb{V}$ such that $z = \Pi_1(z) + \Pi_2(z)$.

3. Controllability Result

Definition 3.1. System (1)-(3) is said to be controllable on the interval $(-\infty, \mathbb{T}]$ if for every initial stochastic process φ defined on $(-\infty, \mathbb{T}]$, there exists a stochastic control $\mathbf{u} \in \mathcal{L}^2([0, \mathbb{T}]; \mathbb{U})$ such that the mild solution $\mathtt{x}(\cdot)$ of (1)-(3) satisfies $\mathtt{x}(\mathbb{T}) = \mathtt{x}_1$.

In order to establish the controllability of (1)-(3), we impose the following hypotheses:
(H3) The resolvent operator $\mathfrak{R}(.)$ is compact and there exist constants $M \geq 1$ such that $\|\mathfrak{R}(\mathrm{t})\|^2 \leq M$.
(H4) The mapping $\Gamma : [0, \mathbb{T}] \times \mathscr{B}_{\mathfrak{H}} \to \mathbb{X}$ satisfies the following conditions and there exist constants $\mathrm{k}_\Gamma > 0$ such that

$$\begin{aligned} \mathbf{E}\left\|\Gamma(\mathrm{t},\mathtt{x}) - \Gamma(\mathrm{t},\mathtt{y})\right\|^2 &\leq \mathrm{k}_\Gamma \left\|\mathtt{x}-\mathtt{y}\right\|^2_{\mathscr{B}_{\mathfrak{H}}}, \mathrm{t} \in [0,\mathbb{T}],\ \mathtt{x},\mathtt{y} \in \mathscr{B}_{\mathfrak{H}}, \\ \bar{\mathrm{k}_\Gamma} &= \sup_{\mathrm{t}\in[0,\mathbb{T}]} \left\|\Gamma(\mathrm{t},0)\right\|^2. \end{aligned}$$

(H5) (i) Let $\mathrm{t} \to \Gamma(\mathrm{t}, \mathtt{x})$ be non-linear continuous measurable function for each $\mathtt{x} \in \mathscr{B}_{\mathfrak{H}}$, which is for a.e $\mathrm{t} \in [0, \mathbb{T}]$,
(ii) $\exists\ p_{\mathrm{k}} \in \mathcal{L}^1([0, \mathbb{T}], \mathbb{R}^+)$, s.t

$$\left\|\Gamma(\mathrm{t},\mathtt{x})\right\|^2 \leq p_{\mathrm{k}}(\mathrm{t}),\ \forall\ \mathrm{t} \in [0,\mathbb{T}],\ \mathrm{k} > 0,$$

and

$$\lim_{\mathrm{k}\to\infty} \frac{1}{\mathrm{k}} \int_0^T p_{\mathrm{k}}(\tau) d\tau = \gamma < \infty.$$

(H6) The impulses functions $\mathrm{I_k}$ for $\mathrm{k} = 1, 2, ..., m$, satisfies, $\exists$ + constants M_{k}, $\widetilde{M_{\mathrm{k}}}$ s.t

$$\left\|\mathrm{I_k}(\mathtt{x}) - \mathrm{I_k}(\mathtt{y})\right\|^2 \leq M_{\mathrm{k}} \left\|\mathtt{x}-\mathtt{y}\right\|^2 \text{ and } \left\|\mathrm{I_k}(\mathtt{x})\right\|^2 \leq \widetilde{M_{\mathrm{k}}} \text{ for all } \mathtt{x},\mathtt{y} \in \mathscr{B}_{\mathfrak{H}}.$$

(H7) The function $\Xi : [0, \infty) \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ satisfies

$$\int_0^{\mathbb{T}} \left\|\Xi(s)\right\|^2_{\mathcal{L}_2^0} ds < \infty, \text{ for } \mathrm{t} > 0.$$

(H8) The linear operator $\mathfrak{W}$ from $\mathbb{U}$ into $\mathbb{X}$ defined by

$$\mathfrak{W}\mathbf{u} = \int_0^{\mathbb{T}} \mathfrak{R}(\mathbb{T}-s)\mathfrak{B}\mathbf{u}(s)ds$$

has an inverse operator $\mathfrak{W}^{-1}$ that takes values in $\mathcal{L}^2([0, \mathbb{T}], \mathbb{U})/\ker \mathfrak{W}$, where $ker\mathfrak{W} = \left\{\mathtt{x} \in \mathcal{L}^2([0, \mathbb{T}], \mathbb{U}) : \mathfrak{W}\mathtt{x} = 0\right\}$ The main result of this paper is given in the next theorem.

Theorem 3.2. *Suppose that* **(H1)**-**(H8)** *hold. Then, the system (1)-(3) is controllable on* $(-\infty, \mathbb{T}]$ *provide that*

$$6l^2\left(1 + 7MM_bM_{\mathfrak{W}}\mathbb{T}^2\right)\left[8[\mathrm{k}_2(1+2\mathrm{k}_1)] + 8M\mathbb{T}^2[\mathrm{k}_3(1+2\mathrm{k}_4)]\right] < 1. \quad (10)$$

Proof. Using **(H8)** for an arbitrary function $\mathtt{x}(\cdot)$, define the control

$$\mathbf{u}_{\mathtt{x}}(\mathrm{t}) = \mathfrak{W}^{-1}\Bigg[\mathtt{x}_1 - \mathfrak{R}(\mathbb{T})\left[\varphi(0) - \Gamma(0,\mathtt{x}_0)\right] - \Gamma(\mathbb{T},\mathtt{x}_{\mathbb{T}}) - \int_0^{\mathbb{T}} \mathfrak{R}(\mathbb{T}-s)\Lambda(s,\mathtt{x}_s)ds$$

$$- \int_0^T \mathfrak{R}(\mathbb{T}-s)\Xi(s)d\mathtt{B}^{\mathrm{H}}(s) - \sum_{0<\mathrm{t_k}<\mathrm{t}} \mathfrak{R}(\mathbb{T}-\mathrm{t_k})\mathrm{I_k}(\mathtt{x}(\mathrm{t_k^-}))\Bigg](\mathrm{t}).$$

Now, we consider the stochastic control system (8) with $\mathbf{u}(\cdot)$ and let nonlinear operator Ψ on $\mathscr{B}_{\mathfrak{D}\mathrm{I}}$ is obtained has follows:

$$\Psi(\mathtt{x})(\mathrm{t}) = \begin{cases} \varphi(\mathrm{t}), \text{ for } \mathrm{t} \in (-\infty, 0], \\ \mathfrak{R}(\mathrm{t})\left[\varphi(0) - \Gamma(0,\varphi,0)\right] + \Gamma(\mathrm{t},\mathtt{x}_\mathrm{t}) + \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\mathfrak{B}\mathbf{u}_\mathtt{x}(s)ds \\ + \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\Lambda(s,\mathtt{x}_s)ds + \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\Xi(s)d\mathtt{B}^{\mathrm{H}}(s) \\ + \sum_{0<\mathrm{t_k}<\mathrm{t}} \mathfrak{R}(\mathrm{t}-\mathrm{t_k})\mathrm{I_k}(\mathtt{x}(\mathrm{t_k^-})), \text{ if } \mathrm{t} \in [0,\mathbb{T}]. \end{cases}$$

Now it is obvious that the mild solution of the system (1)-(3) is similer to obtained a fixed point to operator Ψ .

Let $\mathtt{y} : (-\infty, \mathbb{T}] \to \mathbb{X}$ be the function is,

$$\mathtt{y}(\mathrm{t}) = \begin{cases} \varphi(\mathrm{t}), \text{ if } \mathrm{t} \in (-\infty, 0], \\ \mathfrak{R}(\mathrm{t})\varphi(0), \text{ if } \mathrm{t} \in [0,\mathbb{T}]. \end{cases}$$

if, $\mathtt{y}_0 = \varphi$. For each function $z \in \mathscr{B}_{\mathfrak{D}\mathrm{I}}$, set

$$\mathtt{x}(\mathrm{t}) = z(\mathrm{t}) + \mathtt{y}(\mathrm{t}).$$

It is obviously that $\mathtt{x}$ satisfies the stoachastic control system (8) iff z satisfies $z_0 = 0$ and

$$\begin{aligned} z(\mathrm{t}) &= \Gamma(\mathrm{t}, z_\mathrm{t} + \mathtt{y}_\mathrm{t}) - \mathfrak{R}(\mathrm{t})\Gamma(0,\varphi) + \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\mathfrak{B}_{z+\mathtt{y}}(s)ds \\ &+ \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\Lambda(s, z_s + \mathtt{y}_s)ds + \int_0^\mathrm{t} \mathfrak{R}(\mathrm{t}-s)\Xi(s)d\mathtt{B}^{\mathrm{H}}(s) \\ &+ \sum_{0<\mathrm{t_k}<\mathrm{t}} \mathfrak{R}(\mathrm{t}-\mathrm{t_k})\mathrm{I_k}[z(\mathrm{t_k^-}) - \mathtt{y}(\mathrm{t_k^-})], \text{ if } \mathrm{t} \in [0,\mathbb{T}], \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathbf{u}_{z+\mathtt{y}}(\mathrm{t}) &= \mathrm{W}^{-1}\Bigg[\mathtt{x}_1 - \mathfrak{R}(\mathbb{T})\left[\varphi(0) - \Gamma(0, z_0 + \mathtt{y}_0)\right] - \Gamma(\mathbb{T}, z_\mathbb{T} + \mathtt{y}_\mathbb{T}) \\ &- \int_0^\mathbb{T} \mathfrak{R}(\mathbb{T}-s)\Lambda(s, z_s + \mathtt{y}_s)ds \\ &- \int_0^\mathbb{T} \mathfrak{R}(\mathbb{T}-s)\Xi(s)d\mathtt{B}^{\mathrm{H}}(s) - \sum_{0<t_k<\mathbb{T}} \mathfrak{R}(\mathbb{T}-\mathrm{t_k})\mathrm{I_k}[z(\mathrm{t_k^-}) + \mathtt{y}(\mathrm{t_k^-})]\Bigg](\mathrm{t}). \end{aligned}$$

Set

$$\mathscr{B}^0_{\mathfrak{D}\mathrm{I}} = \{z \in \mathscr{B}_{\mathfrak{D}\mathrm{I}} : z_0 = 0\},$$

for any $z \in \mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$, we have

$$\|z\|_{\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}} = \|z_0\|_{\mathscr{B}_{\mathfrak{H}}} + \sup_{\mathrm{t}\in[0,\mathbb{T}]}(\mathbf{E}\,\|z(\mathrm{t})\|^2)^{\frac{1}{2}} = \sup_{\mathrm{t}\in[0,\mathbb{T}]}(\mathbf{E}\,\|z(\mathrm{t})\|^2)^{\frac{1}{2}}.$$

Then, $(\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}, \|\cdot\|_{\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}})$ is a Banach space. Define the operator $\Theta : \mathscr{B}^0_{\mathfrak{D}\mathrm{I}} \to \mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$ by

$$(\Theta z)(\mathrm{t}) = \begin{cases} 0 \text{ if } \mathrm{t} \in (-\infty, 0], \\ \Gamma(\mathrm{t}, z_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) - \mathfrak{R}(\mathrm{t})\Gamma(0,\varphi) + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{B}_{z+\mathrm{y}}(s)ds \\ + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\Lambda(s, z_s + \mathrm{y}_s)ds + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\Xi(s)d\mathrm{B}^{\mathrm{H}}(s) \\ + \sum_{0<\mathrm{t_k}<\mathrm{t}} \mathfrak{R}(\mathrm{t}-\mathrm{t_k})\mathrm{I_k}[z(\mathrm{t}_\mathrm{k}^-) - \mathrm{y}(\mathrm{t}_\mathrm{k}^-)], \text{ if } \mathrm{t} \in [0,\mathbb{T}], \end{cases} \tag{12}$$

Set

$$\mathscr{B}_{\mathrm{k}} = \left\{ z \in \mathscr{B}^0_{\mathfrak{D}\mathrm{I}} : \|z\|^2_{\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}} \le \mathrm{k} \right\}, \text{ for some } \mathrm{k} \ge 0,$$

then $\mathscr{B}_{\mathrm{k}} \subseteq \mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$ is a bounded closed convex set, and for $z \in \mathscr{B}_{\mathrm{k}}$, we have

$$\begin{aligned} & \|z_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}\|_{\mathscr{B}_{\mathfrak{D}\mathrm{I}}} \\ \le\ & 2\left(\|z_{\mathrm{t}}\|^2_{\mathscr{B}_{\mathfrak{D}\mathrm{I}}} + \|\mathrm{y}_{\mathrm{t}}\|^2_{\mathscr{B}_{\mathfrak{D}\mathrm{I}}}\right) \\ \le\ & 4\left(l^2 \sup_{0\le s\le \mathrm{t}} \mathbf{E}\,\|z(s)\|^2 + \|z_0\|^2_{\mathscr{B}_{\mathfrak{H}}} + l^2 \sup_{0\le s\le \mathrm{t}} \mathbf{E}\,\|\mathrm{y}(s)\|^2 + \|\mathrm{y}_0\|^2_{\mathscr{B}_{\mathfrak{H}}}\right) \\ \le\ & 4l^2\left(\mathrm{k} + M\mathbf{E}\,\|\varphi(0)\|^2\right) + 4\,\|\mathrm{y}\|^2_{\mathscr{B}_{\mathfrak{H}}} \\ :=\ & r^*. \end{aligned}$$

Next,

$$\begin{aligned} \mathbf{E}\left\|\mathbf{u}_{z+\mathrm{y}}\right\|^2 \le\ & 7M_{\mathrm{W}}\Bigg[\|\mathrm{x}_1\|^2 + M\mathbf{E}\,\|\varphi(0)\|^2 + 2M[\mathrm{k}_\Gamma\,\|\mathrm{y}\|^2_{\mathscr{B}_{\mathfrak{H}}} + \bar{\mathrm{k}}_\Gamma] \\ & + M\mathbb{T}\int_0^t pr^*(s)ds + 2M\mathbb{T}^{2\mathrm{H}-1}\int_0^{\mathbb{T}} \|\Xi(s)\|^2_{\mathcal{L}^0_2}\,ds \\ & + mM\sum_{\mathrm{k}=1}^m \widetilde{M}_{\mathrm{k}}\Bigg] := \mathcal{G} \end{aligned} \tag{13}$$

It is clearly proved that the operator Θ has a fixed point iff $\hat{\Theta}$, so it turns to prove that $\hat{\Theta}$ has a fixed point. To this end, we decompose $\hat{\Theta}$ as $\hat{\Theta} = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are defined on $\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$, respectively by

$$(\Theta_1 z)(\mathrm{t}) = \begin{cases} 0 \text{ if } \mathrm{t} \in (-\infty, 0], \\ \Gamma(\mathrm{t}, z_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) - \mathfrak{R}(\mathrm{t})\Gamma(0,\varphi) \\ + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\Xi(s)d\mathrm{B}^{\mathrm{H}}(s), \text{ if } \mathrm{t} \in [0,\mathbb{T}], \end{cases} \tag{14}$$

$$(\Theta_2 z)(\mathrm{t}) = \begin{cases} 0 \text{ if } \mathrm{t} \in (-\infty, 0], \\ \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\Lambda(s, z_s + \mathrm{y}_s)ds + \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{B}\mathbf{u}_{z+\mathrm{y}}(s)ds \\ + \sum_{0<\mathrm{t_k}<\mathrm{t}} \mathfrak{R}(\mathrm{t}-\mathrm{t_k})\mathrm{I_k}(z(\mathrm{t_k^-}) + \mathrm{y}(\mathrm{t_k^-})), \text{ if } \mathrm{t} \in [0, \mathbb{T}], \end{cases} \quad (15)$$

By applying Krasnoselskii fixed point theorem for the operator $\hat{\Theta}$, we show the following conditions:
(1) $\Theta_1(\mathtt{x}) + \Theta_2(\mathtt{x}) \in \mathscr{B}_\mathrm{k}$ whenever $\mathtt{x} \in \mathscr{B}_\mathrm{k}$,
(2) Θ_1 is a contraction,
(3) Θ_2 is continuous and compact map.
For our convenience, the proof will be splitup into three steps:
Step 1: We have to show that $\exists\ \mathrm{k} > 0$, s.t $\Theta_1(\mathtt{x}) + \Theta_2(\mathtt{x}) \in \mathscr{B}_\mathrm{k}$ whenever $\mathtt{x} \in \mathscr{B}_\mathrm{k}$. If it is not true, then for each $\mathrm{k} > 0$, there is a function $z^\mathrm{k}(\cdot) \in \mathscr{B}_\mathrm{k}$, but $\Theta_1(z^\mathrm{k}) + \Theta_2(z^\mathrm{k}) \notin \mathscr{B}_\mathrm{k}$,

$$\mathbf{E}\left\|\Theta_1(z^\mathrm{k})(\mathrm{t}) + \Theta_2(z^\mathrm{k})(\mathrm{t})\right\|^2 > \mathrm{k}.$$

On the other hand,

$$\begin{aligned} \mathrm{k} &< \mathbf{E}\left\|\Theta_1(z^\mathrm{k})(\mathrm{t}) + \Theta_2(z^\mathrm{k})(\mathrm{t})\right\|^2 \\ &\leq 6\Bigg[2M[\mathrm{k}_\Gamma \|\mathrm{y}\|^2_{\mathscr{B}_\mathfrak{H}} + \bar{\mathrm{k}_\Gamma}] + 2[r^* + \bar{\mathrm{k}_\Gamma}]MM_b\mathbb{T}^2\mathcal{G} \\ &\quad + M\mathbb{T}\int_0^\mathbb{T} pr^*(s)ds + 2M\mathbb{T}^{2\mathrm{H}-1}\int_0^\mathbb{T} \|\Xi(s)\|^2_{\mathcal{L}_2^0} ds + M\sum_{\mathrm{k}=1}^m \tilde{M}_\mathrm{k}\Bigg] \\ &\leq 6\left(1 + 6MM_bM_\mathfrak{W}\mathbb{T}^2\right)\Bigg[2M[\mathrm{k}_\Gamma \|\mathrm{y}\|^2_{\mathscr{B}_\mathfrak{h}} + \bar{\mathrm{k}_\Gamma}] + 2[r^* + \bar{\mathrm{k}_\Gamma}]MM_b\mathbb{T}^2\mathcal{G} \\ &\quad + M\mathbb{T}\int_0^\mathbb{T} pr^*(s)ds + 2M\mathbb{T}^{2\mathrm{H}-1}\int_0^\mathbb{T} \|\Xi(s)\|^2_{\mathcal{L}_2^0} ds + M\sum_{\mathrm{k}=1}^m \tilde{M}_\mathrm{k}\Bigg] \\ &\quad + 6MM_bM_\mathfrak{W}\mathbb{T}^2\left[\|\mathtt{x}_1\|^2 + M\mathbf{E}\|\varphi(0)\|^2\right] \\ &\leq \bar{K} + 6\left(1 + 6MM_bM_\mathfrak{W}\mathbb{T}^2\right)\left[2M\mathbb{T}\int_0^\mathbb{T} pr^*(s)ds\right] \end{aligned}$$

where

$$\begin{aligned} \bar{K} &= 6\left(1 + 6MM_bM_\mathfrak{W}\mathbb{T}^2\right)\Bigg[2M[\mathrm{k}_\Gamma \|\mathrm{y}\|^2_{\mathscr{B}_\mathfrak{H}} + \bar{\mathrm{k}_\Gamma}] + 2\bar{\mathrm{k}_\Gamma} \\ &+ 2M\mathbb{T}^{2\mathrm{H}-1}\int_0^\mathbb{T} \|\Xi(s)\|^2_{\mathcal{L}_2^0} ds + M\sum_{\mathrm{k}=1}^m \tilde{M}_\mathrm{k} \\ &+ 6MM_bM_\mathfrak{W}\mathbb{T}^2\left[\|\mathtt{x}_1\|^2 + M\mathbf{E}\|\varphi(0)\|^2\right]\Bigg]. \end{aligned}$$

is independent of k. Dividing both sides by k and taking the lower limit as $\mathrm{k} \to \infty$, we obtain

$$\begin{aligned} r^* &= 4l^2 \left[\mathrm{k} + M\mathbf{E} \left\| \varphi(0) \right\|^2 \right] + 4 \left\| \mathrm{y} \right\|_{\mathscr{B}_{\mathfrak{H}}} \to \infty \text{ as } \mathrm{k} \to \infty. \\ \lim_{\mathrm{k}\to\infty} \inf \frac{\int_0^t pr^*(s)ds}{\mathrm{k}} &= \lim_{\mathrm{k}\to\infty} \inf \frac{\int_0^t pr^*(s)ds}{r^*} \cdot \frac{r^*}{\mathrm{k}} \\ &= 4l^2\gamma. \end{aligned}$$

Thus, we have

$$6l^2 \left(1 + 6MM_bM_{\mathfrak{W}}\mathbb{T}^2\right) \left[8M\mathbb{T}\gamma\right] \geq 1.$$

This contraction (10). Hence $(\Theta_1 + \Theta_2)(\mathscr{B}_{\mathrm{k}}) \subseteq \mathscr{B}_{\mathrm{k}}$.
Step 2: Θ_1 is a contraction.
Let $\mathrm{t} \in [0, \mathbb{T}]$ and $z^1, z^2 \in \mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$

$$\begin{aligned} \mathbf{E} \left\| (\Theta_1 z^1)(\mathrm{t}) - (\Theta_1 z^2)(\mathrm{t}) \right\|^2 &\leq 2\mathbf{E} \left\| \Gamma(\mathrm{t}, z^1_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) - \Gamma(\mathrm{t}, z^2_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) \right\|^2 \\ &\leq \mathrm{k}_\Gamma \left\| z^1_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}} - z^2_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}} \right\|^2 \\ &\leq \mathrm{k}_\Gamma \left\| z^1_{\mathrm{t}} - z^2_{\mathrm{t}} \right\|^2 \\ &\leq \mathrm{k}_\Gamma \times \left[2l^2 \sup_{0\leq s\leq \mathbb{T}} \mathbf{E} \left\| z^1(s) - z^2(s) \right\|^2 \right. \\ &+ \left. 2 \left(\left\| z^1_0 \right\|^2_{\mathscr{B}_{\mathfrak{H}}} + \left\| z^2_0 \right\|^2_{\mathscr{B}_{\mathfrak{H}}} \right) \right] \\ &\leq \Delta \times \sup_{0\leq s\leq \mathbb{T}} \mathbf{E} \left\| z^1_s - z^2_s \right\|^2 \end{aligned}$$

where $\Delta = 2\mathrm{k}_\Gamma l^2 < 1$. Thus Θ_1 is a contraction on $\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$.
Step 3 : Θ_2 is completely continuous on $\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$.
Claim 1. Θ_2 is continuous on $\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$. Let z^n be a sequence s.t $z^n \to z$ in $\mathscr{B}^0_{\mathfrak{D}\mathrm{I}}$. Then $\exists$ a number $\mathrm{k} > 0$ s.t $\|z^n(t)\| \leq \mathrm{k}$, for all n and a.c. $\mathrm{t} \in [0, \mathbb{T}]$, so $z^n \in \mathscr{B}_{\mathrm{k}}$ and $z \in \mathscr{B}_{\mathrm{k}}$. By hypothesis **(H5)-(H6)**,
(1) I_{k}, $\mathrm{k} = 1, 2...m$ is continuous.
(2) $\Lambda(\mathrm{t}, z^n_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) \to \Lambda(\mathrm{t}, z_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}})$ for each $\mathrm{t} \in [0, \mathbb{T}]$. Since

$$\left\| \Lambda(\mathrm{t}, z^n_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) - \Lambda(\mathrm{t}, z_{\mathrm{t}} + \mathrm{y}_{\mathrm{t}}) \right\|^2 \leq 2pr^*(\mathrm{t}).$$

(H4) and by using dominated convergence theorem, we get

$$\begin{aligned} &\mathbf{E} \left\| \Theta_2 z^n(\mathrm{t}) - (\Theta_2 z)(\mathrm{t}) \right\|^2 \\ &\leq 3\mathbf{E} \left\| \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\mathfrak{B} \left[\mathbf{u}_{z^n+\mathrm{y}} - \mathbf{u}_{z+\mathrm{y}} \right] ds \right\|^2 \\ &+ 3\mathbf{E} \left\| \int_0^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s) \left[\Lambda(s, z^n_s + \mathrm{y}_s) - \Lambda(s, z_s + \mathrm{y}_s) \right] ds \right\|^2 \end{aligned}$$

$$
\begin{aligned}
&+3\mathbf{E}\left\|\sum_{0\le \mathrm{t_k}\le \mathrm{t}}\mathfrak{R}(\mathrm{t}-\mathrm{t_k})\left[\mathrm{I_k}(z^n(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))-\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right]\right\|^2\\
&\le 9M_wM_bM\mathbb{T}\int_0^{\mathbb{T}}\bigg[\mathbf{E}\left\|\Gamma(\mathbb{T},z_{\mathbb{T}}^n+\mathrm{y}_{\mathbb{T}})-\Gamma(\mathbb{T},z_{\mathbb{T}}+\mathrm{y}_{\mathbb{T}})\right\|^2\\
&+M\mathbb{T}\int_0^{\mathbb{T}}\mathbf{E}\left\|\Lambda(s,z_s^n+\mathrm{y}_s)-\Lambda(s,z_s+\mathrm{y}_s)\right\|^2ds\\
&+Mm\sum_{\mathrm{k}=1}^{m}\mathbf{E}\left\|\mathrm{I_k}(z^n(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))-\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right\|^2\bigg](\lambda)d\lambda\\
&+3M\mathbb{T}\int_0^{\mathbb{T}}\mathbf{E}\left\|\Lambda(s,z_s^n+\mathrm{y}_s)-\Lambda(s,z_s+\mathrm{y}_s)\right\|^2ds\\
&+3mM\sum_{\mathrm{k}=1}^{m}\mathbf{E}\left\|\mathrm{I_k}(z^n(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))-\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right\|^2\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Thus, Θ_2 is continuous.
Claim 2. Θ_2 maps $\mathscr{B}_{\mathrm{k}}$ into equicontinuous. Let $z\in\mathscr{B}_{\mathrm{k}}$ and $\tau_1,\tau_2\in[0,\mathbb{T}]$, $\tau_1,\tau_2\in \mathrm{t_k}$, $\mathrm{k}=1,...m$,

$$
\begin{aligned}
&\mathbf{E}\left\|(\Theta_2 z)(\tau_2)-(\Theta_2 z)(\tau_1)\right\|^2\\
&\le 6\mathbf{E}\left\|\int_0^{\tau_1}\left[\mathfrak{R}(\tau_2-s)-\mathfrak{R}(\tau_1-s)\right]\Lambda(s,z_s+\mathrm{y}_s)ds\right\|^2\\
&+6\mathbf{E}\left\|\int_0^{\tau_1}\left[\mathfrak{R}(\tau_2-s)-\mathfrak{R}(\tau_1-s)\right]\mathfrak{B}\mathbf{u}(s)ds\right\|^2\\
&+6\mathbf{E}\left\|\sum_{0<\mathrm{t_k}<\mathrm{t}}\left[\mathfrak{R}(\tau_2-\mathrm{t_k})-\mathfrak{R}(\tau_1-\mathrm{t_k})\right]\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right\|^2\\
&+6\mathbf{E}\left\|\int_{\tau_1}^{\tau_2}\mathfrak{R}(\tau_2-s)\Lambda(s,z_s+\mathrm{y}_s)ds\right\|^2\\
&+6\mathbf{E}\left\|\int_{\tau_1}^{\tau_2}\mathfrak{R}(\tau_2-s)\mathfrak{B}\mathbf{u}(s)ds\right\|^2\\
&+6\mathbf{E}\left\|\sum_{\tau_1<\mathrm{t_k}<\tau_2}\left[\mathfrak{R}(\tau_2-\mathrm{t_k})\right]\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right\|^2.
\end{aligned}
$$

From (13), Holder's inequality, it follows that

$$
\mathbf{E}\left\|(\Theta_2 z)(\tau_2)-(\Theta_2 z)(\tau_1)\right\|^2 \quad\le\quad 6\mathbb{T}\left\|\int_0^{\tau_1}\left\|\mathfrak{R}(\tau_2-s)-\mathfrak{R}(\tau_1-s)\right\|\right\|^2 pr^*(s)ds
$$

$$\begin{aligned}
&+ \quad 6\mathbb{T}M_b\mathcal{G}\int_0^{\tau_1}\|\mathfrak{R}(\tau_2-s)-\mathfrak{R}(\tau_1-s)\|^2 ds \\
&+ \quad 6m\sum_{0<\mathrm{t_k}<\tau_1}\|\mathfrak{R}(\tau_2-\mathrm{t_k})-\mathfrak{R}(\tau_1-\mathrm{t_k})\|^2\tilde{M}_{\mathrm{k}} \\
&+ \quad 6\mathbb{T}\int_{\tau_1}^{\tau_2}\|\mathfrak{R}(\tau_2-s)\|^2 pr^*(s)ds \\
&+ \quad 6\mathbb{T}M_b\mathcal{G}\int_{\tau_1}^{\tau_2}\|\mathfrak{R}(\tau_2-s)\|^2 \\
&+ \quad 6mM\sum_{\tau_1<\mathrm{t_k}<\tau_2}\tilde{M}_{\mathrm{k}}.
\end{aligned}$$

The RHS is independent of $z \in \mathscr{B}_{\mathrm{k}}$ and tends to zero as $\tau_2 - \tau_1 \to 0$, since the compactness of $\mathfrak{R}(\mathrm{t})_{\mathrm{t}>0}$ implies the continuity in the uniform operator topology. Thus, Θ_2 maps $\mathscr{B}_{\mathrm{k}}$ into an equicontinuous family of functions. The equicontinuous for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.

Claim 3. $(\Theta_2\mathscr{B}_{\mathrm{k}})(\mathrm{t})$ is precompact set in $\mathbb{X}$. Let $0 < \mathrm{t} \leq \mathbb{T}$ be fixed, $0 < \epsilon < \mathrm{t}$, for $z \in \mathscr{B}_{\mathrm{k}}$,

$$\begin{aligned}
(\Theta_{2,\epsilon}z)(\mathrm{t}) &= \mathfrak{R}(\epsilon)\int_0^{\mathrm{t}-\epsilon}\mathfrak{R}(\mathrm{t}-s-\epsilon)\Lambda(s,z_s+\mathrm{y}_s)ds \\
&+ \mathfrak{R}(\epsilon)\int_0^{\mathrm{t}-\epsilon}\mathfrak{R}(\mathrm{t}-s-\epsilon)\mathfrak{B}\mathbf{u}_{z+\mathrm{y}}(s)ds \\
&+ \mathfrak{R}(\epsilon)\sum_{0<\mathrm{t_k}<\mathrm{t}-\epsilon}\mathfrak{R}(\mathrm{t}-\mathrm{t_k}-\epsilon)\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-})).
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\Theta}_{2,\epsilon}z)(\mathrm{t}) &= \int_0^{\mathrm{t}-\epsilon}\mathfrak{R}(\mathrm{t}-s)\Lambda(s,z_s+\mathrm{y}_s)ds + \int_0^{t-\epsilon}\mathfrak{R}(\mathrm{t}-s)\mathfrak{B}\mathbf{u}_{z+\mathrm{y}}(s)ds \\
&+ \sum_{0<\mathrm{t_k}<\mathrm{t}}\mathfrak{R}(\mathrm{t}-\mathrm{t_k})\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-})).
\end{aligned}$$

Using (13) and the compactness of $\mathfrak{R}(\mathrm{t})_{\mathrm{t}>0}$, we obtain $\mathbb{V}_\epsilon(\mathrm{t}) = \big\{(\Theta_{2,\epsilon}z)(\mathrm{t}) : z \in \mathscr{B}_{\mathrm{k}}\big\}$ is relative compact in $\mathbb{X}\ \forall\ \epsilon, 0 < \epsilon < \mathrm{t}$. And also by Lemma 2.5 , Holder inequality, for each $z \in \mathscr{B}_{\mathrm{k}}$, we get

$$\begin{aligned}
&\mathbf{E}\left\|(\Theta_{2,\epsilon}z)(\mathrm{t})-(\tilde{\Theta}_{2,\epsilon}z)(\mathrm{t})\right\|^2 \\
&\leq 3\mathbb{T}\int_0^{\mathrm{t}-\epsilon}\|\mathfrak{R}(\epsilon)\mathfrak{R}(\mathrm{t}-s-\epsilon)-\mathfrak{R}(\mathrm{t}-s)\|^2_{\mathcal{L}(\mathbb{X})}\mathbf{E}\|\Lambda(s,z_s+\mathrm{y}_s)\|^2 ds \\
&+ 3\mathbb{T}M_b\mathcal{G}\int_0^{\mathrm{t}-\epsilon}\|\mathfrak{R}(\epsilon)\mathfrak{R}(\mathrm{t}-s-\epsilon)-\mathfrak{R}(\mathrm{t}-s)\|^2_{\mathcal{L}(\mathbb{X})}ds \\
&+ 3m\sum_{\mathrm{t}-\epsilon<\mathrm{t_k}<\mathrm{t}}\|\mathfrak{R}(\epsilon)\mathfrak{R}(\mathrm{t}-\mathrm{t_k}-\epsilon)-\mathfrak{R}(\mathrm{t}-\mathrm{t_k})\|^2_{\mathcal{L}(\mathbb{X})}\mathbf{E}\left\|\mathrm{I_k}(z(\mathrm{t_k^-})+\mathrm{y}(\mathrm{t_k^-}))\right\|^2
\end{aligned}$$

So the set $\tilde{\mathbb{V}}_\epsilon(\mathrm{t}) = \left\{(\tilde{\Theta}_{2,\epsilon}z)(\mathrm{t}) : z \in \mathscr{B}_\mathrm{k}\right\}$ is precompact in $\mathbb{X}$ by using the total boundedness. Then for $z \in \mathscr{B}_\mathrm{k}$, we have

$$\begin{aligned}
&\mathbf{E}\left\|(\Theta_2 z)(\mathrm{t}) - (\tilde{\Theta}_{2,\epsilon}z)(\mathrm{t})\right\|^2 \\
&\leq 3\mathbb{T}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}} \|\mathfrak{R}(\mathrm{t}-s)\|^2 \mathbf{E}\left\|\Lambda(s, z_s + \mathtt{y}_s)\right\|^2 ds \\
&+ 3\mathbb{T}M_b\mathcal{G}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}} \|\mathfrak{R}(\mathrm{t}-s)\|^2 ds \\
&+ 3m\sum_{\mathrm{t}-\epsilon<\mathrm{t_k}<\mathrm{t}} \|\mathfrak{R}(\mathrm{t}-\mathrm{t_k})\|^2 \mathbf{E}\left\|\mathrm{I_k}(z(\mathrm{t}_\mathrm{k}^-) + \mathtt{y}(\mathrm{t}_\mathrm{k}^-))\right\|^2 \\
&\leq 3\mathbb{T}M\int_{\mathrm{t}-\epsilon}^{\mathrm{t}} pr^*(s)ds + 3\mathbb{T}M_b\mathcal{G}M\epsilon + 3mM\sum_{\mathrm{t}-\epsilon<\mathrm{t_k}<\mathrm{t}} \tilde{M}_\mathrm{k}.
\end{aligned}$$

Therefore,

$$\mathbf{E}\left\|(\Theta_2 z)(\mathrm{t}) - (\tilde{\Theta}_{2,\epsilon}z)(\mathrm{t})\right\|^2 \to 0, \text{ as } \epsilon \to 0^+,$$

and there are precompact sets arbitrarily close to the set $\mathbb{V}(\mathrm{t}) = \left\{(\Theta_{2,\epsilon}z)(\mathrm{t}) : z \in \mathscr{B}_k\right\}$, Thus the set $\mathbb{V}(\mathrm{t})$ is also precompact in $\mathbb{X}$. Thus, by Arzela-Ascoli theorem Θ_2 is compact. Thus, by Krasnoselskii fixed point theorem there exists a fixed point $z(.)$ for $\hat{\Theta}$ on $\mathscr{B}_\mathrm{k}$. If we define $\mathtt{x}(\mathrm{t}) = z(\mathrm{t}) + \mathtt{y}(\mathrm{t})$, $-\infty < \mathrm{t} \leq \mathbb{T}$, it is easy to say that $\mathtt{x}(.)$ is a mild solution of (1)-(3) satisfing $\mathtt{x}_0 = \varphi$, $\mathtt{x}(\mathbb{T}) = \mathtt{x}_1$. Hence the proof. □

4. Application

We consider the neutral impulsive stochastic functional integrodifferential equations with infinite delays, driven by a fBm of the form:

$$\begin{aligned}
&\frac{\partial}{\partial \mathrm{t}}\left[\mathtt{x}(\mathrm{t},\zeta) - \bar{\Gamma}(\mathrm{t}, \mathtt{x}(\mathrm{t}-\mathrm{k},\zeta)\right] \\
&= \frac{\partial^2}{\partial^2\zeta}\left[\mathtt{x}(\mathrm{t},\zeta) + \bar{\Gamma}(\mathrm{t}, \mathtt{x}(\mathrm{t}-\mathrm{k},\zeta)\right] \\
&+ \int_0^{\mathrm{t}} \bar{\mathfrak{B}}(\mathrm{t}-s)\frac{\partial^2}{\partial^2\zeta}\left[\mathtt{x}(\mathrm{t},\zeta) + \bar{\Gamma}(\mathrm{t}, \mathtt{x}(\mathrm{t}-\mathrm{k},\zeta)ds\right] \\
&+ \bar{\Lambda}(\mathrm{t}, \mathtt{x}(\mathrm{t}-\mathrm{k},\zeta) + c(\zeta)\mathbf{u}(\mathrm{t}) + \bar{\Xi}\frac{d\mathtt{B}^\mathrm{H}}{dt}, \text{ for } \mathrm{t} \neq \mathrm{t_k}, \ \mathrm{t} \geq 0 \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\Delta\mathtt{x}(\mathrm{t_k},\zeta) &= \mathtt{x}(\mathrm{t}_\mathrm{k}^+,\zeta) - \mathtt{x}(\mathrm{t}_\mathrm{k}^-,\zeta) = \int_{-\infty}^{\mathrm{t_k}} \alpha_\mathrm{k}(\mathrm{t}_\mathrm{k}^- - s)\mathtt{x}(s,\zeta)ds, \ \mathrm{k} = 1, 2, ...m, \\
\mathtt{x}(\mathrm{t},0) &= \mathtt{x}(\mathrm{t},\pi) = 0, \ 0 \leq \mathrm{t} \leq \mathbb{T}, \\
\mathtt{x}(s,\zeta) &= \varphi(s,\zeta), \ -\infty < s \leq 0, \ 0 \leq \zeta \leq \pi;
\end{aligned}$$

where $\mathtt{B}^{\mathrm{H}}(\mathrm{t})$ is cylindrical fBm, and $\varphi:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ is a given continuous stochastic process such that $\|\varphi\|^2_{\mathscr{B}_{\mathfrak{H}}}<\infty$. We take $\mathbb{X}=\mathbb{Y}=\mathbb{U}=\mathcal{L}^2([0,\pi])$ with norm $\|\cdot\|$. Define $\mathfrak{A}:\mathfrak{D}(\mathfrak{A})\subset\mathbb{X}\to\mathbb{X}$ given by $\mathfrak{A}=\frac{\partial^2}{\partial^2\zeta}$ with $\mathfrak{D}(\mathfrak{A})=\Big\{\mathtt{y}\in\mathbb{X}:$ $\mathtt{y}^{'}$ is absolutely continuous $\mathtt{y}^{''}\in\mathbb{X},\ \mathtt{y}(0)=\mathtt{y}(\pi)=0\Big\}$, thus

$$\mathfrak{A}\mathtt{x} \quad = \quad \sum_{n=1}^{\infty} n^2 < \mathtt{x}, e_n > e_n,\ \ \mathtt{x}\in\mathfrak{D}(\mathfrak{A}),$$

where $e_n=\sqrt{\frac{2}{\pi}}\sin n\mathtt{x}$, $n=1,2,...$ is an orthogonal set of eigenvector of $\mathfrak{A}$. The phase function $\mathfrak{H}(s)=e^{4s}$, $s<0$, then $l=\int_{-\infty}^0\mathfrak{H}(s)ds=1/4<\infty$ and the phase space $\mathscr{B}_{\mathfrak{H}}$ is Banach space with

$$\|\varphi\|_{\mathscr{B}_{\mathfrak{H}}} \quad = \quad \int_{-\infty}^{0}\mathfrak{H}(s)\sup_{\theta\in[s,0]}\mathbf{E}(\|\varphi(\theta)\|^2)^{1/2}ds.$$

System (16) can written in the abstract formulation of the system (1)-(3) as follows:

For $(\mathrm{t},\varphi)\in[0,\mathbb{T}]\times\mathscr{B}_{\mathfrak{H}}$, where $\varphi(\theta)(\zeta)=\varphi(\theta,\zeta)$, $(\theta,\zeta)\in(-\infty,0]\times[0,\pi]$, we put $\mathtt{x}(\mathrm{t})(\zeta)=\mathtt{x}(\mathrm{t},\zeta)$. The functions $\Gamma,\Lambda:[0,\mathbb{T}]\times\mathscr{B}_{\mathfrak{H}}\times\mathbb{X}\to\mathbb{X}$, $\Xi:[0,\mathbb{T}]\to\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$ are defined by

$$\begin{aligned}\Gamma(\mathrm{t},\mathtt{x}(\mathrm{t}-\mathrm{k},\zeta))ds &= \bar{\Gamma}(\mathrm{t},\mathtt{x}(\mathrm{t}-\mathrm{k},\zeta))ds)\\ \Lambda(\mathrm{t},\mathtt{x}(\mathrm{t}-\mathrm{k},\zeta))ds)+c(\zeta)\mathbf{u}(\mathrm{t}) &= \bar{\Lambda}(\mathrm{t},\mathtt{x}(\mathrm{t}-\mathrm{k},\zeta))ds)+c(\zeta)\mathbf{u}(\mathrm{t}).\end{aligned}$$

Further, we assume that $\mathfrak{B}:\mathbb{U}\to\mathbb{X}$ is defined by

$$\mathfrak{B}\mathbf{u}(\mathrm{t})(\zeta)=c(\zeta)\mathbf{u}(\mathrm{t}),\ \ 0\le\zeta\le\pi,\ \ \mathbf{u}\in\mathcal{L}^2([0,\mathbb{T}];\mathbb{U}).$$

and the linear operator $\mathfrak{W}:\mathcal{L}^2([0,\mathbb{T}];\mathbb{U})\to\mathbb{X}$ by

$$\mathfrak{W}\mathbf{u}(\zeta) \quad = \quad \int_0^{\mathbb{T}}\mathfrak{R}(\mathbb{T}-s)c(\zeta)\mathbf{u}(\mathrm{t})ds,\ \ 0\le\zeta\le\pi.$$

Let $\ker\mathfrak{W}=\Big\{\mathtt{x}\in\mathcal{L}^2([0,\mathbb{T}];\mathbb{U}),\mathfrak{W}\mathtt{x}=0\Big\}$ be the null space of $\mathfrak{W}$ and $[\ker\mathfrak{W}]^{\perp}$ be its orthogonal complement in $\mathcal{L}^2([0,\mathbb{T}];\mathbb{U})$. Let $\widetilde{\mathfrak{W}}=[\ker\mathfrak{W}]^{\perp}\to\mathrm{Range}(\mathfrak{W})$ be the restriction of $\mathfrak{W}$ to $[\ker\mathfrak{W}]^{\perp}$, $\mathfrak{W}$ is necessarily one-to-one. The inverse mapping theorem tells that $\widetilde{\mathfrak{W}}^{-1}$ is bounded since $[\ker\mathfrak{W}]^{\perp}$ and $\mathrm{Range}(\mathfrak{W})$ are Banach spaces. So that $\mathfrak{W}^{-1}$ is bounded and takes values in $\mathcal{L}^2([0,\mathbb{T}];\mathbb{U})\ \ker\mathfrak{W}$, assumption **(H8)** is satisfied. Hence, verify the assumptions on Theorem 3.1 are satisfied and hence, the system (16) is controllable on $(-\infty,\mathbb{T}]$.

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