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ON GENERALIZED LATTICE $\mathcal{B}_2$

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Abstract. This study is on a Boolean $\mathcal{B}$ or Boolean lattice L in abstract algebra with closed binary operation $*$, complement and distributive properties. Both Binary operations and logic properties dominate this set. A lattice sheds light on binary operations and other algebraic structures. In particular, the construction of the elements of this L set from idempotent elements, our definition of k-order idempotent has led to the expanded definition of the definition of the lattice theory. In addition, a lattice offers clever solutions to vital problems in life with the concept of logic. The restriction on a lattice is clearly also limit such applications. The flexibility of logical theories adds even more vitality to practices. This is the main theme of the study. Therefore, the properties of the set elements resulting from the binary operation force the logic theory. According to the new definition given, some properties, lemmas and theorems of the lattice theory are examined. Examples of different situations are given.

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1. Introduction

The term "Boolean algebra" honors George Boole (1815–1864), a self-educated English mathematician. He introduced the algebraic system initially in a small pamphlet, The Mathematical Analysis of Logic, published in 1847 in response to an ongoing public controversy between Augustus De Morgan and William Hamilton, and later as a more substantial book, to let us start with the definition of an idempotent element. Laws of Thought, published in 1854. Boole's formulation differs from that described above in some important respects. For example, conjunction and disjunction in Boole were not a dual pair of operations. Boolean algebra emerged in the 1860s, in papers written by William Jevons and Charles Sanders Peirce. The first systematic presentation of Boolean algebra

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and distributive lattices is owed to the 1890 Vorlesungen of Ernst Schröder. The first extensive treatment of Boolean algebra in English is A. N. Whitehead's 1898 Universal Algebra. Boolean algebra as an axiomatic algebraic structure in the modern axiomatic sense begins with a 1904 paper by Edward V. Huntington. Boolean algebra came of age as serious mathematics with the work of Marshall Stone in the 1930s, and with Garrett Birkhoff's 1940 Lattice Theory. In the 1960s, Paul Cohen, Dana Scott, and others found deep new results in mathematical logic and axiomatic set theory using offshoots of Boolean algebra, namely forcing and Boolean-valued models [5]. In this way, many authors have brought their studies on this subject to the literature.

There are many uses for idempotent matrices, especially regression analysis in statistics, economics and Computer Science. Coined 1870 by American mathematician Benjamin Peirce in context of algebra in [1]. In particular, by taking the variables to represent values of on and off (or 0 and 1), Boolean algebra is used to design and analyze digital switching circuitry, such as that found in personal computers, pocket calculators, cd players, cellular telephones, and a host of other electronic products in [4].

There are many studies on idempotent matrices in the literature. Let us start with the information that takes us to the depth of the study. The set of solutions of the equation $x^2 = 1$ in real numbers is $\{-1, 1\}$. The real number -1 is not an idempotent element. The set of solution of the equation $x^2 = x$ is $\{0, 1\}$. The set of solutions of this equation, namely 0 and 1 are idempontent elements in [2].

The set of all matrices of order n over a field $\mathcal{F}$ is denoted by $\mathbb{M}_n(F)$. The equation $AX = B$ is written for matrices $A, X, C \in \mathbb{M}_n(F)$. If matrix $A = C = X$ is taken in equation $AX = C$, then equation $X^2 = X$ is obtained. Necessary conditions in the solution of this equation are necessary for the following definition.

$$X^2 - X = [0] \implies X(X - I_n) = [0] \implies X = [0], X = I_n.$$

Also, the solutions of the equation $X^2 = X$ are zero dividing matrices,

$$X^2 - X = [0]\{-1, 0, 1\}.X \neq [0], X - I_n \neq 0.$$

2. Main results

Definition 2.1. A group is a set B equipped with a binary operation $* : B \times B \to B$ that associates an element $a * b \in B$ to every to elements $a, b \in B$, and having the following properties: $*$ is associative, has an identity element $e \in B$, and every element in B is invertible (w.r.t. $*$). More explicitly, this means that the following equations hold for all $a, b, c \in B$:

(i) $(a * b) * c = a * (b * c)$ (associativity);
(ii) $a * e = e * a = a$ (identity);
(iii) For every $a \in B$, there is some $a^{-1} \in B$ such that $a * a^{-1} = a^{-1} * a = e$ (inverse).

A set B together with an operation $* : B \times B \to B$ and satisfying only conditions (i) and (ii) is called a *monoid* in [6, 7].

The set all idempotent elements in the monoid $(\mathbb{R}, +)$ is $\{0\}$ and the set of all idempotent elements in the monoid $(\mathbb{R}, \cdot)$ is $\{0,1\}$.
The set of all periodic elements k^{th}degree in the monoid B with a binary operation $*$ is denoted by

$$P^k(B,*) = \{a \in B | \underbrace{a * \cdots * a}_{(k+1)-times} = a, k \in \mathbb{Z}^+\}.$$

The set of all idempotent elements in the monoid B with a binary operation $*$ is denoted by

$$I^2(B,*) = \{a \in B | a * a = a, \}.$$

The set of all $k-$potent elements in the monoid B with a binary operation $*$ is denoted by

$$I^k(B,*) = \{a \in B | \underbrace{a * \cdots * a}_{k-times}, k \in \mathbb{Z}^+\}.$$

Idempotent matrices have been the objects of many studies in matrix theory and its applications. The set of idempotent elements on the monoid $\mathbb{M}_n(\mathbb{R}, \cdot)$ is

$$I^k(\mathbb{M}_n(\mathbb{R}), \cdot) = \{A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R}) | A^k = A, k \in \mathbb{Z}^+\} = \{[0], I_n, \ldots\}.$$

Or it is seen that

$$I^k(\mathbb{M}_n(\mathbb{R}, \cdot) = \{A | A = [0] \ or \ A \in P^{k-1}(\mathbb{M}_n(\mathbb{R}, \cdot)), k \in \mathbb{Z}^+\}.$$

If $n = 2$ and $k = 2$, this set $\mathbb{M}_2(\mathbb{R}), \cdot)$ is

$$I^2(\mathbb{M}_2(\mathbb{R}), \cdot) = \{A = [a_{ij}]_2 \in \mathbb{M}_2(\mathbb{R}) | A^2 = A\}$$

Then, the lattice $\mathcal{B}$ belonging to this monoid is also denoted by

$$k^{I^k(B,*)} = \{A \in B | A^k = A, k \in \mathbb{Z}^+\}.$$

or

$$k^B = \{e_{min.}, ..., A^k = A, ..., I_{max.}\}.$$

If $x \in 2^{(\mathbb{R},.)}$ then

$$x^2 = x \implies x^2 - x \implies x(x-1) = 0 \implies x = 0, x = 1.$$

The complement of x is $x - 1$ and the complement of $x - 1$ is x. Shortly,

$$2^{(\mathbb{R},.)} = \{0, 1\}.$$

If $B = \{x \in \mathbb{R} | x^k = x\}$,then $B = \{-1, 0, 1\}$ and then the k^{th} lattice belonging to B in $\mathbb{R}$ with multiplication is

$$k^B = \{-1, 0, 1\}.$$

And if $k = 2$ then $2^B = \{0, 1\}$.

Definition 2.2. ([8]). A Boolean algebra B is a system $\mathcal{B} = (B, \wedge, \vee, ', 0, 1\ )$ such that $\wedge$ and $\vee$ are binary operations on B, "$'$" is a unary operation on B, and $0, 1 \in B$, and that the following conditions hold for all $x, y, z \in B$:

(i) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;
(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$;
(iii) $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$;
(iv) $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$ and $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$;
(v) $x \wedge x' = 0$ and $x \vee x' = 1$.

Proposition 2.3. *([10, 11]). Let $x, y \in B$, $x \leq y$ then, the followings are equivalent.*

(i) $x \leq y$.
(ii) $x \vee y = y$.
(iii) $x \wedge y = x$.
(iv) $x' = 1 - x$.

An ordered structure $(B, \vee, \wedge, \leq)$ is a Boolean lattice if and only if it satisfies items (i), (ii) and (iii) of Proposition 2.3.

Definition 2.4. A ring in the mathematical sense is a set B together with two binary operators $+$ and $*$ (commonly interpreted as addition and multiplication, respectively) satisfying the conditions $i - v$

(i) Additive associativity: For all $a, b, c \in B$, $(a + b) + c = a + (b + c)$,
(ii) Additive commutativity: For all $a, b \in B$, $a + b = b + a$,
(iii) Additive identity: There exists an element $0 \in B$ such that for all $a \in B$, $0 + a = a + 0 = a$,
(iv) Additive inverse: For every $a \in B$ there exists $-a \in B$ such that $a + (-a) = (-a) + a = 0$,
(v) Left and right distributivity: For all $a, b, c \in B$, $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$,

A ring satisfying all the following properties (vi)-(ix) is called a field and called a division algebra if satisfying the properties vi, vii and ix in [9]

(vi) Multiplicative associativity: For all $a, b, c \in B$, $(a * b) * c = a * (b * c)$ (a ring satisfying this property is sometimes explicitly termed an associative ring). Rings may also satisfy various optional conditions:
(vii) Multiplicative commutativity: For all $a, b, \in B$, $a * b = b * a$ (a ring satisfying this property is termed a commutative ring),
(viii) Multiplicative identity: There exists an element $1 \in B$ such that $a \neq 0$ for all $a \in B$, $1 * a = a * 1 = a$ (a ring satisfying this property is termed a unit ring, or sometimes a "ring with identity"),
(ix) Multiplicative inverse: For each $a \neq 0$ for all $a \in B$, there exists an element $a^{-1} \in B$ that $a \neq 0$ for all $a \in B$, $a * a^{-1} = a^{-1} * a = 1$, where 1 is the identity element.

The power set of a set B is the set of all subsets of B. The power set of B is denoted by $\mathbb{P}(B)$. If the set B is finite with m elements, it is denoted by

$|B| = m$. The effect of the operations defined on the set B is very important. These operations take an effective role in the formation of the power set.

3. Generalized Lattice $\mathcal{B}_k$

In this section, the basic concept of logic is discussed by considering the idempotent elements which are the basic building blocks of the known lattice theory. If B is an arbitrary ring then its set of *central idempotents*, which is the set

$$Cen(B) = \{e \in B | e^2 = e, xe = ex = x \text{ } for \text{ } all \text{ } x \in B\}$$

becomes a Boolean algebra when its operations are defined by

$$x \vee y := x + y - xy = max\{x, y\}, x \wedge y := xy = min\{x, y\}.$$

Let B be an arbitrary ring. It is clear that $Cen(B) \subseteq k^B$.
The current lattice is

$$\mathcal{B}_2 = (B,', \wedge, \vee, 0, 1 \text{ }) \equiv (B,', \wedge, \vee, 2^B \text{ }).$$

If the equation $x^k = x$ is taken into account instead of the equation $x^2 = x$, then

$$\mathcal{B}_k \equiv (B,', \wedge, \vee, k^B \text{ }), where \text{ } k \geq 2, k \in Z^+.$$

In $\mathcal{B}_2$

$$x \in B \Rightarrow x^2 = x \Rightarrow 2^B = \{0, 1\}.$$

If the number of elements of the set B is m, then total number of elements in power set is 2^m in $\mathcal{B}_2$. This number is 3^m in $\mathcal{B}_3$.

Example 3.1. Let $x \in \mathbb{R}$ and $B = \{x\}$. The power set of B in $\mathcal{B}_2$ is $\mathbb{P}(B) = \{\emptyset, \{x\}\}$. But the power set of B in $\mathcal{B}_3$ is $\mathbb{P}(B) = \{\emptyset, \{x\}, \{-x\}\}$.

Proposition 3.2. *Let B be a finite set in $\mathcal{B}_3$ with $|B| = m$. Then*

$$|\mathbb{P}(B)| = 3^m.$$

Proof. Let $|B| = m$ be the finite set in $\mathcal{B}_3$. Then,

$$\sum_{i=0}^{m} \binom{m}{i} \sum_{k=0}^{i} \binom{i}{k} = \sum_{i=0}^{m} \binom{m}{i} 1^{m-i} 2^i = (1 + 2)^m = 3^m.$$

□

Let T be true, F be false, and L be lie, imaginary or obscure in a logical expression in calculations below.
In $\mathcal{B}_3$

$$x \in B \Rightarrow x^3 = x \Rightarrow 3^B = \{-1, 0, 1\}.$$

If $x = 0 \Rightarrow x' = 1 \vee x' = -1$.

$$x \vee x' = \begin{cases} 0, if & x' = 1 \\ 1, if & x' = -1 \end{cases}$$

TABLE 1. In this table, the logic values in $\mathcal{B}_2$ are given [3].

x	x'	$x \wedge x'$	$x \vee x'$
0	1	0	1
1	0	0	1
F	T	F	T
T	F	F	T

$$x \wedge x' = \begin{cases} 0, if & x' = 1 \\ -1, if & x' = -1. \end{cases}$$

$$\mathcal{B}_3 = (B,', \wedge, \vee, -1, 0, 1\ ) \equiv (B,', \wedge, \vee, 3^B\ ).$$

$$\mathcal{B}_2 \subseteq \mathcal{B}_3.$$

TABLE 2. In this table, the logic values in $\mathcal{B}_3$ are given.

x	x_1'	x_2'	$x \vee x_1'$	$x \wedge x_1'$	$x \vee x_2'$	$x \wedge x_2'$
0	-1	1	0	-1	1	0
1	0	-1	1	0	1	-1
-1	0	1	0	-1	1	-1
F	L	T	F	L	T	F
T	F	L	T	F	T	L
L	F	T	F	L	T	L

Proposition 3.3. *Let $\mathcal{B}_3$ be a Boolean Lattice. Then the following properties are equivalent for any $x \in B$.*

(i) $x' = x^2 - 1$.
(ii) *Any one x' of x is in $\mathcal{B}_2$.*
(iii) $x' = 1$,$x' = -1$.

Proof. Let $\mathcal{B}_3$ be a Boolean Lattice and $x \in B$. Then

$$x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0$$

If (i) holds, then

$$x' = x^2 - 1 \Rightarrow (x+1)(x-1) = 0 \Rightarrow x' = 1 \Rightarrow (ii).$$

If (ii) and (i) hold, then

$$x' = -1, x' = -1 \Rightarrow (iii).$$

If (iii) holds, then

$$x' = (x-1), x' = (x+1) \Rightarrow x' = x^2 - 1. \Rightarrow (i).$$

□

Lemma 3.4. *Let $\mathcal{B}_k$ be a Boolean ring. If $k \in \mathbb{Z}^+$ is odd, then*

$$\mathcal{B}_{2k} \subseteq \mathcal{B}_3.$$

Proof. The proof is done by the induction method. For $k = 1$, $\mathcal{B}_2 \subseteq \mathcal{B}_3$. Let us assume $\mathcal{B}_{2k-2} \subseteq \mathcal{B}_3$ is true for $k-1$. We have to prove that $\mathcal{B}_{2k} \subseteq \mathcal{B}_3$.

$$x \in \mathcal{B}_{2k-2} \Rightarrow x^{2k-2} = x.$$

$$x^k = (x^{2k-2})(x^{2-k}) = x(x^2)(x^k)^{-1} = x^3(x^{-1}) = x^2 = x.$$

$$\Rightarrow x \in \mathcal{B}_3 \Rightarrow \mathcal{B}_{2k} \subseteq \mathcal{B}_3.$$

□

Lemma 3.5. *Let $\mathcal{B}_3$ be a Boolean ring. Then*

(i) $x \vee x' = 1, x \vee x' = 0.$
(ii) $x \wedge x' = -1, x \wedge x' = 0.$

Proof. It enough to prove this only for $x = 0$, loss of generality. Therefore, Proof of (i):

$$x = 0 \Rightarrow x \vee x' = 1, x \vee x' = 0.$$

And Proof of (ii):

$$x = 0 \Rightarrow x \wedge x' = -1, x \wedge x' = 0.$$

□

Let us explain that it is necessary to use $\mathcal{B}_3$ in practice. $-1 \equiv 2(mod 3)$ in $\mathcal{B}_3$. Although $-1 \equiv 1(mod 2)$ in $\mathcal{B}_2$, Likewise, $1 \neq 2$ in $\mathcal{B}_3$. Therefor $\mathcal{B}_3$ necessitates from this situation.

4. Conclusion

In existing applications, the logic values in B are operated. This is due to some gaps, concerns, etc. in the future. drives communities. I think that the trust and quality of life of societies will increase when logic values are used in C to sustain vital values.

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