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CERTAIN TOPOLOGICAL METHODS FOR COMPUTING DIGITAL TOPOLOGICAL COMPLEXITY

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Abstract. In this paper, we examine the relations of two closely related concepts, the digital Lusternik-Schnirelmann category and the digital higher topological complexity, with each other in digital images. For some certain digital images, we introduce $\kappa-$topological groups in the digital topological manner for having stronger ideas about the digital higher topological complexity. Our aim is to improve the understanding of the digital higher topological complexity. We present examples and counterexamples for $\kappa-$topological groups.

1. Introduction

The interaction between two popular topics (digital image processing and robotics) can often be very valuable in science. The subject of robotics is rapidly increasing its popularity. In the digital topology, one of the extraordinary fields of mathematics, with using topological properties, we can melt these two topics in one pot. Thus, we are trying to build a theoretical bridge between motion planning algorithms of a robot and digital image analysis. In future studies, we think that the theoretical knowledge will focus on the applications of industry, perhaps in other fields. In more details, an autonomous robot is expected to be able to determine its own direction and route without any help. There are many types of robots using motion planning algorithms for this duty, especially industrial and mobile robots. Industrial robots undertake some tasks in various fields such as assembly and welding works in the industry. As an example of mobile robots, we can consider unmanned aerial vehicles and the room cleaning robots.

Digital topology [26] has been developing and increasing its scientific importance. Many significant invariants of topology, especially homotopy, homology, and cohomology, have a substantial value for digital images. The digital homotopy is, in particular, our fundamental equipment. You can easily have the comprehensive knowledge about

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the digital homotopy from [5–8, 10, 11, 21–23, 25]. We are concerned with topological interpretations of robot motions in digital images. Farber [14] studies the notion topological complexity of motion planning. The construction of motion planning algorithms on topological spaces has been discussed. Different types of topological structures have been expertly placed in the theory of this subject [15]. Rudyak [28] first defines the higher topological complexity for ordinary topological spaces. İs and Karaca [19] has studied some results of Rudyak [28] for digital images. The digital meaning of the Lusternik-Schnirelmann category, denoted by cat, appears in [3] and we frequently use cat having precious results for the computations of TC_2 (in another saying, Farber's TC number) and TC_n. The definitions of cat, TC_2, and TC_n are expressed by the concept of the digital Schwarz genus of some digital fibrations. In a way, we can figure out that these concepts are closely related to all the properties of the notion digital Schwarz genus and each other. Moreover, the $\kappa-$topological group structure of digital images helps us stating one of the strongest relationship between cat and TC, where κ is an adjacency relation of a digital image. We introduce this notion and outline the framework of it.

The structure of the paper is as follows. First, we start by recalling the cornerstones of the digital topology and some previously emphasized properties for the digital higher topological complexity. After Preliminaries, we give basic facts about the digital Schwarz genus of a digital image and the digital Lusternik-Schnirelmann category of a digital image. However, our aim is to obtain a lower or an upper bound for TC_n in digital images. We deal with the digital topological groups in Section 4. After we give the definition of a $\kappa-$topological group for digital images, we present interesting examples for some digital images. Before the last section, we obtain various results using cat and $\kappa-$topological groups. Moreover, we give examples and counterexamples about certain digital images.

2. Preliminaries

For any positive integer r, a *digital image* (Y, c_k) consists of a subset Y of $\mathbb{Z}^r$ and an adjacency relation c_k for the elements of Y such that the relation is defined as follows: Two distinct points y and z in $\mathbb{Z}^r$ are c_k-*adjacent* [21] for a positive integer k with $1 \leq k \leq r$, if there are at most k indices i such that $|y_i - z_i| = 1$, and for all other indices i such that $|y_i - z_i| \neq 1$, $y_i = z_i$. In the one-dimensional case, if we study in $\mathbb{Z}$, then we merely have the 2−adjacency. There are completely two adjacency relations 4 and 8 in $\mathbb{Z}^2$ and completely three adjacency relations 6, 18, and 26 in $\mathbb{Z}^3$. The subsets of $\mathbb{Z}^4$ have 8, 32, 64, and 80 as the adjacency relations. Similarly, $\mathbb{Z}^5$ and $\mathbb{Z}^6$ have the possible adjacency groups $(10, 50, 130, 210, 242)$ and $(12, 72, 232, 472, 664, 728)$, respectively. The notion c_k-adjacency is introduced by Rosenfeld [26, 27] and the generalized version of this notion is mentioned in [16, 17]. Note that the notations κ and λ are often used for the adjacency relation instead of c_k in digital topology.

Let $Y \subset \mathbb{Z}^r$ be a digital image. Then Y is $\lambda-$*connected* [17] if and only if for any $y, z \in Y$ with $y \neq z$, there is a set $\{y_0, y_1, ..., y_m\}$ of points of Y such that $y = y_0$, $z = y_m$ and y_i and y_{i+1} are $\lambda-$adjacent, where $i = 0, 1, ..., m-1$. Let (Y_1, λ_1) and (Y_2, λ_2) be two digital images in $\mathbb{Z}^{r_1}$ and $\mathbb{Z}^{r_2}$, respectively. Let $f : Y_1 \to Y_2$ be a map. Then f is $(\lambda_1, \lambda_2)-$*continuous* [5, 27] if, for any λ_1-connected subset A_1 of Y_1, $f(A_1)$

is also λ_2-connected. The composition of any two digitally continuous maps is again digitally continuous (see Proposition 2.5 in [5]).

Let Y_1 and Y_2 be two digital images. Then we mean a digital map $f : Y_1 \to Y_2$ by any map of digital images from Y_1 to Y_2. A digital map $f : (Y_1, \lambda_1) \to (Y_2, \lambda_2)$ is called a $(\lambda_1, \lambda_2)-$*isomorphism* [8,16] if f is bijective, $(\lambda_1, \lambda_2)-$continuous and also f^{-1} is $(\lambda_2, \lambda_1)-$continuous. A set $[y_1, y_2]_{\mathbb{Z}} = \{z \in \mathbb{Z} : y_1 \leq z \leq y_2\}$ is said to be a *digital interval from* y_1 *to* y_2 [4,7]. Let $[0, r]_{\mathbb{Z}}$ be a digital image with a positive integer r. For any digital image (Y, λ), if $f : [0, r]_{\mathbb{Z}} \to Y$ is a $(2, \lambda)-$continuous map with $f(0) = y_1$ and $f(r) = y_2$, then f is called a *digital path* [26] between the initial point y_1 and the final point y_2. Two digital paths f_1 and f_2 in (Y, λ) are *adjacent paths* [24] if $f_1(t)$ and $f_2(t^{'})$ are $\lambda-$adjacent or equal whenever t and $t^{'}$ are $2-$adjacent.

Suppose that $f_1, f_2 : (Y_1, \lambda_1) \to (Y_2, \lambda_2)$ are $(\lambda_1, \lambda_2)-$continuous maps, where $Y_1 \subset \mathbb{Z}^{r_1}$ and $Y_2 \subset \mathbb{Z}^{r_2}$. f_1 and f_2 are $(\lambda_1, \lambda_2)-$*homotopic* [5,20] in Y (denoted by $f_1 \simeq_{(\lambda_1,\lambda_2)} f_2$), if, for a positive integer m, there is a digitally continuous map $F : Y_1 \times [0, m]_{\mathbb{Z}} \to Y_2$ which admits the following conditions:

- for all $y \in Y_1$, $F(y, 0) = f_1(y)$ and $F(y, m) = f_2(y)$;
- for all $y \in Y_1$ and for all $s \in [0, m]_{\mathbb{Z}}$,
$$F_y : [0, m]_{\mathbb{Z}} \longrightarrow Y_2$$
$$s \longmapsto F_y(s) = F(y, s)$$
is $(2, \lambda_2)-$continuous;
- for all $s \in [0, m]_{\mathbb{Z}}$ and for all $y \in Y_1$,
$$F_s : Y_1 \longrightarrow Y_2$$
$$y \longmapsto F_s(y) = F(y, s)$$
is $(\lambda_1, \lambda_2)-$continuous.

The function F in the definition above is said to be *digital homotopy* between f_1 and f_2. Note that a homotopy relation is equivalence on the set of digitally continuous maps [5].

Let (Y_1, λ_1) and (Y_2, λ_2) be any digital images. Let $f : Y_1 \to Y_2$ be a digitally $(\lambda_1, \lambda_2)-$continuous map. Then f is called $(\lambda_1, \lambda_2)-$*nullhomotopic* [4,20] in Y_2 if f is $(\lambda_1, \lambda_2)-$homotopic to a constant map in Y_2. Assume that the digital map $f : (Y_1, \lambda_1) \to (Y_2, \lambda_2)$ is $(\lambda_1, \lambda_2)-$continuous. If there exists a $(\lambda_2, \lambda_1)-$continuous map $g : (Y_2, \lambda_2) \to (Y_1, \lambda_1)$ for which $g \circ f \simeq_{(\lambda_1,\lambda_1)} id_{Y_1}$ and $f \circ g \simeq_{(\lambda_2,\lambda_2)} id_{Y_2}$, then f is a $(\lambda_1, \lambda_2)-$*homotopy equivalence* [6]. We say that a digital image (Y, λ) is $\lambda-$*contractible* [4,20] if id_Y is $(\lambda, \lambda)-$homotopic to a map c of digital images for some $c_0 \in Y$, where $c : Y \to Y$ is defined with $c(y) = c_0$ for all $y \in Y$.

The adjacency relation varies in several digital images. For instance, an adjacency relation on the set of digital functions is discussed in [24]. For any images X and Y, the *digital function space* Y^X is stated with the set of all maps $X \to Y$ with an adjacency as follows: for any two maps $f, g : X \to Y$, they are called *adjacent* in the set of digital function spaces if $f(x)$ and $g(x^{'})$ are adjacent points in Y whenever x and $x^{'}$ are adjacent points in X. Another crucial example is given on the cartesian product of digital images [2]: Let (Y, λ_1) and (Z, λ_2) be any two digital images such that the points (y, z) and (y', z') belong to $Y \times Z$. Then (y, z) and (y', z') are *adjacent in the cartesian product digital image* $Y \times Z$ if one of the following conditions holds:

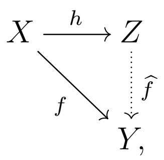
- $y = y'$, and z and z' are λ_2-adjacent; or
- y and y' are λ_1-adjacent, and $z = z'$; or

- y and y' are λ_1-adjacent, and z and z' are λ_2-adjacent.

The adjacency on the cartesian product is known as *the normal product adjacency* or *the strong product adjacency*. Boxer and Karaca [10] show that the normal product adjacency need not to be a c_k-adjacency. The normal product adjacency is completely determined by the adjacencies of the factors. Proposition 3.1 of [10] says that the coincidence of the normal product adjacency with a c_k-adjacency is also possible: Let (X_i, c_{k_i}) be any digital images for $i \in \{1,2\}$. Then the normal product adjacency coincides with the $c_{k_1+k_2}-$adjacency for $X_1 \times X_2$. See [10] for more information on products in digital images.

The wedge of digital images is studied in [8, 16]: Let (X, κ_1) and (Y, κ_2) be any digital images such that $X \cap Y = \{*\}$. The union of X and Y is *the wedge of* X and Y, denoted $X \vee Y$, if $x \in X - \{*\}$ and $y \in Y - \{*\}$ implies x is adjacent to $*$, y is adjacent to $*$, and x and y are not adjacent. If a map $p : (X, \kappa_1) \to (Y, \kappa_2)$ has the digital homotopy lifting property for every digital image, then p is called a *digital fibration* [13].

Definition 2.1. [18] Let (X, κ_1) and (Y, κ_2) be any digitally connected images. A *digital fibrational substitute* of a map $f : (X, \kappa_1) \to (Y, \kappa_2)$ is a fibration $\widehat{f} : (Z, \kappa_3) \to (Y, \kappa_2)$ for which $\widehat{f} \circ h = f$, i.e.,



where h is the digital homotopy equivalence in the sense of digital homotopy.

Lemma 2.2 is first given in [18] with its proof. We restate this result with its proof in order to clarify the facts that are missing or need to be corrected in its proof.

Lemma 2.2. *In the digital setting, any map has a fibrational substitute.*

Proof. Let $f : X \to Y$ be a digital map and $m \in \mathbb{N}$. We define the digital set Z as $\{(x, \alpha) : x \in X,\ \alpha \text{ is a digital path on } Y \text{ with } \alpha(0) = f(x)\}$. Let λ be the normal product adjacency on $X \times Y^{[0,m]_{\mathbb{Z}}}$ and defined as follows: Given any two points (x_1, α_1), (x_2, α_2) in $X \times Y^{[0,m]_{\mathbb{Z}}}$, if x_1 and x_2 are adjacent or equal points, and α_1 and α_2 are adjacent or equal paths, then (x_1, α_1) and (x_2, α_2) are adjacent or equal pairs. Note that $Z \subset X \times Y^{[0,m]_{\mathbb{Z}}}$ implies that Z has $\lambda-$adjacency. We consider the digitally continuous map

$$g : Z \longrightarrow Y$$
$$(x, \alpha) \longmapsto \alpha(1),$$

and define $h : X \to Z$ with $h(x) = (x, \alpha_x)$, where $\alpha_x(t) = f(x)$ for all $t \in [0, m]_{\mathbb{Z}}$. Then we find

$$g \circ h(x) = g(x, \alpha_x) = \alpha_x(1) = f(x).$$

In order to show that g is a digital fibration, let $G : X \times [0, m]_{\mathbb{Z}} \to Y$ be a digital homotopy with the condition $g \circ h = G \circ i$ for the inclusion map $i : X \to X \times [0, m]_{\mathbb{Z}}$. When we define $\widetilde{G}$ as $\widetilde{G}(x, t) := h(x)$ for $t \in [0, m]_{\mathbb{Z}}$, we get

$$\widetilde{G} \circ i(x) = \widetilde{G}(x, t) = h(x),$$

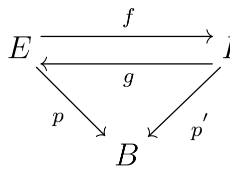
and

$$g \circ \widetilde{G}(x,t) = g \circ h(x) = G \circ i(x) = G(x,t).$$

This shows that g is a digital fibration. From the definition of digital fibrational substitutes, it is enough to say that $h : X \to Z$ is a digital homotopy equivalence. Since the digital map $k : Z \to X$, $k(x,\alpha) = x$, for any $(x,\alpha) \in Z$ is the digital homotopy inverse of h, we conclude that g is a digital fibrational substitute of f. □

Recall that the digital fiber homotopy equivalence over a digital image is defined in [18] as follows:

Definition 2.3. [18] Let $p : (E,\lambda_1) \to (B,\lambda_2)$ and $p^{'} : (E^{'},\lambda_1^{'}) \to (B,\lambda_2)$ be two digital fibrations. Then $f : (E,\lambda_1) \to (E^{'},\lambda_1^{'})$ is said to be a digital fiber homotopy equivalence fibration over B if there is a digital map $g : (E^{'},\lambda_1^{'}) \to (E,\lambda_1)$ such that $g \circ f$ and $f \circ g$ are digitally fiber homotopic to the respective identity maps $1_{(E,\lambda_1)}$ and $1_{(E^{'},\lambda_1^{'})}$.



A *digital cover* of (X,κ) is a collection of the subsets $V_1, V_2, \cdots, V_l$ of X with the condition

$$\bigcup_{j=1}^{l} V_j = X.$$

Definition 2.4. [18] The digital Schwarz genus of a digital fibration

$$p : (E,\lambda_1) \to (B,\lambda_2),$$

denoted by $genus_{\lambda_1,\lambda_2}(p)$, is defined as a minimum number k for which $\{V_1, V_2, ..., V_k\}$ is a cover of B with the property that for all $1 \leq i \leq k$, there is a continuous map of digital images $s_i : (V_i,\lambda_1) \to (E,\lambda_2)$ such that $p \circ s_i = id_{V_i}$.

In the digital meaning, we note that the Schwarz genus of a digital map is invariant from the chosen fibrational substitute (see Lemma 3.4 in [18]).

Let (X,κ) and $([0,m]_{\mathbb{Z}},2)$ be two digital images. Recall that $X^{[0,m]_{\mathbb{Z}}}$ is a digital function space and consists of all digital paths in X. $X^{[0,m]_{\mathbb{Z}}}$ has an adjacency relation λ as follows [19,24]. For any two digital paths α and β in X, the fact a and b are 2-adjacent points in $[0,m]_{\mathbb{Z}}$ implies that $\alpha(a)$ and $\beta(b)$ are $\kappa-$adjacent points in X. Thus $X^{[0,m]_{\mathbb{Z}}}$ is a digital image with the adjacency relation λ.

Definition 2.5. [18] Let $X^{[0,m]_{\mathbb{Z}}}$ be a digital function space of all continuous functions from $[0,m]_{\mathbb{Z}}$ to a digitally connected image (X,κ) for any positive integer m. Then the *topological complexity of a digital image* X is

$$\mathrm{TC}(X,\kappa) = genus_{\lambda,\kappa_*}(p),$$

where $p : (X^{[0,m]_{\mathbb{Z}}},\lambda) \to (X \times X,\kappa_*)$, $p(w) = (w(0), w(m))$ is a fibration of digital images for any $w \in X^{[0,m]_{\mathbb{Z}}}$ with the normal product adjacency κ_* based on κ for the digital image $X \times X$.

For any n digital maps (paths) $f_1, f_2, \cdots, f_n : [0,m]_{\mathbb{Z}} \to X$, the product map $f : [0,m]_{\mathbb{Z}}^n \to X^n$ is defined by $f(t_1, t_2, \cdots, t_n) = (f_1(t_1), f_2(t_2), \cdots, f_n(t_n))$.

Definition 2.6. [18] Let n be a positive integer. Let J_n denote the wedge of the digital intervals $[0,m_1]_{\mathbb{Z}}, \cdots, [0,m_n]_{\mathbb{Z}}$, where $0_i \in [0,m_i]_{\mathbb{Z}}$ for $i = 1, \cdots, n$ are identified. Let X be a digitally connected space. Then the *higher topological complexity of a digital image* X is

$$\mathrm{TC}_n(X,\kappa) = genus_{\lambda,\kappa_*}(e_n),$$

where $e_n : (X^{J_n}, \lambda) \to (X^n, \kappa_*)$, $e_n(f) = (f_1(m_1), ..., f_n(m_n))$, is a fibration of digital images for a product map $f = (f_1, \cdots, f_n)$.

In Definition 2.6, κ_* and λ are adjacency relations on respective images X^n and X^{J_n}. Note that the digital higher topological complexity has significant rules [18]. One of them is that TC_1 is always equal to 1. Another is the coincidence of TC_2 with TC, when $n = 2$. Moreover, the number TC_n is not greater than TC_{n+1} anymore.

Proposition 2.7. [18] *Let $d_n : (X,\kappa) \to (X^n, \kappa_*)$ be a diagonal map such that κ_* is the normal product adjacency for the image X^n. Then*

$$TC_n(X,\kappa) = genus_{\kappa,\kappa_*}(d_n).$$

We finish this section with the digital Lusternik-Schnirelmann category $\mathrm{cat}_\kappa(X)$ of the image (X,κ).

Definition 2.8. [3] *The digital Lusternik-Schnirelmann category* of a digital image X (denoted by $\mathrm{cat}_\kappa(X)$) is defined to be the minimum number k for which there is a cover $\{U_1, U_2, ..., U_{k+1}\}$ of X that satisfies each inclusion map from U_i to X, for $i = 1, ..., k$, is $\kappa-$nullhomotopic in X.

In [3], X is covered with $(k+1)$ sets $U_1, U_2, ..., U_{k+1}$ in the definition of the digital L-S category. According to this definition, the result $\mathrm{cat}_\kappa(X) = 0$ is hold when X is $\kappa-$contractible. Considering the strong relationship of cat and TC, we prefer to cover X with k subsets in the definition similar to Farber [14]. This leads that, for instance, a $\kappa-$contractible space X admits that $\mathrm{cat}_\kappa(X) = 1$. In general, for the digital Lusternik-Schnirelmann category, we admits one more than the number computed in [3].

Example 2.9. Let $H = \{a,b,c,d,e,f,g,h\}$ be an image in $\mathbb{Z}^2$ with the $4-$adjacency (see Figure 1) such that

$$a = (0,-1),\ \ b = (0,0),\ \ c = (0,1),\ \ d = (1,1),$$
$$e = (2,1),\ \ f = (2,0),\ \ g = (2,-1),\ \ h = (1,-1).$$

Since H is not $4-$contractible [9], $\mathrm{cat}_4(H) > 1$. By Theorem 2.10 of [29], we conclude that $\mathrm{cat}_4(H) = 2$.

The notion domination between any two adjacencies is given by Boxer [12]. Let κ and λ be two adjacencies on a set X. An adjacency κ *dominates* λ, denoted by $\kappa \geq_d \lambda$, if the fact that x_1 and x_2 are $\kappa-$adjacent implies that x_1 and x_2 are $\lambda-$adjacent for any x_1, $x_2 \in X$. There are some papers such as [19] and [3] that use the notations $\kappa \geq \lambda$ (or $\lambda \geq \kappa$). Let $u \leq v$. Then this means that $c_u \leq c_v$. On the other hand, from the definition in [12], it is appropriate to use $c_v \leq c_u$ for maintaining the consistency.

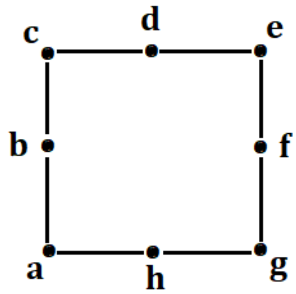


FIGURE 1. The Digital Image H.

THEOREM 2.10. [3] *Let κ and λ be different adjacency relations with $\kappa \leq \lambda$ on a digital image X. Then*

$$cat_{\kappa}(X) \leq cat_{\lambda}(X).$$

THEOREM 2.11. [19] *Let (X, κ) be a digitally connected space such that $X \times X$ has the normal product κ_*-adjacency. Then*

$$cat_{\kappa}(X) \leq TC(X, \kappa) \leq cat_{\kappa_*}(X \times X).$$

3. The Digital LS-Category For The Digital Higher Topological Complexity

PROPOSITION 3.1. *Let $f : (X_1, \kappa_1) \to (Y_1, \lambda_1)$ and $g : (X_2, \kappa_2) \to (Y_2, \lambda_2)$ be two digitally continuous digital maps. Let $f \times g : (X_1 \times X_2, \kappa_*) \to (Y_1 \times Y_2, \lambda_*)$ be the digital product map. Then we have that*

$$genus_{\kappa_*,\lambda_*}(f \times g) \leq genus_{\kappa_1,\lambda_1}(f) + genus_{\kappa_2,\lambda_2}(g).$$

Proof. We first consider the case that f and g are digital fibrations and shall show the desired result. Let $genus_{\kappa_1,\lambda_1}(f) = k$ and $genus_{\kappa_2,\lambda_2}(g) = l$. We shall show that $genus_{\kappa_*,\lambda_*}(f \times g) \leq k + l$. Since $genus_{\kappa_1,\lambda_1}(f) = k$, we may divide the digital image Y_1 into the k subsets $U_1, U_2, ..., U_k$ such that for all $i = 1, ..., k$, there exist digitally continuous maps

$$s_i : (U_i, \tau_i) \to (X_1, \kappa_1)$$

and $f \circ s_i$ is an identity map on the digital image (U_i, λ_i). Similarly, if $genus_{\kappa_2,\lambda_2}(g) = l$, then we may divide the digital image Y_2 into l subsets $V_1, V_2, ..., V_l$ such that there exist digitally continuous maps

$$t_j : (V_j, \sigma_j) \to (X_2, \kappa_2)$$

for all $j = 1, ..., l$, and $g \circ t_j$ is an identity map on the digital image (V_j, σ_j). Consider the digitally continuous map

$$f \times g : X_1 \times X_2 \to (U_1 \cup ... \cup U_k) \times (V_1 \cup ... \cup V_l).$$

We rewrite this map in 2 different ways:

$$f \times g : X_1 \times X_2 \to (U_1 \times Y_2) \cup ... \cup (U_k \times Y_2) \tag{1}$$

and

$$(2) \qquad f \times g : X_1 \times X_2 \to (Y_1 \times V_1) \cup ... \cup (Y_1 \times V_l).$$

Let $h_i : (A_i, \sigma_i^{'}) \to (B_i, \tau_i^{'})$ be digital functions for $i = 1, \cdots, k$. Then

$$h_1 \cup \cdots \cup h_k : A_1 \cup \cdots \cup A_k \to B_1 \cup \cdots \cup B_k$$

is the union of digital functions $h_1, \cdots, h_k$ if and only if

$$h_1|_{(A_1 \cap \cdots \cap A_k)} = \cdots = h_k|_{(A_1 \cap \cdots \cap A_k)}.$$

Consider the map (1). Using this fact, there exists a digitally continuous map

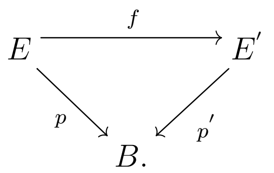
$$w_i : (U_i \times Y_2) \to X_1 \times X_2$$

such that $f \circ w_i$ is the identity on $Y_1 \times Y_2$. Indeed, w_i is the union of digital functions $s_i \times t_1, \cdots, s_i \times t_l$. Similarly, for the map (2), we have a digitally continuous map

$$v_j : (Y_1 \times V_j) \to X_1 \times X_2$$

such that $f \circ v_j$ is the identity on $Y_1 \times Y_2$. Here v_j is the union of digital functions $s_1 \times t_j, \cdots, s_k \times t_j$. Moreover, some of $U_i \times V_j$, for each i and j, can be the same in the union of sets. So we conclude that $genus_{\kappa*,\lambda*}(f \times g)$ must be less than or equal to $k + l$. When f and g are not fibrations in the digital sense, by Lemma 3.3 of [18], we use their digital fibrational substitutes to show that the desired inequality holds and this completes the proof. □

Proposition 3.2. *Let $p : (E, \lambda_1) \to (B, \lambda_2)$ and $p^{'} : (E^{'}, \lambda_1^{'}) \to (B, \lambda_2)$ be two digital fibrations such that the following diagram commutes:*



If f is a digital fiber homotopy equivalence over B, then $genus_{\lambda_1,\lambda_2}(p) = genus_{\lambda_1^{'},\lambda_2}(p^{'})$.

Proof. Let $\text{genus}_{\lambda_1,\lambda_2}(p) = r$. Then $s_i : U_i \to E$ is a digital section of p over U_i for each $i = 1, ..., r$. It follows that $f \circ s_i$ is a digital section of $p^{'}$ over each U_i. Therefore, we get $\text{genus}_{\lambda_1,\lambda_2}(p^{'}) \leq r$. Moreover, if we assume that f is a digital fiber homotopy equivalence over B, then there is a digital homotopy inverse $f^{'} : (E^{'}, \lambda_1^{'}) \to (E, \lambda_1)$ satisfying the fact $r \leq \text{genus}_{\lambda_1^{'},\lambda_2}(p^{'})$. This shows that $\text{genus}_{\lambda_1^{'},\lambda_2}(p^{'}) = r$. □

Proposition 3.3. *For a digital fibration $p : (E, \lambda_2) \to (B, \lambda_3)$,*

$$genus_{\lambda_2,\lambda_3}(p) \leq cat_{\lambda_3}(B).$$

Moreover, if (E, λ_2) is digitally contractible, then $genus_{\lambda_2,\lambda_3}(p) = cat_{\lambda_3}(B)$.

Proof. First, we show that $genus_{\lambda_2,\lambda_3}(p) \leq \text{cat}_{\lambda_3}(B)$. Let $\text{cat}_{\lambda_3}(B) = k$. Then we have k digital covering made by k subset $\{U_1, U_2, ..., U_k\}$ of B, where each inclusion $U_i \to B$ for $i = 1, ..., k$ is digitally λ_3-nullhomotopic in B. Assume $U \subseteq B$, where

U is one of the sets in the covering of B and consider the following diagram for the positive integer m:

$$\begin{array}{ccc} U \times \{0\} & \xrightarrow{c_0} & E \\ {\scriptstyle i}\big\downarrow & \overset{G}{\nearrow} & \big\downarrow{\scriptstyle p} \\ U \times [0,m]_{\mathbb{Z}} & \xrightarrow{H} & B, \end{array}$$

where i is the digital inclusion map. For any $t \in [0,m]_{\mathbb{Z}}$ and $b \in U$, c_0 is the digital constant map defined by $c_0(b,0) = e_0$, where e_0 is a chosen point in $p^{-1}(b_0)$ for any basepoint $b_0 \in B$. H is a digital contracting homotopy between the digital constant map at the basepoint b_0 and the digital inclusion map $U \hookrightarrow B$. Using the digital homotopy lifting property, there is a digitally continuous map G for which $G \circ i = c_0$ and $p \circ G = H$. It follows that

$$p \circ G(x,m) = H(x,m) = id_U.$$

If we take $G(x,m)$ as $G_m(x)$, then G_m is a digital section of p over U. Hence, we get the desired result. We shall prove the second claim. Let E be a digitally λ_2-contractible digital image. Let $genus_{\lambda_2,\lambda_3}(p) = n$. Then there exists $\{A_1, A_2, ..., A_n\}$ of B and, for each A_i, $s_i : A_i \to E$ is digitally continuous having that $p \circ s_i = id_{A_i}$, where $1 \leq i \leq n$. Since E is digitally contractible, id_E is homotopic to the constant map on E in digital images. Let us denote this digital homotopy with H. For any arbitrary $A_i \subset B$, we have the following construction:

$$G : A_i \times [0,m]_{\mathbb{Z}} \xrightarrow{s_i \times id} E \times [0,m]_{\mathbb{Z}} \xrightarrow{H} E \xrightarrow{p} B.$$

For all $a \in A$ and $t \in [0,m]_{\mathbb{Z}}$, conditions for being a digital homotopy of G are held:

$$G(a,0) = p \circ H \circ (s \times id)(a,0) = p \circ H(s(a),0) = p \circ s(a) = id_A(a), \text{ and}$$
$$G(a,1) = p \circ H \circ (s \times id)(a,1) = p \circ H(s(a),1) = p \circ c_{s(a)}(a) = c_{p \circ s(a)}(a),$$

where $c_{s(a)}$ is a constant digital map on E at the point $s(a) \in E$ and $c_{p \circ s(a)}$ is a constant digital map on B at the point $p \circ s(a) \in B$ (Note that constant digital maps are digitally continuous maps.). Moreover, the digital maps $G|_a : [0,m]_{\mathbb{Z}} \to B$ and $G|_t : A \to B$ are digitally continuous. As a consequence, for all $1 \leq i \leq n$, the digital map $A_i \to B$ is digitally nullhomotopic and thus we obtain $\text{cat}_{\lambda_3}(B) = n$. □

By Proposition 3.3, we immediately obtain the following:

PROPOSITION 3.4. *For any connected digital image (X, κ_1) such that X^n has the normal product κ_*-adjacency, we have that*

$$TC_n(X, \kappa_1) \leq cat_{\kappa_*}(X^n).$$

PROPOSITION 3.5. *Let (X, κ_1) be a connected digital image. Then we have*

$$cat_{\lambda_*}(X^{n-1}) \leq TC_n(X, \kappa_1),$$

where λ_* is the normal product adjacency relation on X^{n-1}.

The proof can be modified in digital images with Proposition 3.1 of [1]. One can easily adapt the proof from topological spaces to digital images. The last two results give bounds for TC_n using cat in digital images.

Corollary 3.6. *Let (X, κ_1) be a connected digital image. Then*

$$cat_{\lambda_*}(X^{n-1}) \leq TC_n(X, \kappa_1) \leq cat_{\kappa_*}(X^n),$$

where λ_* and κ_* are normal product adjacencies on X^{n-1} and X^n, respectively.

4. $\kappa-$Topological Groups In Digital Images

We now have a new approach to compute TC_n numbers of some of digital images. Our main equipment is the notion of topological groups in the digital sense.

Definition 4.1. Let (H, c_k) be a digital image, and $(H, *)$ be a group. Assume that $H \times H$ has also c_k-adjacency. If

$$\alpha : H \times H \to H \qquad \text{and} \qquad \beta : H \to H,$$

defined by $\alpha(y, z) = y * z$ and $\beta(y) = y^{-1}$ for all $y,\, z \in H$, respectively, are digitally continuous, then $(H, c_k, *)$ is called a digital version of a topological group.

We simply denote the digital version of a topological group as $(H, \kappa, *)$, and read the triple $(H, \kappa, *)$ as $\kappa-$topological group. Notice that the hypothesis on taking c_k-adjacency for $H \times H$ is necessary. If we ignore this hypothesis, we observe that $(\mathbb{Z}, 2, +)$, one of the simplest construction, cannot be a $2-$topological group. Indeed, consider the digitally continuous map

$$\begin{aligned} \alpha : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} \\ (x, y) &\longmapsto \alpha(x, y) = x + y \end{aligned}$$

and choose k as 2, i.e., $c_2 = 8-$adjacency for $\mathbb{Z}\times\mathbb{Z}$. $(3, 5)$ and $(4, 6)$ are $8-$adjacent but 8 and 10 are not $2-$adjacent in $\mathbb{Z}$. It shows that α cannot be a digitally continuous map. However, if we choose $c_1 = 4-$adjacency for $\mathbb{Z} \times \mathbb{Z}$, then α is a digitally continuous map. Hence, $(\mathbb{Z}, 2, +)$ is a $2-$topological group. A discretization of any topological group need not always be a digital topological group: In topological spaces, $(\mathbb{R}^*, \tau_s, \cdot)$ is a topological group, where $\mathbb{R}^*$ denotes the set $\mathbb{R} - \{0\}$. This does not give a response in digital images. Consider the triple $(\mathbb{Z}^*, 2, \cdot)$, where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. Even $\mathbb{Z}^*$ is not a monoid under the group operation $\cdot$ because the inverse of 3 does not exist. As a result, $(\mathbb{Z}^*, 2, \cdot)$ does not have a $2-$topological group structure.

We consider the same c_k-adjacency for H and $H\times H$ in Definition 4.1. This means that if the digital image H is a subset of $\mathbb{Z}$, then there is only one option: H and $H\times H$ have 2 and 4 adjacencies, respectively. For a subset H with $4-$adjacency in $\mathbb{Z}^2$, we have $8-$adjacency on $H \times H \subset \mathbb{Z}^4$. If H has $c_2 = 8-$adjacency in $\mathbb{Z}^2$, then $H \times H$ has $c_2 = 32-$adjacency in $\mathbb{Z}^4$. Similarly, when we consider $6-$adjacency on $H \subset \mathbb{Z}^3$, $H \times H \subset \mathbb{Z}^6$ has $12-$adjacency. If $H \subset \mathbb{Z}^3$ has $18-$adjacency, then $H \times H \subset \mathbb{Z}^6$ has $72-$adjacency. The fact H has $26-$adjacency implies that $H\times H$ has $232-$adjacency. For any digital image $H \subset \mathbb{Z}^n$ with $n > 3$, the process continues in the same way.

We begin with a trivial example of $\kappa-$topological groups. We also give another example with a different construction.

Example 4.2. Let $G = \{-1, 1\} \subset \mathbb{Z}$ be a digital image. Then G is a group under $\cdot$. Consider the digital maps

$$\begin{aligned} \alpha : G \times G &\longrightarrow G \\ (x, y) &\longmapsto \alpha(x, y) = x \cdot y \end{aligned}$$

and

$$\begin{aligned} \beta : G &\longrightarrow G \\ x &\longmapsto \beta(x) = x. \end{aligned}$$

In the domains of α and β, there does not exist any adjacent pair of points. It means that α and β are trivially digitally continuous. Consequently, $(G, 2, \cdot)$ is a $2-$topological group.

Example 4.3. Given an integer m, let $H = [m, m+1]_{\mathbb{Z}} \subset \mathbb{Z}$. For having a group construction on H, take a binary operation $*$ such that for all $a, b \in H$,

$$*(a, b) = \begin{cases} m, & a = b \\ m+1, & a \neq b. \end{cases}$$

The digital map

$$\begin{aligned} \alpha : H \times H &\longrightarrow H \\ (a, b) &\longmapsto \alpha(a, b) = a * b \end{aligned}$$

is digitally continuous because of the fact that $a*b = m$ or $m+1$. In addition, another digital map

$$\begin{aligned} \beta : H &\longrightarrow H \\ m &\longmapsto \beta(m) = m \\ m+1 &\longmapsto \beta(m+1) = m+1 \end{aligned}$$

is clearly digitally continuous. It shows that $(H, 2, *)$ is a $2-$topological group.

Theorem 4.4. *Let m be any integer. Then there is no $2-$topological group structure on the digital interval $[m, m+p-1]_{\mathbb{Z}}$ for all prime $p \geq 3$.*

Proof. Let $p = 3$. Assume that $[m, m+2]_{\mathbb{Z}}$ has $2-$topological group structure with any group operation $*$ and the $2-$adjacency relation. It means that $([m, m+2]_{\mathbb{Z}}, *)$ is a group in the algebraic sense. Moreover, the digital maps

$$\alpha : [m, m+2]_{\mathbb{Z}} \times [m, m+2]_{\mathbb{Z}} \to [m, m+2]_{\mathbb{Z}} \text{ and } \beta : [m, m+2]_{\mathbb{Z}} \to [m, m+2]_{\mathbb{Z}},$$

defined by $\alpha(x, y) = x*y$ and $\beta(x) = x^{-1}$, respectively, are digitally continuous. Then there are three cases for identity element of the group: $e_{[m,m+2]_{\mathbb{Z}}}$ is equal to only one of $m, m+1$ and $m+2$. Assume that m is the identity element. Since 3 is prime, every group of 3 elements is the cyclic group of order 3. Moreover, the set $\{m, m+1, m+2\}$ is an abelian group and every element different from the identity is a generator. This gives us the following properties:

$$\begin{aligned} &(m+2) * (m+2) = (m+1), \\ &(m+1) * (m+2) = (m+2) * (m+1) = e_{[m,m+2]_{\mathbb{Z}}}, \\ &(m+1)^{-1} = m+2 \quad \text{and} \quad (m+2)^{-1} = m+1. \end{aligned}$$

If $e_{[m,m+2]_{\mathbb{Z}}} = m$, then we find $\beta(m) = m$ and $\beta(m+1) = m+2$. This means that β is not digitally continuous. This is a contradiction. Now consider the second case. In other words, let $m+1$ be an identity element of the group. Then α is not digitally continuous because we get

$$\alpha(m, m+1) = m \quad \text{and} \quad \alpha(m+1, m+2) = m+2.$$

This is again a contradiction. Consider the third case, i.e., $m+2$ is the identity element of the group. The case is symmetric to the case $e_{[m,m+2]_{\mathbb{Z}}} = m$ since the map that swaps m and $m+2$ is an isomorphism of digital images. As a consequence, $([m, m+2]_{\mathbb{Z}}, 2, *)$ cannot be a $2-$topological group. If p is a prime with $p > 3$, then the idea can be generalized because we have two elements, namely the endpoints m and $m+p-1$, that have only one adjacent element, while, by the symmetry induced by the group action, each element have precisely two adjacent elements. □

PROPOSITION 4.5. *Let $(H, \kappa, *)$ be a $\kappa-$topological group and $(H,' \lambda, \circ)$ a $\lambda-$topological group, respectively. Then their cartesian product $H \times H^{'}$ is also a $(\kappa+\lambda)$-topological group.*

Proof. Let H be a $\kappa-$topological group. Then the digital maps

$$\alpha_1 : H \times H \to H \qquad \text{and} \qquad \beta_1 : H \to H,$$

defined by $\alpha_1(y_1, z_1) = y_1 * z_1$ and $\beta_1(y_1) = y_1^{-1}$ for all $y_1,\ z_1 \in H$, respectively, are digitally continuous. Similarly, for the $\lambda-$topological group $H,'$ we have that the digital maps

$$\alpha_2 : H^{'} \times H^{'} \to H^{'} \qquad \text{and} \qquad \beta_2 : H^{'} \to H,^{'}$$

defined by $\alpha_2(y_2, z_2) = y_2 \circ z_2$ and $\beta_2(y_2) = y_2^{-1}$ for all $y_2,\ z_2 \in H,'$ are digitally continuous. Define a digital map

$$\begin{aligned} \alpha = \alpha_1 \times \alpha_2 : H \times H \times H^{'} \times H^{'} &\to H \times H^{'} \\ ((y_1, z_1), (y_2, z_2)) &\longmapsto (y_1 * z_1, y_2 \circ z_2). \end{aligned}$$

We shall show that α is a digitally continuous map. The product of digitally continuous maps is again digitally continuous. Let $((y_1, z_1), (y_2, z_2))$ and $(({y_1}', {z_1}'), ({y_2}', {z_2}'))$ be adjacent points. Then (y_1, z_1) is adjacent or equal to $({y_1}', {z_1}')$ and (y_2, z_2) is adjacent or equal to $({y_2}', {z_2}')$. Since α_1 is digitally continuous, $y_1 * z_1$ is adjacent or equal to ${y_1}' * {z_1}'$. Similarly, for the digital continuity of α_2, we have that $y_2 \circ z_2$ is adjacent or equal to ${y_2}' \circ {z_2}'$. The cartesian product adjacency gives that α is digitally continuous. In order to satisfy the other condition, we define the digital map

$$\begin{aligned} \beta = \beta_1 \times \beta_2 : H \times H^{'} &\to H \times H^{'} \\ (y_1, z_1) &\longmapsto (y_1^{-1}, z_1^{-1}). \end{aligned}$$

Let (y_1, z_1) and (y_2, z_2) be adjacent points for the cartesian product. Then we have that y_1 is adjacent or equal to y_2 and z_1 is adjacent or equal to z_2. Since β_1 is digitally continuous, we obtain that y_1^{-1} is adjacent or equal to y_2^{-1}. Similary, for the digital continuity of β_2, we obtain that z_1^{-1} is adjacent or equal to z_2^{-1}. Using the definition of the adjacency for the cartesian product, we conclude that β is digitally continuous. This gives the required result. □

DEFINITION 4.6. Let $(H, \kappa, *)$ and $(H', \lambda, \circ)$ be a $\kappa-$topological group and a $\lambda-$topological group, respectively. Then a digital map $\gamma : (H, \kappa, *) \to (H', \lambda, \circ)$ is a $(\kappa, \lambda)-$homomorphism between $\kappa-$topological group and $\lambda-$topological group if γ is both digitally continuous and group homomorphism. Moreover, a $(\kappa, \lambda)-$isomorphism between $\kappa-$topological group and $\lambda-$topological group is both digital isomorphism and group homomorphism.

EXAMPLE 4.7. It is easy to see that $(\mathbb{Z}^2, 4, +)$ is $4-$topological group by Proposition 4.5. Consider the digital projection map

$$\begin{aligned} \alpha : (\mathbb{Z}^2, 4, +) &\longrightarrow (\mathbb{Z}, 2, +) \\ (m, n) &\longmapsto m \end{aligned}$$

We prove that α is a $(4, 2)-$homomorphism in the sense of topological groups but it is not a $(4, 2)-$topological group isomorphism. α is a digitally continuous map because m_1 and m_2 are $2-$adjacent or $m_1 = m_2$ whenever (m_1, n_1) and (m_2, n_2) are $4-$adjacent points in $\mathbb{Z}^2$. Using the fact that the projection maps associated with a product of groups are always group homomorphisms, we have that α is a group homomorphism. Hence, we prove that α is a $(4, 2)-$topological group homomorphism. On the other hand, the projection maps are not injective. Therefore, we show that α is not a $(4, 2)-$topological group isomorphism.

Note that the digital isomorphism of two topological groups is stronger than simply requiring a digitally continuous group isomorphism. The inverse of the digital function must also be digitally continuous. The next example shows that two topological groups in digital images are not digitally isomorphic in the sense of topological groups whenever they are isomorphic as ordinary groups.

EXAMPLE 4.8. Consider the $2-$topological group $(G, 2, \cdot)$ given in Example 4.2. Let $(H, 2, *)$ be another $2-$topological group for which $H = [8, 9]_{\mathbb{Z}} \subset \mathbb{Z}$ and $*$ is the same group operation given in Example 4.3. Then the digital map $f : (G, 2, \cdot) \to (H, 2, *)$, defined by $f(1) = 8$ and $f(-1) = 9$, is an isomorphism of algebraic groups but not a $(2, 2)-$isomorphism of topological groups. It is clear that f is bijective. Further, f preserves the group operation:

$$\begin{aligned} f(1 \cdot 1) &= f(1) = 8 = 8 * 8 = f(1) * f(1) \\ f(1 \cdot -1) &= f(-1) = 9 = 8 * 9 = f(1) * f(-1) \\ f(-1 \cdot 1) &= f(-1) = 9 = 9 * 8 = f(-1) * f(1) \\ f(-1 \cdot -1) &= f(1) = 8 = 9 * 9 = f(-1) * f(-1). \end{aligned}$$

There is no pair of adjacent points in G. So, f is digitally continuous. Contrarily, 8 and 9 are $2-$adjacent but $f^{-1}(8) = 1$ and $f^{-1}(9) = -1$ are not $2-$adjacent. Hence, the inverse of f is not digitally continuous.

THEOREM 4.9. *If H is a subgroup of a $\kappa-$topological group $(G, \kappa, *)$, then $(H, \kappa, *)$ is a $\kappa-$topological group.*

Proof. Suppose that $(G, \kappa, *)$ is a $\kappa-$topological group. Then

$$\begin{aligned} \alpha : G \times G &\to G & \text{and} \qquad \beta : G &\to G \\ (x, y) &\longmapsto x * y & x &\longmapsto x^{-1} \end{aligned}$$

are digitally continuous. To show that the digital maps

$$\begin{aligned} \alpha_1 : H \times H &\to H & \text{and} \qquad \beta_1 : H &\to H \\ (a, b) &\longmapsto a * b & a &\longmapsto a^{-1} \end{aligned}$$

are digitally continuous, it is enough to demonstrate that α and β are digitally continuous. Indeed, for two adjacent points in $H \times H$, they are also adjacent in $G \times G$ and their images are adjacent in G. The adjacency relation in H is the same for G.

Therefore, their images are also adjacent in H. It shows that α_1 is digitally continuous. Similarly, β_1 is digitally continuous. The continuity of α_1 and β_1 gives the desired result. □

5. Some Results For The Digital Higher Topological Complexity

THEOREM 5.1. *Let $(H, \kappa, \cdot)$ be a $\kappa-$topological group such that (H, κ) is digitally connected and $n > 1$. Then*

$$TC_n(H, \kappa) = cat_{\kappa_*}(H^{n-1}),$$

where κ_* is the normal product adjacency for H^{n-1}.

Proof. By Corollary 3.6, it is enough to show that $\mathrm{TC}_n(H, \kappa) \leq r$ when r equals $\mathrm{cat}_{\kappa_*}(H^{n-1})$. Let $\{M_1, M_2, ..., M_r\}$ be a covering of H^{n-1}, where M_i is a digitally contractible in H^{n-1}, for each $i = 1, ..., r$. In another saying, M_i contracts to an element $(h_1, h_2, ..., h_{n-1})$ in H^{n-1} for each i. Since H is a $\kappa-$topological group, it has the identity element e_H. Each contracting homotopy on M_i can be extended to $e_H^{(n-1)} = (e_H, e_H, ..., e_H)$ because H is $\kappa-$connected. For $i = 1, ..., r$, we define

$$N_i = \{(h, hm_1, ..., hm_{n-1}) : (m_1, ..., m_{n-1}) \in M_i, \;\; h \in H\}.$$

We shall show that $e_H^{(n)}$ admits a digitally continuous section over each N_i. Let $m = (m_1, ..., m_{n-1})$. The digital contractibility of M_i gives a digital path α_m and this path joins $e_H^{(n-1)}$ to each $m \in M_i$ in H^{n-1}. Therefore, we define a new digital path ${\alpha_m}'$ from $e_H^{(n)}$ to $(e_H, m_1, ..., m_{n-1})$ in N_i. Then for any $h \in H$, $h{\alpha_m}'$ is a digital path in H^n from $(h, h, ..., h) = he_H^{(n)}$ to $(h, hm_1, ..., hm_{n-1})$. Finally, we construct a map

$$s_i : N_i \to H^{J_n}$$

such that $s_i(h, hm_1, ..., hm_{n-1})$ is the $j-$th element of $h{\alpha_m}'$ on the $j-$th digital interval of J_n. More clearly, $s_i(h, hm_1, ..., hm_{n-1})$ is a digital multipath in H from $(h, h, ..., h)$ to $(h, hm_1, ..., hm_{n-1})$. s_i is a digitally continuous map. Indeed, for any elements h, $h' \in H$, the fact $(h, hm_1, ..., hm_{n-1})$ is adjacent to $(h', h'm_1', ..., h'm_{n-1}')$ implies that h is adjacent to h' and hm_j is adjacent to $h'm_j'$ for each $j \in \{1, ..., n-1\}$. It follows that $s_i(h, hm_1, ..., hm_{n-1})(t)$ is adjacent to $s_i(h', hm_1', ..., h'm_{n-1}')(t)$ for all $t \in J_n$. Consider any $(c_1, ..., c_n) \in H^n$, and put $h = c_1$ and $m_i = h^{-1}c_i$. Then there exists j such that $(m_1, ..., m_{n-1}) \in M_j$. This means that $(c_1, ..., c_n) \in N_j$. Hence, we get $H^n = N_1 \cup ... \cup N_r$. As a result, $\mathrm{TC}_n(H, \kappa) \leq r$. □

EXAMPLE 5.2. Consider the digital image H given in Example 2.9. $(H, 4, \circ)$ is a $4-$topological group, where $\circ$ is a group operation (See Table 1). Note that H is a cyclic group where b is the identity, and a is a generator. H is not $4-$contractible digital image, so it is true that $\mathrm{TC}_n(H, 4) = 1$ only when $n = 1$. To compute $\mathrm{TC}_2(H, 4)$, we use Theorem 5.1. By Example 2.9, we obtain $\mathrm{cat}_4(H) = 2$. As a result, we get $\mathrm{TC}_2(H, 4) = 2$.

∘	a	b	c	d	e	f	g	h
a	h	a	b	c	d	e	f	g
b	a	b	c	d	e	f	g	h
c	b	c	d	e	f	g	h	a
d	c	d	e	f	g	h	a	b
e	d	e	f	g	h	a	b	c
f	e	f	g	h	a	b	c	d
g	f	g	h	a	b	c	d	e
h	g	h	a	b	c	d	e	f.

Table 1. The group operation ∘ for H.

6. Conclusion

We first consider a relation between the Lusternik-Schnirelmann theory and the higher topological complexity more conceretely in digital images. Second, our task is to include $\kappa-$topological groups in the study of digital manner of topological robotics. While doing theoretical modeling, we also observe that examples of digital images may be useful in future works. We try to get the properties in terms of the digital higher topological complexity. Some theoretical infrastructure needs to be established before accessing the applications of motion planning algorithms in digital images. This shows the importance of our results. We wish to progress to the wide application area of motion planning algorithms by proceeding step by step. We intend to make an impact on at least one application area for the future works. For example, in computer games, virtual characters have to use motion planning algorithms to determine their direction and find a way between two locations in the virtual environment. In addition to this, we can encounter motion planning problem in almost every aspect of our life such as military simulations, probability and economics, artificial intelligence, urban design, robot-assisted surgery, and the study of biomolecules.

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