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# RICCI CURVATURE OF WARPED PRODUCT POINTWISE BI-SLANT SUBMANIFOLDS

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**Abstract.** In this paper, we study doubly warped product point-wise bi-slant submanfolds of $S$-space form. Here we try to establish an inequality containing the Ricci curvature, squared norm of mean curvature and the warping function.

## 1. Introduction

The most fundamental problem in submanifold theory is to establish the connection between the intrinsic and extrinsic invariants of submanifolds, and also immersibility and non-immersibility of submanifolds. Bishop and O'Neill [4] introduced the concept of warped products to study manifolds of negative sectional curvature. O'Neill discussed warped products and explored curvature formulas of warped products in terms of curvatures of components of warped products. Motivated by this approach, Chen [6, 7, 8] studied immersibility and non-immersibility of Riemannian warped products in Riemannian manifolds, especially in Riemannian space forms. Since then, the study of warped product submanifolds has been investigated by many geometers (see, e.g., [1, 2, 12, 13] among many others). Meraj Ali Khan in [13] studied warped product pointwise semi-slant submanifold in complex space form. The authors also studied in [22] warped product bi-slant submanifolds. In [12], the author studied Ricci curvature inequalities for skew CR-warped product submanifolds in complex space forms. In [1], A. Ali studied warped product immersions into an $m$-dimensional unit sphere $S^m$ and complex space form. The same author also studied Ricci curvature on warped product submanifolds of complex space forms and its applications.

Doubly warped products can be considered as a generalization of singly warped products which were mainly studied in [20, 21]. Olteanu [18], Sular and Özgür [19], Matsumoto [15] and Faghfouri and Majidi in [10] extended

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some properties of warped product submanifolds and geometric inequalities in warped product manifolds for doubly warped product submanifolds into Riemannian manifolds. Many authors studied Ricci curvature on submanifolds in different space forms such as complex space form [13], Kenmotsu space form [3], cosymplectic space form [23], Sasakian space form [16] and generalized Sasakian space form [11]. Recently, [14] studied Ricci curvature on warped product pointwise bi-slant submanifolds of Sasakian space form and obtained several inequalities between intrinsic invariant and extrinsic invariant.

In this paper, we have extended some results of [14] for doubly warped product pointwise bi-slant submanifolds immersed in $S$-space form.

## 2. Preliminaries

Consider a manifold $\mathcal{M}^{2n+s}$ along an $f$ -structure of rank $2n$. We take $s$ structural vector fields $\xi_1, \xi_2, \ldots, \xi_s$ on $\mathcal{M}$ such as:

$$f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0, \quad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha,$$

where $\eta_\alpha$ and $\xi_\alpha$ are the dual forms to each other, therefore complemented frames exist on $f$-structure. For $f$-manifold we define a Riemannian metric $g$ as

$$g(Y, X) = g(fY, fZ) + \sum \eta_\alpha(Y)\eta_\alpha(Z),$$

for vector fields $Y$ and $Z$ on $\mathcal{M}$ [17]. If $\mathcal{M}$ is an $S$-manifold, then we consider the formulas [17]:

$$\widetilde{\nabla}_Y \xi_\alpha = -fY, \quad Y \in T(\mathcal{M}), \quad \alpha = 1, \ldots, s \tag{1}$$

$$\left(\widetilde{\nabla}_Y f\right) Z = \textstyle\sum_\alpha \left\{g(fY, fZ)\xi_\alpha + \eta_\alpha(Z) f^2 Y\right\}, \quad Y, Z \in, T(\mathcal{M}), \tag{2}$$

where $\widetilde{\nabla}$ is the Riemannian connection of g. The projection tensor $-f^2$ determines the distribution $\mathfrak{L}$ and $f^2 + I$ determines the complementary distribution $\mathfrak{M}$ which is determined and spanned by $\xi_1, \ldots, \xi_s$. It can be observed that if $Y \in \mathfrak{L}$ then $\eta_\alpha(Y) = 0$ for all $\alpha$, and for $Y \in \mathfrak{M}$, we have $fY = 0$. A plane section $\Pi$ on $M$ is said to be $f$-section if it is established by a vector $Y \in \mathfrak{L}(p), p \in \mathcal{M}$, such that $\{Y, fY\}$ spans the section. We take the sectional curvature of $\Pi$ as the $f$-sectional curvature. If $\mathcal{M}$ is an $S$-manifold of constant $f$-sectional curvature $k$, then its curvature tensor is as:

$$\begin{aligned} \widetilde{R}(Y, Z)U &= \sum_{\alpha,\beta} \{\eta^\alpha(Y)\eta^\beta(U) f^2 Z - \eta^\alpha(Z)\eta^\beta(U) f^2 Y \\ &- g(fY, fU)\eta^\alpha(Z)\xi_\beta. + g(fZ, fU)\eta^\alpha(Y)\xi_\beta\} \\ &+ \frac{k+3s}{4}\left\{-g(fZ, fU) f^2 Y + g(fY, fU) f^2 Z\right\} \\ &+ \frac{k-s}{4}\{g(Y, fU) fZ - g(Z, fU) fY + 2g(Y, fZ) fU\}, \end{aligned} \tag{3}$$

$Y,\ Z,\ U \in T(\mathcal{M})$. Such a manifold $\mathcal{M}(k)$ will be called an $S$-space form. Examples of $S$-space forms are the Euclidean space $E^{2n+s}$ and the hyperbolic space $H^{2n+s}$.

Let $\mathcal{M}$ be an $m+s$-dimensional submanifold of an $n$-dimensional Riemannian manifold $\widetilde{\mathcal{M}}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle\cdot,\cdot\rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{\mathcal{M}}$ and the induced metric $g$ on the submanifold $\mathcal{M}$. The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in T\mathcal{M}$ and $N \in T^{\perp}\mathcal{M}$, where $\widetilde{\nabla}, \nabla$ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{\mathcal{M}}, \mathcal{M}$ and the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively, and $\sigma$ is the second fundamental form related to the shape operator $A$ by $\langle\sigma(X,Y),N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle, \tag{4}$$

for all $X, Y, Z, W \in T\mathcal{M}$, where $\widetilde{R}$ and $R$ are the Riemann curvature tensors of $\widetilde{M}$ and $\mathcal{M}$, respectively.

For any $X \in \Gamma(T\mathcal{M})$, we can write

$$\phi X = PX + QX, \tag{5}$$

where $PX$ and $QX$ are tangential and normal component of $\phi X$. $\mathcal{M}$ is said to be totally geodesic, totally umbilical and minimal according as $\sigma(X,Y) = 0, \sigma(X,Y) = g(X,Y)H$ and $H = 0$, where $X, Y \in \Gamma(T\mathcal{M})$ and $H$ is the mean curvature.

The mean curvature vector $H$ is given by $H = \frac{1}{m+s}\operatorname{trace}(\sigma)$. The submanifold $\mathcal{M}$ is totally geodesic in $\widetilde{\mathcal{M}}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in T\mathcal{M}$, then $\mathcal{M}$ is totally umbilical.

**Definition 2.1.** [5] *An immersion $\phi : \mathcal{M} \to \widetilde{\mathcal{M}}^{2n+s}(k)$ is called pointwise slant if, any point $p \in \mathcal{M}$, the Wirtinger angle $\theta(X)$ between $\phi X$ and $T_p\mathcal{M}$ is independent of the choice of a non-zero tangent vector $X \in T_p\mathcal{M}$ which is linearly independent of $\xi$. The function $\theta$ on $\mathcal{M}$ is called the slant function. A pointwise slant submanifold is called pointwise proper slant if it contains no points $p \in \mathcal{M}$ such that $\cos\theta = 0$ at $p$.*

In a pointwise slant submanifold, we have the following relations [5]:

$$P^2 = \cos^2\theta[-I + \eta\otimes\xi],$$
$$g(PU, PV) = \cos^2\theta g(U,V),$$
$$g(FU, FV) = \sin^2\theta g(U,V)$$

for $U, V \in \chi(\mathcal{M})$.

**Definition 2.2.** [9] *A submanifold $\mathcal{M}$ of $\widetilde{\mathcal{M}}^{2n+s}(k)$ is said to be pointwise bi-slant if there exist two distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ at any point $p \in \mathcal{M}$ satisfying the following conditions:*

- *$\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are orthogonal.*
- *$TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$.*
- *$\phi\mathcal{D}^{\theta_1} \perp \mathcal{D}^{\theta_2}$ and $D^{\theta_1} \perp \phi D^{\theta_2}$.*
- *The distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are pointwise slant with slant functions $\theta_1$ and $\theta_2$ from $T\mathcal{M}$ to $\mathbb{R}$.*

**Remark 2.3.**

- *If $\theta_1$ and $\theta_2$ from $T\mathcal{M}$ to $\mathbb{R}$ are constant functions, then $\mathcal{M}$ is called bi-slant submanifold.*
- *If one of $\theta_1$ and $\theta_2$ is $\frac{\pi}{2}$, then $\mathcal{M}$ is called pointwise pseudo-slant submanifold.*
- *If one of $\theta_1$ and $\theta_2$ is $0$, then $\mathcal{M}$ is called pointwise semi-slant submanifold.*
- *If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ or $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then $\mathcal{M}$ is called $CR$ submanifold.*

If $\mathcal{M}$ is a pointwise bi-slant submanifold of $\widetilde{\mathcal{M}}^{2n+s}(k)$, then for any $X \in \Gamma(T\mathcal{M})$ we have

$$X = T_1X + T_2X, \tag{6}$$

where $T_1, T_2$ are projections from $T\mathcal{M}$ onto $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ respectively. Using $P_1 = T_1 \circ P$ and $P_2 = T_2 \circ P$ in (5), we get

$$\phi X = P_1X + P_2X + QX \tag{7}$$

for all $X \in \Gamma(T\mathcal{M})$. From (6), we have

$$P_i^2X = \cos^2\theta_i(-X + \eta(X)\xi), X \in \Gamma(T\mathcal{M}), i = 1, 2. \tag{8}$$

Let $p \in \mathcal{M}$ and $\{e_1, \cdots, e_m, e_{m+1} = \xi_1, \cdots, e_{m+s} = \xi_s\}$ be an orthonormal basis of $T_p\mathcal{M}$. Then the mean curvature vector $H(p)$ is defined as

$$H(p) = \frac{1}{m+s} \sum_{i=1}^{m+s} \sigma(e_i, e_i).$$

Also we define

$$\|\sigma\|^2 = \sum_{i,j=1}^{m+s} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{9}$$

The gradient of a smooth function $f$ on $\mathcal{M}$ is denoted by $\nabla f$ and defined by

$$g(\nabla f, X) = Xf. \tag{10}$$

Also we get

$$\|\nabla f\|^2 = \sum_{j=1}^{m+s} (e_j(f))^2. \tag{11}$$

The warped product of two smooth manifolds $N_1$ and $N_2$ with Riemannian metric $g_1, g_2$ is denoted by $N_1 \times_f N_2$ and defined by another Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where $g = g_1 + f^2 g_2$ and $f : N_1 \to \mathbb{R}^+$ is a smooth function. If $f$ is constant then the warped product is said to be trivial warped product. In $\mathcal{M} = N_1 \times_f N_2$, we have [4]

$$\nabla_X \mathrm{Z} = \nabla_{\mathrm{Z}} \mathrm{X} = (X \ln f)\mathrm{Z} \tag{12}$$

for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. Let $\mathfrak{L}$ be a $k$-plane section of $T_p\mathcal{M}$ and let $X$ be a unit vector in $\mathfrak{L}$. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_k\}$ of $\mathfrak{L}$ such that $X = e_A \in \{e_1, e_2, \cdots, e_k\}$. The Ricci curvature, denoted by $\mathrm{Ric}_{\mathfrak{L}}(X)$, is defined by

$$\mathrm{Ric}_{\mathfrak{L}}(X) = \sum_{\substack{i=1\\ i\neq A}}^{k} K_{iA}, \tag{13}$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $\{e_i, e_j\}$. Such a curvature is called $k$-Ricci curvature. The scalar curvature of the $k$-plane section $\mathfrak{L}$ is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{14}$$

Let $\mathcal{M} = \mathcal{M}_{\theta_1} \times_f \mathcal{M}_{\theta_2}$ be a warped product pointwise bi-slant submanifold of $\widetilde{\mathcal{M}}^{2n+s}(k)$ such that $\xi \in \Gamma\left(D^{\theta_1}\right)$. $K_{ij}$ and $\widetilde{K}_{ij}$ are the sectional curvatures of the plane sections in $\mathcal{M}^{m+s}$ and $\widetilde{\mathcal{M}}^{2n+s}(k)$, respectively. Let dim $\mathcal{M}_{\theta_1} = 2n_1 + s$ and $\dim \mathcal{M}_{\theta_2} = 2n_2$ such that $\dim \mathcal{M} = m + s = 2n_1 + 2n_2 + s$. Then, from Gauss equation we get

$$\begin{aligned} &\sum_{1\leq i<j\leq m+s} K_{ij} \\ &= \sum_{1\leq i<j\leq m+s} \widetilde{K}_{ij} + \sum_{r=m+s+1}^{2n+s} \sum_{1\leq i<j\leq m+s} \left\{\sigma_{ii}^r \sigma_{jj}^r - \left(\sigma_{ij}^r\right)^2\right\}. \end{aligned} \tag{15}$$

From (14) and (15), we have

$$\tau\left(T_p\mathcal{M}_{\theta_1}\right) = \widetilde{\tau}\left(T_p\mathcal{M}_{\theta_1}\right) + \sum_{r=m+s+1}^{2n+s} \sum_{1\leq i<j\leq 2n_1+s} \left\{\sigma_{ii}^r \sigma_{jj}^r - \left(\sigma_{ij}^r\right)^2\right\}, \tag{16}$$

$$\tau\left(T_p\mathcal{M}_{\theta_2}\right) = \widetilde{\tau}\left(T_p\mathcal{M}_{\theta_2}\right) + \sum_{r=m+s+1}^{2n+s} \sum_{2n_1+s+1\leq i<j\leq 2n_2} \{\sigma_{ii}^r \sigma_{jj}^r - \left(\sigma_{ij}^r\right)^2\}. \tag{17}$$

Since $\mathcal{M}^{m+s}$ is a warped product submanifold, we have [8]

$$\sum_{1\leq i<j\leq m+s} K_{ij} = 2n_2\left(\Delta(\ln f) - \|\nabla \ln f\|^2\right) = \frac{2n_2 \Delta(f)}{f}, \tag{18}$$

where $\Delta(f)$ is the Laplacian of the warping function $f$.

## 3. Main Results

**Theorem 3.1.** *Let $\psi : \mathcal{M}^{m+s} = \mathcal{M}_1 \times_f \mathcal{M}_2 \to \widetilde{\mathcal{M}}^{2n+s}(k)$ be a $\mathcal{D}^{\theta_1}$-minimal isometric immersion of a $(m+s)$-dimensional warped product pointwise bi-slant submanifold $\mathcal{M}^{m+s}$ into $\widetilde{\mathcal{M}}^{2n+s}(k)$, where $\mathcal{M}_{\theta_1}, \mathcal{M}_{\theta_2}$ are pointwise slant submanifolds with distinct slant functions $\theta_1, \theta_2$, respectively, and $\xi \in \Gamma\left(\mathcal{D}^{\theta_1}\right)$. Then we have*

$$\begin{aligned}(19)\quad \mathrm{Ric}(X) + 2n_2\Delta(\ln f) &\leq 2n_2\|\nabla \ln f\|^2 + \frac{k+3s}{2}(2n_1 s + 2n_2 s \\ &- 4n_1n_2 - 2n_1 - 2n_2 + 2 + 5s^2 - 4s) \\ &- \frac{k+3s-4}{2}(s^2 - 3s) + 3\frac{k-s}{2}(\cos^2\theta_1) \\ &+ \frac{(m+s)^2}{4}\|H\|^2\end{aligned}$$

*if $X \in \Gamma\left(\mathcal{D}^{\theta_1}\right)$*

$$\begin{aligned}(20)\ \mathrm{Ric}(X) + 2n_2(\Delta \ln f) &\leq 2n_2\|\nabla(\ln f)\|^2 - \frac{k+3s}{2}(4n_1 s - 8n_1n_2 - 4n_1 \\ &- 4n_2 - s + 2 + 4s^2) - \frac{k+3s-4}{2}(2s^2 - 5s) \\ &+ 6\frac{k-s}{4}(\cos^2\theta_2) + \frac{(m+s)^2}{4}\|H\|^2\end{aligned}$$

*if $X \in \Gamma\left(\mathcal{D}^{\theta_2}\right)$.*

*Proof.* Let $X \in T_p\mathcal{M}$ be a unit tangent vector at $p \in \mathcal{M}$. Consider a local orthonormal frame $\{e_1, e_2, \cdots, e_m, e_{m+1}, \cdots, e_{m+s}, e_{m+s+1}, \cdots, e_{2n+s}\}$ of $\bar{\mathcal{M}}^{2n+s}(c)$ such that

$$\{e_1, e_2, \cdots, e_{2n_1}, e_{2n_1+1} = \xi_1, \cdots, e_{2n_1+s} = \xi_s, \cdots, e_{2n_1+2n_2+s} = e_{m+s}\}$$

is tangent to $\mathcal{M}$. Assume that

$$\begin{aligned}\{&e_1, e_2 = \sec\theta_1 P_1 e_1, e_3, e_4 = \sec\theta_1 P_1 e_3, \cdots, e_{2n_1-1}, \\ &e_{2n_1} = \sec\theta_1 P_1 e_{2n_1-1}, e_{2n_1+1} = \xi\}\end{aligned}$$

is an orthogonal frame of $\mathcal{D}^{\theta_1}$, the distribution corresponding to $\mathcal{M}_{\theta_1}$ and

$$\begin{aligned}\{&e_{2n_1+2}, e_{2n_1+3} = \sec\theta_2 P_2 e_{2n_1+2}, e_{2n_1+4}, e_{2n_1+5} = \sec\theta_2 P_2 e_{2n_1+4}, \\ &\cdots, e_{2n_1+2n_2}, e_{2n_1+2n_2+1} = \sec\theta_2 P_2 e_{2n_1+2n_2}\}\end{aligned}$$

is an orthonormal frame of $\mathcal{D}^{\theta_2}$, the distribution corresponding to $\mathcal{M}_{\theta_2}$. Set

$$X = e_A \in \{e_1, e_2, \cdots, e_m, e_{m+1} = \xi_1, \cdots, e_{m+s} = \xi_s\}.$$

Now, from Gauss equation we have

$$(m+s)^2\|H\|^2 = 2\tau\left(T_p\mathcal{M}\right) + \|\sigma\|^2 - 2\bar{\tau}\left(T_p\mathcal{M}\right). \tag{21}$$

We expand (21) for our constructed frame as follows:

$$\begin{aligned}(m+s)^2\|H\|^2 &= 2\tau\left(T_p\mathcal{M}\right) + \sum_{r=m+s+1}^{2n+s}\{(\sigma_{11}^r + \cdots + \sigma_{m+s}^r - \sigma_{AA}^r)^2 \\ &+ (\sigma_{AA}^r)^2\} - 2\sum_{r=m+s+1}^{2n+s}\sum_{\substack{1\le i<j\le m+s\\ i,j\neq A}}\sigma_{ii}^r\sigma_{jj}^r \\ &- 2\bar{\tau}(T_p\mathcal{M}) + 2\sum_{r=m+s}^{2n+s}\sum_{i<j\le m+s}(\sigma_{ij}^r)^2.\end{aligned} \tag{22}$$

From (22) we get

$$\begin{aligned}(m+s)^2\|H\|^2 &= 2\tau\left(T_p\mathcal{M}\right) + \frac{1}{2}\sum_{r=m+s}^{2n+s}[(\sigma_{11}^r + \cdots + \sigma_{m+s}^r)^2 \\ &+ \{2\sigma_{AA}^r - (\sigma_{11}^r + \cdots + \sigma_{m+s}^r)\}^2] \\ &+ 2\sum_{r=m+s+1}^{2n+s}\sum_{1\le i<j\le m+s}\sum_{m+s}(\sigma_{ij}^r)^2 \\ &- 2\sum_{r=m+s+1}^{2n+s}\sum_{\substack{i\le j\le m+s,\\ i,j\neq A}}\sigma_{ii}^r\sigma_{jj}^r - 2\bar{\tau}(T_p\mathcal{M}).\end{aligned} \tag{23}$$

Using the $\mathcal{D}^{\theta_1}$ -minimality of $M^{m+s}$ in (23), we get

$$\begin{aligned}(m+s)^2\|H\|^2 &= 2\tau(T_p\mathcal{M}) + \frac{1}{2}(m+s)^2\|H\|^2 \\ &+ \frac{1}{2}\sum_{r=m+s+1}^{2n+s}[\{2\sigma_{AA}^r - (\sigma_{2n_1+s+12n_1+s+1}^r \\ &+ \cdots + \sigma_{m+sm+s}^r)\}^2] - 2\bar{\tau}(T_p\mathcal{M}) \\ &+ \sum_{r=m+s+1}^{2n+s}[\sum_{1\le i<j\le m+s}(\sigma_{ij}^r)^2 - \sum_{1\le i<j\le m+s,}\sigma_{ii}^r\sigma_{jj}^r \\ &+ \sum_{\substack{i=1\\ i\neq A}}^{m+s}(\sigma_{iA}^r)^2 + \sum_{1\le i<j\le m+s}(\sigma_{ij}^r)^2 \\ &- \sum_{1\le i<j\le m+s,}\sigma_{ii}^r\sigma_{ij}^r].\end{aligned} \tag{24}$$

By virtue of (15) in (24), we get

$$
\begin{aligned}
\frac{1}{2}(m+s)^2\|H\|^2 &= 2\tau(T_p\mathcal{M}) - 2\widetilde{\tau}(T_p\mathcal{M}) \\
&+ \frac{1}{2}\sum_{r=m+s+1}^{2n+s}[\{2\sigma^r_{AA} \\
&- (\sigma^r_{2n_1+s+1}2n_1+s+1+\cdots+\sigma^r_{m+s})\}^2 \\
&+ \sum_{\substack{1\le i<j\le m+s,\\ i,j\neq A}}\widetilde{K}_{ij} - \sum_{\substack{1\le i<j\le m+s,\\ i,j\neq A}}K_{ij} \\
&+ \sum_{\substack{i=1\\ i\neq A}}^{m+s}(\sigma^r_{iA})^2 + \sum_{1\le i<j\le m+s}(\sigma^r_{ij})^2 \\
&- \sum_{\substack{1\le i<j\le m+s,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj}].
\end{aligned}
\tag{25}
$$

Using $(16), (17)$ and $(18)$ in $(25)$, we have

$$
\begin{aligned}
\frac{1}{2}(m+s)^2\|H\|^2 &= \mathrm{Ric}(X) + \widetilde{\tau}(T_p\mathcal{M}_{\theta_1}) + \widetilde{\tau}(T_p\mathcal{M}_{\theta_2}) \\
&- 2\bar{\tau}(T_p\mathcal{M}) + 2n_2\frac{\Delta f}{f} + \sum_{1\le i<j\le m+s}\widetilde{K}_{ij} \\
&+ \sum_{r=m+s+1}^{2n+s}[\sum_{1\le i<j\le 2n_1+s+1}\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\} \\
&+ \sum_{2n_1+s+1\le i<j\le m+s}\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma_{ij})^2\} + \sum_{\substack{i=1\\ i\neq A}}^{m+s}(\sigma^r_{iA})^2 \\
&+ \frac{1}{2}\sum_{r=m+s+1}^{2n+s}\{(2\sigma^r_{AA} - (\sigma^r_{2n_1+s+1}2n_1+s+1 \\
&+ \cdots + \sigma^r_{m+s})\}^2 \\
&+ \sum_{1\le i<j\le m+s}(\sigma^r_{ij})^2 - \sum_{\substack{1\le i<j\le m+s,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj}].
\end{aligned}
\tag{26}
$$

Case-1: $e_A$ is tangent to $\mathcal{M}_{\theta_1}$, i.e.,

$$e_A \in \{e_1, e_2, \cdots, e_{2n_1+s+1}\}.$$

We choose $e_A = e_1 = X$. Then from (26), we get

$$
\begin{aligned}
(27)\ \frac{1}{2}(m+s)^2\|H\|^2 &= \mathrm{Ric}(X) + \widetilde{\tau}(T_p\mathcal{M}_{\theta_1}) + \widetilde{\tau}(T_p\mathcal{M}_{\theta_2}) \\
&- 2\bar{\tau}(T_p\mathcal{M}) + 2n_2\frac{\Delta f}{f} + \sum_{2\le i<j\le m+s}\widetilde{K}_{ij} + \sum_{i=2}^{m+s}(\sigma_{1i}^r)^2 \\
&+ \sum_{r=m+s+1}^{2n+1}[\sum_{1\le i<j\le 2n_1+s+1}\sigma_{ii}^r\sigma_{jj}^r + \sum_{2n_1+2\le i<j\le m+1}\sigma_{ii}^r\sigma_{jj}^r \\
&- \sum_{2\le i<j\le m+1}\sigma_{ii}^r\sigma_{jj}^r] - \sum_{r=m+s+1}^{2n+s}[\sum_{1\le i<j\le 2n_1+s}(\sigma_{ij}^r)^2 \\
&+ \sum_{2n_1+s+1\le i<j\le m+s}(\sigma_{ij}^r)^2 - \sum_{1\le i<j\le m+s}(\sigma_{ij}^r)^2] \\
&+ \frac{1}{2}\sum_{r=m+2}^{2n+s}\{2\sigma_{11}^r - (\sigma_{2n_1+s+12n_1+s+1}^r + \cdots + \sigma_{m+s}^r)\}^2.
\end{aligned}
$$

Now, from (3), we get

$$
\begin{aligned}
(28)\ \sum_{1\le i<j\le m+s}\widetilde{R}(e_i,e_j,e_j,e_i) &= -\frac{k+3s-4}{4}\{(m+s)s - 2s + s^2\} \\
&+ \frac{k+3s}{4}\{(m+s)^2 - (m+s)s - (m+s) + s^2\} \\
&+ \frac{k-s}{4}\{3\sum_{1\le i<j\le m+s}g^2(e_i,\phi e_j)\}.
\end{aligned}
$$

For our constructed frame field, we obtained

$$
g^2(e_i,\phi e_j) = \begin{cases} \cos^2\theta_1, & \text{for } 1\le i,j\le 2n_1, \\ \cos^2\theta_2, & \text{for } 1\le i,j\le 2n_2. \end{cases}
$$

Therefore

$$
\sum_{1\le i<j\le m+s} g^2(e_i,\phi e_j) = 2\left(n_1\cos^2\theta_1 + n_2\cos^2\theta_2\right).
$$

Thus we obtain

$$
\begin{aligned}
(29)\qquad \widetilde{\tau}(T_p\mathcal{M}_{\theta_1}) &+ \widetilde{\tau}(T_p\mathcal{M}_{\theta_2}) + \sum_{2\le i<j\le m+s}\widetilde{K}_{ij} - 2\widetilde{\tau}(T_p\mathcal{M}) \\
&= -\frac{k+3s}{2}(2n_1s + 2n_2s - 4n_1n_2 - 2n_1 - 2n_2 + 2 \\
&+ 5s^2 - 4s) + \frac{k+3s-4}{2}(s^2-3s) - 3\frac{k-s}{2}(\cos^2\theta_1).
\end{aligned}
$$

Also, we calculate

$$\sum_{r=m+s+1}^{2n+s}[\sum_{1\leq i<j\leq 2n_1+s}\sigma^r_{ii}\sigma^r_{jj}+\sum_{2n_1+s+1\leq i<j\leq m+1}\sigma^r_{ii}\sigma^r_{jj}-\sum_{2\leq i<j\leq m+s}\sigma^r_{ii}\sigma^r_{jj}]$$
$$=\sum_{r=m+s+1}^{2n+s}[\sum_{j=2}^{2n_1+s}\sigma^r_{11}\sigma^r_{jj}-\sum_{i=2}^{2n_1+s}\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{ii}\sigma^r_{jj}]. \tag{30}$$

Since $M$ is $\mathcal{D}^{\theta_1}$ -minimal, we have

$$\sum_{j=2}^{2n_1+s}\sigma^r_{jj}=-\sigma^r_{11} \tag{31}$$

Therefore, from (30) and (31), we get

$$\sum_{r=m+s+1}^{2n+s}[\sum_{1\leq i<j\leq 2n_1+s}\sigma^r_{ii}\sigma^r_{jj}+\sum_{2n_1+s+1\leq i<j\leq m+s}\sigma^r_{ii}\sigma^r_{jj}-\sum_{2\leq i<j\leq m+s}\sigma^r_{ii}\sigma^r_{jj}]$$
$$=-\sum_{r=m+s}^{2n+s}(\sigma^r_{11})^2+\sum_{r=m+s}^{2n+s}\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{11}\sigma^r_{jj}. \tag{32}$$

Again we find

$$\sum_{r=m+s+1}^{2n+s}[\sum_{1\leq i<j\leq 2n_1+s}(\sigma^r_{ij})^2+\sum_{2n_1+s+1\leq i<j\leq m+1}(\sigma^r_{ij})^2$$
$$-\sum_{1\leq i<j\leq m+s}(\sigma^r_{ij})^2]$$
$$=-\sum_{r=m+s+1}^{2n+s}\sum_{i=1}^{2n_1+s}\sum_{j=2n_1+s+1}^{m+s}(\sigma^r_{ij})^2. \tag{33}$$

Also

$$\sum_{r=m+s+1}^{2n+s}\{2\sigma^r_{11}-(\sigma^r_{2n_1+s+12n_1+s+1}+\cdots+\sigma^r_{m+s})\}^2$$
$$=4\sum_{r=m+s+1}^{2n+s}(\sigma^r_{11})^2-4\sum_{r=m+s+1}^{2n+s}\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{11}\sigma^r_{jj}$$
$$+(m+s)^2\|H\|^2. \tag{34}$$

Thus, using $(29), (32), (33)$ and $(34)$ in $(27)$ we get

$$
\begin{aligned}
(35)\quad \frac{1}{2}(m+s)^2\|H\|^2 &= \mathrm{Ric}(X)+2n_2\frac{\Delta f}{f}\\
&- \frac{k+3s}{2}(2n_1s+2n_2s-4n_1n_2-2n_1\\
&- 2n_2+2+5s^2-4s)+\frac{k+3s-4}{2}(s^2-3s)\\
&- 3\frac{k-s}{2}(\cos^2\theta_1)+\sum_{r=m+s+1}^{2n+s}\sum_{i=1}^{2n_1+s}\sum_{j=2n_1+s+1}^{m+s}(\sigma_{ij}^r)^2\\
&+ \sum_{r=m+s+1}^{2n+s}\sum_{j=2n_1+s+1}^{m+s}\sigma_{11}^r\sigma_{jj}^r\\
&+ \sum_{i=2}^{m+s}(\sigma_{1i}^r)^2\sum_{r=m+s}^{2n+s}(\sigma_{11}^r)^2-2\sum_{r=m+s+1}^{2n+s}\sum_{j=2n_1+s+1}^{m+s}\sigma_{11}^r\sigma_{jj}^r\\
&+ \frac{1}{2}(m+s)^2\|H\|^2.
\end{aligned}
$$

Neglecting the positive terms $\sum_{i=2}^{m+s}\left(\sigma_{1i}^r\right)^2$ and

$$
\sum_{r=m+s+1}^{2n+s}\sum_{i=1}^{2n_1+s}\sum_{j=2n_1+s+1}^{m+s}\left(\sigma_{ij}^r\right)^2
$$

from the right hand side of (35), we get

$$
\begin{aligned}
(36)\ \mathrm{Ric}(X)+2n_2\frac{\Delta f}{f} &\le \frac{k+3s}{2}(2n_1s+2n_2s-4n_1n_2-2n_1-2n_2+2\\
&+ 5s^2-4s)-\frac{k+3s-4}{2}(s^2-3s)+3\frac{k-s}{2}(\cos^2\theta_1)\\
&- \sum_{r=m+s+1}^{2n+s}(\sigma_{11}^r)^2+\sum_{r=m+s+1}^{2n+s}\sum_{j=2n_1+s+1}^{m+1}\sigma_{11}^r\sigma_{jj}^r
\end{aligned}
$$

Using (18) in (36), we get

$$
\begin{aligned}
(37)\quad \mathrm{Ric}(X)+2n_2\Delta(\ln f) &\le 2n_2\|\nabla\ln f\|^2\\
&+ \frac{k+3s}{2}[2(n_1+n_2+\frac{5}{2}s-4)s\\
&+ 2(1-n_1-2n_1n_2-n_2)]-\frac{k+3s-4}{2}(s^2-3s)\\
&+ 3\frac{k-s}{2}(\cos^2\theta_1)\\
&- \sum_{r=m+s+1}^{2n+s}\left\{\sigma_{11}^r-\frac{1}{2}\sum_{j=2n_1+s+1}^{m+1}\sigma_{jj}^r\right\}^2\\
&+ \frac{(m+s)^2}{4}\|H\|^2.
\end{aligned}
$$

Neglecting the term $\sum_{r=m+s+1}^{2n+s}\left\{\sigma_{11}^r-\frac{1}{2}\sum_{j=2n_1+s+1}\sigma_{jj}^r\right\}^2$ from (37), we get the required equation (19).

Case-2. $e_A$ is tangent to $\mathcal{M}_{\theta_2}$ i.e., $e_A\in\{e_{2n_1+s+1},\cdots,e_{m+s}\}$. We choose $e_A=e_{m+s}=X$. Then from (26), we get

$$
\begin{aligned}
(38)\ \frac{1}{2}(m+s)^2\|H\|^2 \;=\;& \mathrm{Ric}(X)+\bar{\tau}(T_p\mathcal{M}_{\theta_1})+\bar{\tau}(T_p\mathcal{M}_{\theta_2})-2\bar{\tau}(T_p\mathcal{M})+2n_2\frac{\Delta f}{f}\\
&+\frac{1}{2}\sum_{r=m+s+1}^{2n+s}\{(2\sigma_{m+s}^r-(\sigma_{2n_1+s+1\ 2n_1+s+1}^r+\cdots+\sigma_{m+s}^r)\}^2\\
&-\sum_{r=m+s+1}^{2n+s}[\sum_{1\le i<j\le 2n_1+s}(\sigma_{ij}^r)^2+\sum_{2n_1+s+1\le i<j\le m+s}(\sigma_{ij}^r)^2\\
&-\sum_{1\le i<j\le m+s}(\sigma_{ij}^r)^2]+\sum_{1\le i<j\le m+s-1}\bar{K}_{ij}\sum_{i=1}^{m+s-1}(\sigma_{im+s}^r)^2\\
&+\sum_{r=m+s+1}^{2n+s}[\sum_{1\le i<j\le 2n_1+s}\sigma_{ii}^r\sigma_{jj}^r+\sum_{2n_1+s+1\le i<j\le m+s}\sigma_{ii}^r\sigma_{jj}^r\\
&-\sum_{1\le i<j\le m}\sigma_{ii}^r\sigma_{jj}^r].
\end{aligned}
$$

Now, we calculate

$$
\begin{aligned}
(39)\quad \bar{\tau}\left(T_p\mathcal{M}_{\theta_1}\right) &+ \bar{\tau}\left(T_p\mathcal{M}_{\theta_2}\right)+\sum_{1\le i<j\le m+s-1}\bar{K}_{ij}-2\bar{\tau}\left(T_p\mathcal{M}\right)\\
&= \frac{k+3s}{2}\left(4n_1s-8n_1n_2-4n_1-4n_2-s+2+4s^2\right)\\
(40)\qquad &- \frac{k+3s-4}{2}(2s^2-5s)+6\frac{k-s}{4}\left(\cos^2\theta_2\right).
\end{aligned}
$$

Again we find

$$
\begin{aligned}
(41)\quad \sum_{r=m+s+1}^{2n+s}&\left\{\sum_{1\le i<j\le 2n_1+s}\sigma_{ii}^r\sigma_{jj}^r+\sum_{2n_1+s+1\le i<j\le m+s}\sigma_{ii}^r\sigma_{jj}^r-\sum_{1\le i<j\le m}\sigma_{ii}^r\sigma_{jj}^r\right\}\\
&=-\sum_{r=m+s+1}^{2n+s}\left(\sigma_{m+s\ m+s}^r\right)^2+\sum_{r=m+s+1}^{2n+s}\sum_{j=2n_1+s+1}^{m+s}\sigma_{m+s\ m+s}^r\sigma_{jj}^r.
\end{aligned}
$$

Also we obtain

$$
\begin{aligned}
(42)\qquad &\textstyle\sum_{r=m+s+1}^{2n+s}\left\{2\sigma_{m+s}^r-\left(\sigma_{2n_1+s+1\ 2n_1+s+1}^r+\cdots+\sigma_{m+sm+s}^r\right)\right\}^2\\
&=4\textstyle\sum_{r=m+s+1}^{2n+s}[(\sigma_{m+s\ m+s}^r)^2-\sum_{2n_1+s+1\le i<j\le m+s}\sigma_{m+s\ m+s}^r]\\
&\qquad +(m+s)^2\|H\|^2.
\end{aligned}
$$

Using $(33), (39), (41)$ and $(42)$ in $(38)$, we get

$$\begin{aligned}
(43)\qquad \frac{1}{2}(m+s)^2\|H\|^2 &= \mathrm{Ric}(X)+\frac{2n_2\Delta f}{f}-\frac{k+3s}{2}(4n_1s-8n_1n_2-4n_1-4n_2\\
&- s+2+4s^2)-\frac{k+3s-4}{2}(2s^2-5s)+6\frac{k-s}{4}(\cos^2\theta_2)\\
&+ \sum_{r=m++s+1}^{2n+s}\sum_{i=1}^{m}(\sigma^r_{im+s})^2-\sum_{r=m+s+1}^{2n+s}(\sigma^r_{m+s\ m+s})^2\\
&+ \sum_{r=m+s+1}^{2n+s}[\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{m+s\ m+s}\sigma^r_{jj}+\sum_{1\le i<j\le m+s}(\sigma^r_{ij})^2.\\
&+ 2(\sigma^r_{m+s}m+s)^2-2\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{m+s}m+s\sigma^r_{jj}\\
&+ \frac{1}{2}(m+s)^2\|H\|^2.
\end{aligned}$$

Neglecting the positive terms $\sum_{r=m+s+1}^{2n+s}\sum_{i=1}^{m}\left(\sigma^r_{im+s}\right)^2$ and $\sum_{r=m+s+1}^{2n+s}\sum_{i\le i<j\le m+s}\left(\sigma^r_{ij}\right)^2$ from the right hand side of (43), we get

$$\begin{aligned}
(44)\qquad \mathrm{Ric}(X)+\frac{2n_2\Delta f}{f} &\le \frac{(c+3)}{4}(5n_1+3n_2+4n_1n_2)+\frac{(c-1)}{4}\left(3\cos^2\theta_2-2\right)\\
&- \sum_{r=m+2}^{2n+1}(\sigma^r_{m+1m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{m+1m+1}\sigma^r_{jj}.
\end{aligned}$$

Using (18) in (44), we get

$$\begin{aligned}
(45)\qquad \mathrm{Ric}(X)+2n_2\Delta(\ln f) &\le 2n_2\|\nabla\ln f\|^2-\frac{k+3s}{2}(4n_1s-8n_1n_2-4n_1\\
&- 4n_2-s+2+4s^2)-\frac{k+3s-4}{2}(2s^2-5s)\\
&+ 6\frac{k-s}{4}(\cos^2\theta_2)-\sum_{r=m+s+1}^{2n+s}\{\sigma^r_{m+s\ m+s}\\
&- \frac{1}{2}\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{jj}\}^2+\frac{(m+s)^2}{4}\|H\|^2.
\end{aligned}$$

Neglecting the $\sum_{r=m+s+1}^{2n+s}\left\{\sigma^r_{m+s\ m+s}-\frac{1}{2}\sum_{j=2n_1+s+1}^{m+s}\sigma^r_{jj}\right\}^2$ from (45), we get (34). □

For the equality case:

(1) If $H(p)=0$, then a unit vector $X\in T_pM$ orthogonal to $\xi$ satisfies the equality case of (33) or (34) if and only if $X\in\mathcal{N}_p$, the relative null space at $p$.

(2) The equality case of (33) holds identically for all unit vectors tangent to $M_{\theta_1}$ and orthogonal to $\xi$ at $p$ if and only if $p$ is a mixed totally geodesic and $\mathcal{D}^{\theta_1}$ -totally geodesic point at $p$.

(3) The equality case of (34) holds identically for all unit vectors tangent to $M_{\theta_2}$ at $p$ if and only if $p$ is a mixed totally geodesic point and either $\mathcal{D}^{\theta_2}$-totally geodesic or $\mathcal{D}^{\theta_2}$-totally umbilical point with $n_2 = 1$

For (1) we assume that $H(p) = 0$. Then for any unit tangent vector $e_A \in \{e_1, e_2, \cdots, e_{2n_1+s}, e_{2n_1+s+1}, \cdots, e_{m+s}\}$, the equality of (33) and (34) holds if and only if the following conditions hold

(a) $\sum_{\substack{i=1,\\ i\neq A}}^{2n_1+s} \left(\sigma_{iA}^r\right)^2 = 0$

(b) $\sum_{i=1}^{2n+s} \sum_{j=2n_1+s+1}^{m+s} \left(\sigma_{ij}^r\right)^2 = 0$

(c) $2\sigma_{AA}^r = \sum_{j=2n_1+s+1}^{m+s} \sigma_{jj}$ such that $r \in \{m+s+1, \cdots, 2n+s\}$.

The condition (a) implies that $p$ is a mixed totally geodesic point. Thus using $\mathcal{D}^{\theta_1}$-minimality of $M^{m+s}$ and combining the conditions of (b) and (c), it is clear that $X = e_A \in \mathcal{N}_p$, the relative null space at $p$. The converse is trivial. For (2) the equality condition of (33) holds if and only if the following conditions hold

(a) $\sum_{i=1}^{2n+s} \sum_{j=2n_1+s+1}^{m+s} \left(\sigma_{ij}^r\right)^2 = 0$

(b) $\sum_{\substack{i=1,\\ i\neq \alpha}}^{2n_1+s} \left(\sigma_{\alpha i}^r\right)^2 = 0$

(c) $2\sigma_{\alpha\alpha}^r = \sum_{j=2n_1+s+1}^{m+s} \sigma_{jj}, \alpha \in \{1, \cdots, 2n_1\}, r \in \{m+s+1, \cdots, 2n+s\}$.

By virtue of $\mathcal{D}^{\theta_1}$-minimality of $M^{m+s}$ and condition (c), we get $\sigma_{\alpha\alpha} = 0, \alpha \in \{1, \cdots, 2n_1+s\}$ and then combining conditions (a) and (b), we get $\sigma_{ij}^r = 0$ for all $i \neq j, i, j \in \{1, 2, \cdots, 2n_1+s\}$, from which we get the desired result. For (3), the equality case of (34) holds if and only if

(a) $\sum_{r=m+s}^{2n+s} \sum_{j=2n_1+s+1}^{m+s} \left(\sigma_{ij}^r\right)^2 = 0,$

(b) $\sum_{\substack{i=1\\ i\neq \alpha}}^{2n_1+s} \left(\sigma_{\alpha i}^r\right)^2 = 0,$

(c) $2\sigma_{AA}^r = \sum_{j=2n_1+s+1}^{m+s} \sigma_{jj}, A \in \{2n_1+s+1, \cdots, m+s\}, r \in \{m+s+1, \cdots, 2n+s\}$.

Using $\mathcal{D}^{\theta_1}$-minimality of $M^{m+s}$ with the condition (c), we get $\sigma_{mm}^r = \sigma_{m+s}^r$ for $n_2 = 1$ and $\sigma_{AA}^r = 0 \forall A \in \{2n_1+s+1, \cdots, m+s\}$. Combining the conditions (a) and (b), we get $\sigma_{ij}^r = 0$ for all $i \neq j, i, j \in \{2n_1+s+1, \cdots, m+s\}$. Thus we get the desired results. □

**Corollary 3.2.** *Let $\psi : M^{m+s} = M_\theta \times_f M_\perp \to \widetilde{M}^{2n+s}(c)$ be a $\mathcal{D}^\theta$-minimal isometric immersion of a$(m+s)-$ dimensional warped product pointwise pseudo-slant submanifold $M^{m+s}$ into $\widetilde{M}^{2n+s}(c)$, where $M_\theta$ and $M_\perp$ are* *a pointwise slant submanifold with slant function $\theta$ and an anti-invariant submanifold, respectively, and $\xi \in \Gamma\left(\mathcal{D}^{\theta}\right)$. Then we have*

$$\begin{aligned}\operatorname{Ric}(X)+2n_2\Delta(\ln f) &\leq 2n_2\|\nabla \ln f\|^2+\frac{k+3s}{2}(2n_1s+2n_2s-4n_1n_2-2n_1-2n_2\\ &+ 2+5s^2-4s)-\frac{k+3s-4}{2}(s^2-3s)+3\frac{k-s}{2}(\cos^2\theta)\\ &+ \frac{(m+s)^2}{4}\|H\|^2, \quad if\ X\in\Gamma\mathcal{D}_{\theta},\end{aligned}$$

$$\begin{aligned}\operatorname{Ric}(X)+n_2(\Delta\ln f) &\leq n_2\|\nabla(\ln f)\|^2-\frac{k+3s}{2}(4n_1s-8n_1n_2-4n_1-4n_2\\ &-s+2+4s^2)-\frac{k+3s-4}{2}(2s^2-5s)\\ &+\frac{(m+s)^2}{4}\|H\|^2,\ if\ X\in\Gamma(\mathcal{D}^{\perp}),\end{aligned}$$

*where* $\dim(M_\theta)=2n_1+s$ *and* $\dim(M_\perp)=2n_2$.

*Proof.* Substituting $\theta_1=\theta$ for simplification and $\theta_2=\frac{\pi}{2}$ in (19) and (20) we get the result. □

**Corollary 3.3.** *Let $\psi: M^{m+s}=M_T\times_f M_\theta\to \widetilde{M}^{2n+s}(c)$ be a $\mathcal{D}^T$ -minimal isometric immersion of a $(m+s)$ dimensional warped product pointwise semi-slant submanifold $M^{m+s}$ into $\widetilde{M}^{2n+s}(c)$, where $M_T$ and $M_\theta$ are an invariant submanifold and a pointwise slant submanifold with slant function $\theta$, respectively, and $\xi\in\Gamma\left(\mathcal{D}^T\right)$. Then we have*

$$\begin{aligned}\operatorname{Ric}(X)+2n_2(\Delta\ln f) &\leq 2n_2\|\nabla(\ln f)\|^2+\frac{k+3s}{2}[2(n_1+n_2+\frac{5}{2}s-4)s\\ &+ 2(1-n_1-2n_1n_2-n_2)]-\frac{k+3s-4}{2}(s^2-3s)\\ &+ 3\frac{k-s}{2}+\frac{(m+s)^2}{4}\|H\|^2,\end{aligned}$$

*if* $X\in\Gamma\left(\mathcal{D}^T\right)$,

$$\begin{aligned}\operatorname{Ric}(X)+2n_2(\Delta\ln f) &\leq 2n_2\|\nabla(\ln f)\|^2-\frac{k+3s}{2}(4n_1s-8n_1n_2-4n_1\\ &- 4n_2-s+2+4s^2)-\frac{k+3s-4}{2}(2s^2-5s)\\ &+ 6\frac{k-s}{4}(\cos^2\theta_2)+\frac{(m+s)^2}{4}\|H\|^2,\end{aligned}$$

*if* $X\in\Gamma\left(\mathcal{D}^{\theta}\right)$,

*where* $\dim(M_T)=2n_1+s$ *and* $\dim(M_\theta)=2n_2$.

**Corollary 3.4.** *Let $\psi: M^{m+s}=M_T\times_f M_\perp\to\widetilde{M}^{2n+s}(c)$ be a $\mathcal{D}^T$-minimal isometric immersion of a $(m+s)$ dimensional warped product CR submanifold*

*$M^{m+s}$ into $\widetilde{M}^{2n+s}(c)$, where $M_T$ and $M_\perp$ are invariant and anti-invariant submanifolds, respectively, and $\xi \in \Gamma\left(\mathcal{D}^T\right)$. Then we have*

$$\begin{aligned}
\operatorname{Ric}(X)+n_2(\Delta \ln f) &\leq n_2\|\nabla(\ln f)\|^2+\frac{k+3s}{2}[2(n_1+n_2+\frac{5}{2}s-4)s \\
&+ 2(1-n_1-2n_1n_2-n_2)]-\frac{k+3s-4}{2}(s^2-3s) \\
&+ 3\frac{k-s}{2}+\frac{(c-1)}{4}+\frac{(m+s)^2}{4}\|H\|^2,
\end{aligned}$$

*if* $X \in \Gamma\left(\mathcal{D}^T\right)$,

$$\begin{aligned}
\operatorname{Ric}(X)+n_2(\Delta \ln f) &\leq n_2\|\nabla(\ln f)\|^2-\frac{k+3s}{2}(4n_1s-8n_1n_2-4n_1-4n_2-s \\
&+ 2+4s^2)-\frac{k+3s-4}{2}(2s^2-5s)+\frac{(m+s)^2}{4}\|H\|^2,
\end{aligned}$$

*if* $X \in \Gamma\left(\mathcal{D}^\perp\right)$,

*where* $\dim(M_T)=2n_1+s$ *and* $\dim(M_\perp)=2n_2$.

*Proof.* Substituting $\theta_1=0$ and $\theta_2=\frac{\pi}{2}$ in (19) and (20), we get the result. □

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