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ON A GENERALIZATION OF $\oplus$-CO-COATOMICALLY SUPPLEMENTED MODULES

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Abstract. In this paper, we define $\oplus_\delta$-co-coatomically supplemented and co-coatomically δ-semiperfect modules as a strongly notion of $\oplus$-co-coatomically supplemented and co-coatomically semiperfect modules with the help of Zhou's radical. We say that a module A is $\oplus_\delta$-co-coatomically supplemented if each co-coatomic submodule of A has a $\delta-$supplement in A which is a direct summand of A. And a module A is co-coatomically δ-semiperfect if each coatomic factor module of A has a projective δ-cover. Also we define co-coatomically amply δ-supplemented modules and we examined the basic properties of these modules. Furthermore, we give a ring characterization for our modules. In particular, a ring R is δ-semiperfect if and only if each free R-module is co-coatomically δ-semiperfect.

1. Introduction

In this study, we admit that all rings are with identity and all modules are unitary left modules unless otherwise stated. Let R be such a ring and A be such a module. By the notation $X \leq A$, we mean that X is a submodule of A. A submodule X of A is called *small* in A if $X+Y \neq A$ for any proper submodule Y of A, denoted by $X \ll A$, and we point with $Rad(A)$, the sum of whole small submodules of A. Dual to this concept, a submodule X of A is called *essential* in A, by $X \trianglelefteq A$, if the intersection of X is non-zero with the other submodules of A, except for $\{0\}$. It is known that the set $Z(A) = \{a \in A | \ Ann(a) \trianglelefteq R\}$ is the singular submodule of A, where $Ann(a)$ is an annihilator of a. The module A is entitled *singular* in case $Z(A) = A$. A submodule X of A is called *cofinite* whenever A/X is finitely generated. A module A is called *coatomic* if every proper submodule of A is contained in a maximal submodule of A. In addition to these, in [3] co-coatomic submodules are defined as a generalization of cofinite submodules as follows. If the factor module A/X is coatomic, then we say that $X \leq A$ is *co-coatomic*. A supplement submodule T of X in A is

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minimal element of the set $\{Y \leq A| \ A = X + Y\}$ that equivalents $A = X + T$ and $X \cap T \ll T$. A module A is called *supplemented* if each submodule of A has a supplement in A [24]. If each submodule of A has a supplement in A that is a direct summand of A, then the module A is called $\oplus-$*supplemented* [10]. Besides, cofinitely supplemented and $\oplus$-cofinitely supplemented modules are introduced by [2], [6] respectively, as follows. If every cofinite submodule of M has a supplement in A (that is a direct summand of A), then A is entitled a *($\oplus$-)cofinitely supplemented module*.

In [23], the author generalized the concept of small submodules to the concept of δ-small submodules. $X \ll_\delta A$ denotes that X is a δ-small submodule of A which means $X + Y$ is proper in A for any proper submodule Y of A with A/Y singular. Furthermore, the sum of all δ-small submodules of A shown by $\delta(A)$. Following, δ-supplemented modules as a general version of supplemented modules are introduced in [9]. A module A is entitled *δ-supplemented* if each submodule X of A has a δ-supplement T in A, i.e. $A = X+T$ and $X \cap T \ll_\delta T$. In [1] and [15], a module A is called *($\oplus$-)cofinitely δ-supplemented*, if each cofinite submodule of A has a δ-supplement in A (which is a direct summand of A).

In the paper [3], a generalization of $\oplus$-supplemented modules, defined as $\oplus$-co-coatomically supplemented modules, is given by the authors, which is also a restriction of $\oplus$-cofinitely supplemented modules. And also a module A is entitled *co-coatomically supplemented* if each co-coatomic submodule of A has a supplement in A.

Inspired from the definitions given above, in Section 2 we introduce $\oplus_\delta$-co-coatomically supplemented modules and co-coatomically δ-supplemented modules as follows. We say that a module A is $\oplus_\delta$-co-coatomically supplemented if each co-coatomic submodule of A has a δ-supplement in A which is a direct summand of A. And, a module A is entitled co-coatomically δ-supplemented if each co-coatomic submodule of A has a δ-supplement in A. We give main results related with these concepts. In general, any factor module of a δ-supplemented module is δ-supplemented, but this claim is not valid for $\oplus_\delta$-co-coatomically supplemented modules (see in Example 2.5). A factor module of a $\oplus_\delta$-co-coatomically supplemented module, which is constructed with respect to a fully invariant submodule of the module, is $\oplus_\delta$-co-coatomically supplemented. Being $\oplus_\delta$-co-coatomically supplemented module is inherited for the submodules of a module A which are co-coatomic, fully invariant and also a direct summand in A. Each direct summand of a $\oplus_\delta$-co-coatomically supplemented module with SSP (*summand sum property*) is $\oplus_\delta$-co-coatomically supplemented. Besides, each co-coatomic direct summand of a $\oplus_\delta$-co-coatomically supplemented module with the property (D_3) is $\oplus_\delta$-co-coatomically supplemented. Any finite direct sum of a $\oplus_\delta$-co-coatomically supplemented module is $\oplus_\delta$-co-coatomically supplemented. A module A is $\oplus_\delta$-co-coatomically supplemented if and only if every maximal submodule of A has a δ-supplement which is a direct summand of A if and only if A is δ-radical or δ-local. At the end of this section, we give a ring characterization for our modules such that a ring R is δ-semiperfect if and only if every finitely generated free R-module is $\oplus_\delta$-co-coatomically supplemented.

In Section 3, we define co-coatomically δ-semiperfect modules as a generalization of δ-semiperfect modules and also a restriction of cofinitely δ-semiperfect modules. If each coatomic factor module of a module A has a projective δ-cover, then A is called co-coatomically δ-semiperfect. The concepts of $\oplus_\delta$-co-coatomically supplemented modules and co-coatomically δ-semiperfect modules coincide for projective modules. Every homomorphic image (and δ-cover) of a co-coatomically δ-semiperfect module is co-coatomically δ-semiperfect. If A is a projective δ-semiperfect module, then every A-generated module is co-coatomically δ-semiperfect. Owing to this fact, we give a ring characterization for our modules. A ring R is δ-semiperfect if and only if each free R-module is co-coatomically δ-semiperfect.

2. $\oplus_\delta$-co-coatomically and co-coatomically δ-supplemented modules

Definition 2.1. *Let A be a module. If every coatomic submodule of A has a δ-supplement in A, then A is called co-coatomically δ-supplemented.*

Let A be a co-coatomically δ-supplemented module and K be any cofinite submodule of A. Since the factor module A/K is finitely generated then it is also coatomic. Thus, K is coatomic in A. Hence K has a δ-supplement in A. It means that A is cofinitely δ-supplemented.

Since each factor module of a coatomic module is coatomic, then a coatomic module A is co-coatomically δ-supplemented if and only if A is δ-supplemented.

Definition 2.2. *Let A be a module. If every co-coatomic submodule of A has a δ-supplement which is a direct summand of A, then A is entitled $\oplus_\delta$-co-coatomically supplemented.*

We can write the below hierarchy for a module M.

$\oplus_\delta$-supp. module $\Rightarrow$ $\oplus_\delta$-co-coatomically supp. module $\Rightarrow$ $\oplus_\delta$-cofinitely supp. module

It is clear that $\oplus_\delta$-supplemented modules are $\oplus_\delta$-co-coatomically supplemented in general. Besides it can be seen that the converse statement need not to be true.

Example 2.3. *The $\mathbb{Z}$-module $\mathbb{Q}$ is a $\oplus_\delta$-co-coatomically supplemented module as it has no proper co-coatomic submodule. However ${}_{\mathbb{Z}}\mathbb{Q}$ is not $\oplus_\delta$-supplemented.*

Additionally, $\oplus_\delta$-co-coatomically supplemented modules are also $\oplus$-cofinitely supplemented. Now, let us show an example verifying the converse may not be true. Also, own to this example, it can be noticed that the direct sum of $\oplus_\delta$-co-coatomically supplemented modules may not be $\oplus_\delta$-co-coatomically supplemented.

Recall from that a ring R is called δ-perfect (δ-semiperfect) if every R-module (every simple R-module) has a projective δ-cover.

Example 2.4. *Let F be a field and*

$$R = F\left[[x]\right] = \left\{ f\left(x\right) = \sum_{k=0}^{\infty} a_k x^k,\ a_k \in F \right\}.$$

Here R is a local ring which is also δ-semiperfect but not δ-perfect. As every free R-module is $\oplus$-cofinitely δ-supplemented over a δ-semiperfect ring, in particular, ${}_RR^{(\mathbb{N})}$ is also $\oplus$-cofinitely δ-supplemented. However, the coatomic submodule $Rad\left({}_RR^{(\mathbb{N})}\right) = \delta\left({}_RR^{(\mathbb{N})}\right)$ does not have a δ-supplement as R is local by [17, Proposition 2.5] *and* [4, Theorem 1], *respectively. Hence ${}_RR^{(\mathbb{N})}$ is not $\oplus_\delta$-co-coatomically supplemented.*

During the below exercise, we point that a factor module of a $\oplus_\delta$-co-coatomically supplemented module need not to be $\oplus_\delta$-co-coatomically supplemented.

Example 2.5. [8, Example 2.2] *Let R be a commutative local ring that is not a valuation ring and assume that $s \geq 2$. Then, it can be found a finitely presented indecomposable module $A = R^{(s)}/K$ that can not be generated by fewer than s elements. With* [7, Corollary 1], *$R^{(s)}$ is a $\oplus_\delta$-co-coatomically supplemented. Nevertheless, A is not $\oplus_\delta$-co-coatomically supplemented by* [20, Example 2.1] *and* [17, Example 2.8].

Recall from [11] that a submodule X of A is entitled *fully invariant submodule* of A if $g\left(X\right) \leq X$ for every $g \in End(A)$, where $End(A) = \{g \mid g : A \to A$ is a homomorphism$\}$. If every submodule of A is fully invariant, then A is called a *duo module*.

Theorem 2.6. *Let A be a $\oplus_\delta$-co-coatomically supplemented module and X be a fully invariant submodule of A. Then A/X is $\oplus_\delta$-co-coatomically supplemented.*

Proof. Let N/X be a co-coatomic submodule of A/X. Then

$$\left(A/X\right)/\left(N/X\right) \cong A/N$$

is coatomic and N is a co-coatomic submodule of A. Therefore, there exists a $\delta-$supplement K of N that is a direct summand in A. Then it can be written that $A = N + K$, $N \cap K \ll_\delta K$ and $A = K \oplus K_1$. Following, it is obvious that $\left(K + X\right)/X$ is a δ-supplement of N/X in A/X. Note that [11, Lemma 2.1], $X = \left(K \cap X\right) \oplus \left(K_1 \cap X\right)$ as X is fully invariant. Using this, it can be seen that $A/X = \left(\left(K + X\right)/X\right) \oplus \left(\left(K_1 + X\right)/X\right)$. Hence, A/X is $\oplus_\delta$-co-coatomically supplemented. □

Corollary 2.7. *If A is a $\oplus_\delta$-co-coatomically supplemented module, then $A/\delta(A)$ is also a $\oplus_\delta$-co-coatomically supplemented module.*

Corollary 2.8. *If A is a $\oplus_\delta$-co-coatomically supplemented duo module and $X \leq A$, then A/X is $\oplus_\delta$-co-coatomically supplemented.*

Theorem 2.9. *Let A be a $\oplus_\delta$-co-coatomically supplemented module and X be a co-coatomic fully invariant submodule of A which is also a direct summand in A. Then, X is a $\oplus_\delta$-co-coatomically supplemented module.*

Proof. Let Y be a co-coatomic submodule of X. By the hypothesis, it can be found a coatomic submodule X_1 of A where $A = X \oplus X_1$ with X_1 is coatomic. Following, the factor module $A/Y = [(X \oplus X_1)/Y] \oplus X_1 \cong (X/Y) \oplus X_1$ is coatomic as a direct summand of two coatomic modules. As A is $\oplus_\delta$-co-coatomically supplemented, it can be found a δ-supplement Z of Y in A where $A = Z \oplus Z_1$, $A = Y + Z$, $Y \cap Z \ll_\delta Z$. Now, using modular law we can write $X = (Y + Z) \cap X = Y + (Z \cap X)$. Also, we have $X = (X \cap Z) \oplus (X \cap Z_1)$ as X is fully invariant. Thus, $X \cap Z$ is a direct summand of X. Furthermore, $Y \cap (X \cap Z) = Y \cap Z \ll_\delta Z$ and so $Y \cap (X \cap Z) \ll_\delta X \cap Z$ by [14, Lemma 1.2.(3)]. Hence, X is $\oplus_\delta$-co-coatomically supplemented. □

Theorem 2.10. *Let A be a $\oplus_\delta$-co-coatomically supplemented module, $X \leq A$. If $(X + Y)/X$ is a direct summand of A/X for every direct summand of Y of A, then A/X is a $\oplus_\delta$-co-coatomically supplemented module.*

Proof. Suppose that N/X is a co-coatomic submodule of A/X where N is a co-coatomic submodule of A and $X \leq N$. Since A is a $\oplus_\delta$-co-coatomically supplemented module, it can be found a direct summand T of A where $A = N+T$, $N \cap T \ll_\delta T$ and $A = T \oplus T'$ where $T'$is any submodule of A. Now, we have $A/X = N/X + [(X + T)/X]$. Also, by the hypothesis, $(X + T)/X$ is a direct summand of A/X. Let $f : A \to A/X$ be a canonical epimorphism. Since $N \cap T \ll_\delta T$ and $(N/X) \cap ((X + T)/X) = (N \cap (X + T))/X = (X + (N \cap T))/X = f(N \cap T) \ll_\delta (X + T)/X$ by [23, Lemma 1.5], it follows that $(X + T)/X$ is a δ-supplement of N/X in A/X which is a direct summand. □

An R-module A has *SSP* (*Summand Sum Property*) if the sum of two direct summand of A is again a direct summand of A [22].

Theorem 2.11. *If A is a $\oplus_\delta$-co-coatomically supplemented module with SSP, then each direct summand of A is $\oplus_\delta$-co-coatomically supplemented.*

Proof. For any direct summand X_1 of A, we have $A = X_1 \oplus X'$ for some $X' \leq A$. Suppose that Y is a direct summand of A. Since A has SSP, we have $A = (X_1 + Y) \oplus T$ for some $T \leq A$. Therefore, the equality $A/X' = (X_1 + Y)/X' \oplus (T + X')/X'$ implies that A/X' is a $\oplus_\delta$-co-coatomically supplemented module by Theorem 2.10. □

Recall from [18] that an R-module A is entitled *distributive* if lattice of its submodules is a distributive lattice, equivalently for submodules X, Y and Z of A, $Z+(X\cap Y)=(Z+X)\cap(Z+Y)$ or $Z\cap(X+Y)=(Z\cap X)+(Z\cap Y)$.

Theorem 2.12. *Let A be a $\oplus_\delta$-co-coatomically supplemented distributive module. Then A/X is a $\oplus_\delta$-co-coatomically supplemented module for every submodule X of A.*

Proof. Suppose that Y is a direct summand of A. Then $A=Y\oplus T$ for some submodule T of A and we can write $A/X=[(X+Y)/X]+[(X+T)/X]$. By distributive property of A, we have $X=X+(Y\cap T)=(X+Y)\cap(X+T)$. This implies that $A/X=[(X+Y)/X]\oplus[(X+T)/X]$ and therefore A/X is a $\oplus_\delta$-co-coatomically supplemented module by Theorem 2.10. □

Recall from [10] that a module A is called a (D_3)-module if, for the submodules A_1,$A_2\leq_\oplus A$ with $A=A_1+A_2$, A satisfies $A_1\cap A_2\leq_\oplus A$.

Proposition 2.13. *Let A be a $\oplus_\delta$-co-coatomically supplemented module with (D_3). Then each co-coatomic direct summand of A is $\oplus_\delta$-co-coatomically supplemented.*

Proof. Assume that X is a co-coatomic direct summand of A and Y is a co-coatomic submodule of X. By the hypothesis, $A=X\oplus X_1$ and $A/X=X_1$ is coatomic. Therefore, $A/Y=((X\oplus X_1)/Y)=(X/Y)\oplus X_1$ is coatomic as a direct sum of two coatomic modules, [7, Corollary 5]. Since A is $\oplus_\delta$-co-coatomically supplemented, there exists a $\delta-$supplement Z of Y in A which is a direct summand of A. Following, we have $X=X\cap A=X\cap(Y+Z)=(X\cap Z)+Y$. As A has the property (D_3), $X\cap Z$ is also a direct summand of X. So $Y\cap(X\cap Z)=Y\cap Z\ll_\delta X\cap Z$ by [14, Lemma 1.2.(3)]. Hence, X is $\oplus_\delta$-co-coatomically supplemented. □

Recall from [5] that a module A is called $\delta-$*local* if $\delta(A)\ll_\delta A$ and $\delta(A)$ is a maximal submodule of A. It is well known that a ring R is a left δ-V-ring if and only if $\delta(A)=0$ for each left R-module A (see [19]).

Proposition 2.14. *Let A be a module over the δ-V-ring R. Then A is $\oplus_\delta$-co-coatomically supplemented if and only if A is semisimple.*

Proof. The sufficiency is clear. For the necessity, note that A is also cofinitely δ-supplemented module because it is $\oplus_\delta$-co-coatomically supplemented. Then $A/Cof_\delta(A)$ has no maximal submodule by [1, Theorem 2.9] where $Cof_\delta(A)$ is the sum of all submodules of A that are δ-supplements of maximal submodules of A. Following, we have $A/Cof_\delta(A)=Rad(A/Cof_\delta(A))=0$ and this implies that $A=Cof_\delta(A)$. Write $A=\sum S_i$, where each S_i is a $\delta-$supplement of a maximal submodule P_i of A. Then by [16, Lemma 2.22] each S_i is either δ-local or semisimple projective. Assume that S_i is semisimple projective. Here, as R is a δ-V-ring, we have $\delta(S_i)=S_i=0$ which contradicts with the maximality of P_i in A. Hence, A is only the sum of δ-local submodules S_i of

A. Thus, $\delta(S_i) = 0$ is maximal in S_i and so each S_i is simple. Hence, A is semisimple as a sum of simple submodules. □

Now we give a useful lemma to evidence that the finite sum of $\oplus_\delta$-co-coatomically supplemented modules are also $\oplus_\delta$-co-coatomically supplemented.

Lemma 2.15. *Let A be module and X,Y be submodules of A where X is co-coatomically δ-supplemented, Y is co-coatomic and $X+Y$ has a δ-supplement S in A. Then $X \cap (Y+S)$ has a δ-supplement T in X and $S+T$ is a δ-supplement of Y in A.*

Proof. By the hypothesis, we have that $A = (X+Y)+S$, $(X+Y) \cap S \ll_\delta S$. Furthermore,

$$X/[X \cap (Y+S)] \cong (X+Y+S)/(Y+S) = A/(Y+S) \\ \cong (A/Y)/[(Y+S)/Y]$$

is coatomic. Thus, $X \cap (Y+S) \leq X$ is co-coatomic. Therefore, there exists a $\delta-$supplement T of $X \cap (Y+S)$ in X, i.e., $[X \cap (Y+S)] + T = X$ and $[X \cap (Y+S)] \cap T = (Y+S) \cap T \ll_\delta T$. Then, $A = X+Y+S = Y+S+T$ and

$$Y \cap (S+T) \leq S \cap (Y+T) + T \cap (S+Y) \leq S \cap (Y+X) + T \cap (S+Y) \\ \ll_\delta S+T.$$

Hence, $S+T$ is a δ-supplement of Y in A. □

Proposition 2.16. *Any finite direct sum of $\oplus_\delta$-co-coatomically supplemented modules is $\oplus_\delta$-co-coatomically supplemented.*

Proof. Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where each A_i is $\oplus_\delta$-co-coatomically supplemented. We claim that A is $\oplus_\delta$-co-coatomically supplemented. To complete the proof, it is enough to show that the assertion is true in case $n = 2$. Let $A = A_1 \oplus A_2$ and X be a co-coatomic submodule of A. Then $A = A_1 + A_2 + X$ and 0 is a δ-supplement of $A_1 + A_2 + X$ in A. For the submodule $A_2 \cap (A_1 + X)$ of A_2,

$$A_2/[A_2 \cap (A_1+X)] \cong (A_1+A_2+X)/(A_1+X) \cong (A/X)/[(A_1+X)/X]$$

is coatomic as a factor module of a coatomic module A/X where $X \leq A$ is co-coatomic. Hence, $A_2 \cap (A_1+X) \leq A_2$ is co-coatomic. By the hypothesis, there exists a $\delta-$supplement D of $A_2 \cap (A_1+X)$ which is a direct summand of A_2. Thus, D is a δ-supplement of $A_1 + X$ by Lemma 2.15. By the same way given above, it can be shown that $A_1/A_1 \cap (X+D)$ is coatomic and $A_1 \cap (X+D) \leq A_1$ is co-coatomic. By the assumption, $A_1 \cap (X+D)$ has a δ-supplement S in X that is a direct summand of A_1. Again using Lemma 2.15 $D+S$ is a δ-supplement of X in A, where $D \oplus S$ is a direct summand of A. Finally, $A = A_1 \oplus A_2$ is $\oplus_\delta$-co-coatomically supplemented. □

Recall from [13] that a module A is called δ-radical if $\delta(A) = A$, and denote the sum of all δ-radical submodules of the module A by $P_\delta(A)$, that is, $P_\delta(A) = \sum \{U \leq A : \delta(U) = U\}$.

Proposition 2.17. *The following statements are equivalent for an indecomposable module A.*

(1) Each co-coatomic submodule of A has a $\delta-$supplement which is a direct summand.

(2) Each maximal submodule of A has a $\delta-$supplement which is a direct summand.

(3) A is δ-local or δ-radical.

Proof. (1) $\Rightarrow$ (2) It is obvious that as every maximal submodule is co-coatomic.

(2) $\Rightarrow$ (3) Assume that A is not δ-radical. Then, $\delta(A) \neq A$, i.e. there exists an essential maximal submodule P of A which has a δ-supplement T that is a direct summand of A. As A is indecomposable, then $T = 0$ or $T = A$.

Case 1: Let $T = 0$. This contradicts with the maximality of P.

Case 2: Let $T = A$. By [16, Lemma 2.22], T is either projective semisimple or δ-local. If T is projective semisimple, then $\delta(T) = \delta(A) = A$ which contradicts with the case that A is δ-radical. From here, it forces A to be δ-local.

(3) $\Rightarrow$ (1) Let X be any co-coatomic submodule of A. As A/X is coatomic, there exists a maximal submodule of A/X containing all proper submodule of A/X. Therefore, A has a maximal submodule P containing X. Since A is indecomposable, the intersection of P with the other non-zero submodules of A is non-zero, that is, the submodule $P \leq A$ is essential maximal and so A is not δ-radical. It forces A to be δ-local from the assumption. It follows that A is $\oplus_\delta$-supplemented by [17, Proposition 3.1]. Finally, A is $\oplus_\delta$-co-coatomically supplemented. □

Corollary 2.18. *Let A be an indecomposable module that is not δ-radical. A is δ-local if and only if A is $\oplus_\delta$-co-coatomically supplemented.*

Theorem 2.19. *A ring R is δ-semiperfect if and only if each finitely generated free R-module is $\oplus_\delta$-co-coatomically supplemented.*

Proof. ($\Rightarrow$) Let R be a δ-semiperfect ring and A be a finitely generated free R-module. By Lemma 3.5 in [12], A is $\oplus - \delta$-supplemented. Hence, A is $\oplus_\delta$-co-coatomically supplemented.

($\Leftarrow$) By the hypothesis, ${}_RR$ is $\oplus_\delta$-co-coatomically supplemented. Hence, R is a δ-semiperfect ring by Lemma 3.5 in [12]. □

Corollary 2.20. *For an arbitrary ring R, the following conditions are equivalent:*

(1) R is δ-semiperfect.

(2) ${}_RR$ is $\oplus - \delta$-supplemented.

(3) $_RR$ is $\oplus_\delta$-co-coatomically supplemented
(4) $_RR$ is $\oplus$-cofinitely δ-supplemented.

Proof. (1) $\Rightarrow$(2) It follows from [12, Lemma 3.5].
(2) $\Rightarrow$(3) It is clear.
(3) $\Rightarrow$(4) Since every cofinite submodule is also co-coatomic, the proof is evident.
(4) $\Rightarrow$(1) It is obvious by [1, Theorem 3.9]. □

3. Co-coatomically δ-semiperfect modules

In the paper [21], the concept of δ-semiperfect modules and the connections between δ-supplemented modules and δ-semiperfect modules are investigated. Let A and N be modules, an epimorphism $f : A \to N$ is entitled a *δ-cover* in case $ker\,(f) \ll_\delta A$. A δ-cover $f : A \to N$ is entitled a *projective δ-cover* in case A is a projective module (see [23]).

Definition 3.1. *Let A be a module. If each coatomic factor module of A has a projective δ-cover, then A is called co-coatomically δ-semiperfect.*

Proposition 3.2. *Let A be a projective module. Then A is co-coatomically δ-semiperfect if and only if A is $\oplus_\delta$-co-coatomically supplemented.*

Proof. ($\Rightarrow$) Let A/K be a coatomic factor module of A. As A is $\oplus_\delta$-co-coatomically supplemented, it can be found submodules N and N_1 where $A = N \oplus N_1$, $A = K + N$ and $N \cap K \ll_\delta N$. Here, N is projective because A is projective. For the inclusion homomorphism $i : N \to A$ and the canonical epimorphism $\varphi : A \to A/K$, we have $\varphi \circ i : N \to A/K$ is an epimorphism and $\ker(\varphi \circ i) = N \cap K \ll_\delta N$.
($\Leftarrow$) Let K be a co-coatomic submodule of A. So A/K is coatomic. There exists a projective δ-cover $\sigma : P \to A/K$ by the hypothesis. Then there are submodules X, Y of A where $A = X \oplus Y$ with $X \leq K$ and $Y \cap K \ll_\delta A$ by [23, Lemma 2.4]. If we use [23, Lemma 1.3(2)], we obtain that $Y \cap K \ll_\delta Y$, i.e., Y is a δ-supplement of K. □

Recall that in [15] A is entitled *cofinitely δ-semiperfect* if each finitely generated factor module of A has a projective δ-cover.

Definition 3.3. *Let A be a module. If each co-coatomic submodule of A has ample δ-supplements in A, A is called co-coatomically amply δ-supplemented.*

It is clear that every co-coatomically amply δ-supplemented module is co-coatomically δ-supplemented.

Proposition 3.4. *The following statements are equivalent for a projective module A:*
(1) A is co-coatomically δ-semiperfect.

(2) *A is $\oplus_\delta$-co-coatomically supplemented.*

(3) *Each co-coatomic submodule K of A, A has a decomposition $A = T \oplus T''$ where $T'' \leq K$ and $K \cap T \ll_\delta T$.*

(4) *A is co-coatomically amply δ-supplemented by δ-supplements that have projective δ -covers.*

(5) *A is co-coatomically δ-supplemented by δ-supplements that have projective δ-covers.*

Proof. (1) $\Leftrightarrow$ (2) The proof follows from Proposition 3.2.

(2) $\Rightarrow$ (3)Assume that K is a co-coatomic submodule of A. By the assumption, there exist submodules T and T' of A where $A = K + T$, $K \cap T \ll_\delta T$ and $A = T \oplus T'$. Since A is projective, there exists a submodule T'' of A such that $A = T \oplus T''$ such that $T'' \leq K$ from [22, 41.14].

(3) $\Rightarrow$ (2) The proof is clear.

(1) $\Rightarrow$ (4) Suppose that K be a co-coatomic submodule of A and $A = K+T$ for some submodule T of A. By the assumption, we have a projective δ-cover $\mu : P \to A/K$, for a projective module P. Since P is projective and $A/K \cong T/(K \cap T)$, there exists a homomorphism $h : P \to T$. Since $\ker(\mu) \ll_\delta P$ and $h(\ker \mu) = Im(h) \cap K \cap T = Im(h) \cap K$, $Im(h) \cap K \ll_\delta Im(h)$. And so $T = Im(h) + (K \cap T)$ because μ is an epimorphism. Thus $Im(h)$ is a δ-supplement of $K \cap T$ in T. From here, $A = K + T = K + Im(h) + (K \cap T) = K + Im(h)$ and $Im(h) \cap K \ll_\delta Im(h)$, i.e. $Im(h)$ is a $\delta-$supplement of K in A and $Im(h) \subseteq T$. Finally P is projective δ-cover of $Im(h)$ because $\ker(h) \leq \ker(\mu)$ and $\ker(h) \ll_\delta P$.

(4) $\Rightarrow$ (5) The proof is clear.

(5) $\Rightarrow$ (1) Let K be a co-coatomic submodule of A and T be a δ-supplement of K in A. Then T is a δ-cover of $T/(K \cap T)$. Hence, each projective δ-cover of T is also projective δ-cover of $T/(K \cap T)$. Finally, we say that A/K has a projective δ-cover because $A/K \cong T/(K \cap T)$ and so A is co-coatomically δ-semiperfect. □

Theorem 3.5. *Each homomorphic image of a co-coatomically δ-semiperfect module is co-coatomically δ-semiperfect.*

Proof. Let A be a co-coatomically δ-semiperfect module. We consider a homomorphism $\sigma : A \to K$. Suppose that $\sigma(A)/N$ be a coatomic factor module of $\sigma(A)$. There exists an homomorphism $\mu : A \to \sigma(A)/N$, $\mu(a) = \sigma(a) + N$ for every $a \in A$. Since A is co-coatomically δ-semiperfect, $A/\sigma^{-1}(N) \cong \sigma(a)/N$ that is $\sigma(A)/N$ has a projective δ-cover. As a result $\sigma(A)$ is co-coatomically δ-semiperfect. □

Corollary 3.6. *Each factor module of a co-coatomically $\delta-$semiperfect module is co-coatomically δ-semiperfect.*

Corollary 3.7. *If A is a projective co-coatomically δ-semiperfect module, then each factor module of A is also $\oplus_\delta$-co-coatomically supplemented.*

Proof. The proof follows from Corollary 3.6 and Proposition 3.4. $\square$

Theorem 3.8. *Every $\delta-$cover of a co-coatomically δ-semiperfect module is co-coatomically δ-semiperfect.*

Proof. Suppose that K is a δ-cover of a module A and $\sigma : A \to K$ be an epimorphism with $\ker(\sigma) \ll_\delta A$. For a co-coatomic submodule N of A, the homomorphism $\phi : A/N \to K/\sigma(N)$, defined by $\phi(a+N) = \sigma(a) + \sigma(N)$ is an epimorphism. From here, we say that $K/\sigma(N)$ is an epimorphic image of A/N and $\ker(\phi) = (N + \ker(\sigma))/N$. Let $X/N \leq A/N$ such that $[(N + \ker(\sigma))/N] + X/N = A/N$ and $(A/N)/(X/N)$ is singular. Then $X + \ker(\sigma) = A$ and $A/X \cong (A/N)/(X/N)$ is singular. Since $\ker(\sigma) \ll_\delta A$, $A = X$. It follows that $\ker(\phi) \ll_\delta A/N$. If we consider $K/\sigma(N) = \phi(A/N) \cong (A/N)/(N + \ker(\sigma))/N$, then we say that $K/\sigma(N)$ is coatomic By the assumption, $K/\sigma(N)$ has a projective δ-cover, i.e., $\mu : P \to K/\sigma(N)$. As P is projective, it can be found a homomorphism $h : P \to A/N$ such that the next diagram is commutative

$$\begin{array}{ccc} & & P \\ & \overset{h}{\swarrow} & \downarrow_{\mu} \\ A/N & \overset{\phi}{\longrightarrow} & K/\sigma(N), \end{array}$$

i.e., $\phi \circ h = \mu$. So $A/N = h(P) + \ker(\phi)$. Since $\ker(\phi) \ll_\delta A/N$, there exists a semisimple projective submodule T of $\ker(\phi)$ where $A/N = h(P) + T$. We take a homomorphism $\varphi : P \oplus T \to A/N$, defined by $\varphi(p,t) = h(p) + N$. It is an epimorphism and $\ker(\varphi) = \ker(h) \oplus 0$. Since $\ker(h) \leq \ker(\mu) \ll_\delta P$, then $\ker(h) \oplus 0 \ll_\delta P \oplus T$. Finally, $P \oplus T$ is projective δ-cover of the module A/N. $\square$

Corollary 3.9. *Let $N \ll_\delta A$ and A/N be co-coatomically δ-semiperfect. The module A is co-coatomically δ-semiperfect.*

Corollary 3.10. *If $f : P \to A$ be a projective δ-cover of a module A, then the following conditions are equivalent:*
(1) A is co-coatomically δ-semiperfect.
(2) P is co-coatomically δ-semiperfect.
(3) P is $\oplus_\delta$-co-coatomically supplemented.

Proof. (1)$\Rightarrow$(2) It is clear that by Theorem 3.8.
(2)$\Rightarrow$(1) It is obvious that by Theorem 3.5.
(2)$\Leftrightarrow$(3) It is obvious that by Proposition 3.2. $\square$

Theorem 3.11. *Let A_i be a projective module for every $i \in I$ where I is a finite index set.Then every direct summand A_i is co-coatomically δ-semiperfect if and only if $A = \bigoplus_{i \in I} A_i$ is a co-coatomically δ-semiperfect module.*

Proof. (⇒) Since every A_i is projective and co-coatomically δ-semiperfect, then every A_i is $\oplus_\delta$-co-coatomically supplemented and so A is $\oplus_\delta$-co-coatomically supplemented by Proposition 3.2, Proposition 2.16, respectively. Therefore, A is a co-coatomically δ-semiperfect module by Proposition 3.2.

(⇐) Suppose that $A = \bigoplus_{i\in I} A_i$ be a co-coatomically δ-semiperfect module. With Corollary 3.6, A_i is co-coatomically δ-semiperfect because $A_j \cong A/\left(\bigoplus_{i\in I\setminus\{j\}} A_i\right)$ for every $i \in I$. □

Let A be an R-module. Recall from [22] that an R-module N is called (finitely) *A-generated* if there is an epimorphism $h : A^{(I)} \to N$ for some (finite) index set I.

Lemma 3.12. *Let A be a projective module. If A is δ-semiperfect, then each finitely A-generated module is co-coatomically δ-semiperfect. Moreover, if A is finitely generated, the converse holds.*

Proof. Assume that X be a finitely A-generated module. Since A is a δ-semiperfect projective module, A is co-coatomically δ-semiperfect and so $\oplus_\delta$-co-coatomically supplemented. It follows from Proposition 2.16 that a finite direct sum of A, i.e., for any finite set I, $A^{(I)}$ is $\oplus_\delta$-co-coatomically supplemented. Also by Proposition 3.2, $A^{(I)}$ is co-coatomically δ-semiperfect. Therefore X is co-coatomically δ-semiperfect by Corollary 3.6. Since every finitely generated module is coatomic, the converse is clear. □

Theorem 3.13. *For an arbitrary ring R, the following conditions are equivalent:*

(1) R is δ-semiperfect.

(2) Each finitely generated free R-module is co-coatomically δ-semiperfect.

(3) Each finitely generated free R-module is δ-semiperfect.

Proof. (1)⇒(2) It follows from Proposition 3.2 and Theorem 3.11 that R is co-coatomically δ-semiperfect.

(2)⇒(3) The proof is clear.

(3)⇒(1) By Proposition 3.2. □

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