Honam Mathematical J. **45** (2023), No. 1, pp. 71–81
https://doi.org/10.5831/HMJ.2023.45.1.71

THE HARDY SPACE OF RAMANUJAN-TYPE ENTIRE FUNCTIONS

Erhan Deniz* and Murat Çağlar

Abstract. In this paper, we deal with some geometric properties including starlikeness and convexity of order β of Ramanujan-type entire functions which are natural extensions of classical Ramanujan entire functions. In addition, we determine some conditions on the parameters such that the Ramanujan-type entire functions belong to the Hardy space and to the class of bounded analytic functions.

1. Introduction

Ramanujan introduced a function $A_q(z)$, which is also called Ramanujan function or $q-$Airy function in the literature given by (3) and studied many of its properties in the lost notebooks (see [5]). Indeed the function $A_q(z)$ is also a generalization of the many numerous Rogers–Ramanujan-type identities. Especially $A_q(1)$ and $A_q(q)$ are well known of them. In 2018, Ismail and Zhang [6] defined and studied the function $A_q^{(\alpha)}(z)$ (say: Ramanujan-type entire function), which is a generalization of $A_q(z)$ and the Stieltjes–Wigert polynomial. In the same year, Zhang [16] proved the reality of the zeros of the function $A_q^{(\alpha)}(z)$. In 2020, Deniz [2] determined the radii of starlikeness and convexity of order β and also bounds of them.

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and let $\mathcal{H}$ be the set of all analytic functions in $\mathbb{D}$. Let $\mathcal{A}$ be the class of analytic functions f in $\mathbb{D}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The classes of starlike and convex functions in $\mathbb{D}$ are two important subclasses of $\mathcal{S}$. Analytically, for $\beta \in [0, 1)$ the classes of starlike and convex functions of order β in $\mathbb{D}$ are defined by $\mathcal{S}^*(\beta) := \{f : f \in \mathcal{S} \text{ and } \operatorname{Re}(zf'(z)/f(z)) > \beta\}$ and $\mathcal{C}(\beta) := \{f : f \in \mathcal{S} \text{ and } 1 + \operatorname{Re}(zf''(z)/f'(z)) > \beta\}$, respectively. The familiar classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are known, respectively, as the classes

Received April 18, 2022. Accepted September 30, 2022.
2020 Mathematics Subject Classification. 33C45, 33E99, 30C45.
Key words and phrases. analytic function, Ramanujan-type entire functions, starlike and convex functions, Hardy space.
*Corresponding author

of starlike and convex functions in $\mathbb{D}$. In [1], for $\gamma < 1$, the author introduced the classes

$$\mathcal{P}(\gamma) := \left\{ p \in \mathcal{H} : \exists \eta \in \mathbb{R} \text{ such that } p(0) = 1,\ \operatorname{Re}\left[e^{i\eta} p(z) \right] > \gamma,\ z \in \mathbb{D} \right\}$$

and $\mathcal{R}(\gamma) := \{ g \in \mathcal{A} :\ g' \in \mathcal{P}(\gamma) \}$.

When $\eta = 0$, the classes $\mathcal{P}(\gamma)$ and $\mathcal{R}(\gamma)$ will be denoted by $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$, respectively. Also, for $\gamma = 0$ we denote $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$ simply by $\mathcal{P}$ and $\mathcal{R}$, respectively.

Recall that the Hadamard product (or convolution) of two power series $f(z) = \sum\limits_{n \geq 0} a_n z^n$ and $g(z) = \sum\limits_{n \geq 0} b_n z^n$ is defined as

$$(f * g)(z) = \sum_{n \geq 0} a_n b_n z^n.$$

Let $\mathcal{H}^p$ $(0 < p \leq \infty)$ denote the Hardy space of all analytic functions $f(z)$ in $\mathbb{D}$ and define the integral means $M_p(r, f)$ by

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}} & (0 < p < \infty) \\ \sup\limits_{0 \leq \theta \leq 2\pi} \left| f(re^{i\theta}) \right| & (p = \infty) \end{cases}.$$

An analytic function $f(z)$ in $\mathbb{D}$, is said to belong to the Hardy space $\mathcal{H}^p$ $(0 < p \leq \infty)$, if the set $\{ M_p(r,f) : r \in [0,1) \}$ is bounded. It is important to remind here that $\mathcal{H}^p$ is a Banach space with the norm defined by (see [3, p. 23])

$$\|f\|_p = \lim_{r \to 1^-} M_p(r,f)$$

for $1 \leq p \leq \infty$. On the other hand, we know that $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathbb{D}$, while $\mathcal{H}^2$ is the class of power series $\sum a_n z^n$ such that $\sum |a_n|^2 < \infty$. In addition, it is known from [3] that $\mathcal{H}^s$ is a subset of $\mathcal{H}^p$ for $0 < p \leq s \leq \infty$. Also, two well-known results about the Hardy space $\mathcal{H}^p$ are the following (see [3]):

$$\operatorname{Re} f'(z) > 0 \Rightarrow \begin{cases} f' \in \mathcal{H}^s & (s < 1) \\ f^{\frac{s}{1-s}} \in \mathcal{H}^s & (s \in (0,1)) \end{cases}. \tag{1}$$

2. Preliminaries

In [6] Ismail and Zhang defined and studied the entire function (say: Ramanujan-type entire function)

$$A_q^{(\alpha)}(a;z) = \sum_{n \geq 0} \frac{(a;q)_n q^{\alpha n^2}}{(q;q)_n} z^n \quad (z \in \mathbb{C}), \tag{2}$$

where $\alpha > 0,\ 0 < q < 1,\ a \in \mathbb{C}$ and

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{j=0}^{k-1}(1-aq^j) \quad (k \geq 1).$$

In the special cases of the parameters a and α, we have the following interesting functions

$$A_q^{(\frac{1}{2})}(q^{-n};z) = \sum_{k\geq 0} \frac{(q^{-n};q)_k q^{\frac{k^2}{2}}}{(q;q)_k} z^k = (q;q)_n S_n(zq^{\frac{1}{2}-n};q)$$

and

$$A_q^{(1)}(0;z) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} z^n = A_q(-z), \quad A_q^{(1)}(q;z) = \sum_{n\geq 0} q^{n^2} z^n \tag{3}$$

where $A_q(z)$ and $S_n(z;q)$ are the Ramanujan entire function and the Stieltjes–Wigert polynomial respectively (see [5]). Consequently, $A_q^{(\alpha)}(a;z)$ generalizes both $A_q(z)$ and $S_n(z;q)$. In [16], Zhang proved that $A_q^{(\alpha)}(-a;z)$ has infinitely many negative zeros for $a \geq 0,\ \alpha > 0$ and $0 < q < 1$ by using Pólya frequency sequences. Since the function $A_q^{(\alpha)}(a;z)$ does not belong to $\mathcal{A}$, first we form some natural normalizations. In this paper, we focus on the following normalized form

$$R_{\alpha,q}(a;z) = zA_q^{(\alpha)}(a;z) = z + \sum_{n\geq 2} \frac{(a;q)_{n-1} q^{\alpha(n-1)^2}}{(q;q)_{n-1}} z^n \quad (z \in \mathbb{D}) \tag{4}$$

where $a \in \mathbb{C},\ \alpha > 0,\ 0 < q < 1$. Obviously this function belongs to $\mathcal{A}$. We also say that the function $R_q(z) = zA_q(z)$ is the normalized Ramanujan entire function.

In recent years, the authors in [1, 7, 8, 9, 10, 11, 14, 15] studied the Hardy space of some special functions such as normalized; hypergeometric, Bessel, Struve, Lommel, Wright and Mittag-Leffler. Motivated by the above studies, our main aim is to determine some conditions on the parameters such that the Ramanujan-type entire function $A_q^{(\alpha)}(z)$ is starlike of order β and convex of order β, respectively. Also, we find some conditions for the Hadamard products $R_{\alpha,q}(a;z) * f(z)$ to belong to $\mathcal{R}_0(\gamma)$. Moreover, we investigate the Hardy space of the above mentioned normalized Ramanujan-type entire function $R_{\alpha,q}(a;z)$.

In order to prove the main results we need the following preliminary results.

Lemma 2.1. *(Silverman* [12]*). Let* $f(z) = z + \sum_{n\geq 2} a_n z^n \in \mathcal{A}$. *If*

$$\sum_{n\geq 2}(n-\beta)\,|a_n| \leq 1-\beta,$$

then the function $f(z)$ *is in the class* $\mathcal{S}^*(\beta)$.

Lemma 2.2. *(Silverman [12]). Let* $f(z)=z+\sum_{n\geq 2}a_nz^n\in\mathcal{A}$. *If*

$$\sum_{n\geq 2} n(n-\beta)\left|a_n\right|\leq 1-\beta,$$

then the function $f(z)$ *is in the class* $\mathcal{C}(\beta)$.

Lemma 2.3. *(Eenigenburg and Keogh, [4]). Let* $\beta\in[0,1)$. *If the function* $f\in\mathcal{C}(\beta)$ *is not of the form*

$$\left\{\begin{array}{ll} f(z)=k+lz\left(1-ze^{i\theta}\right)^{2\beta-1} & \left(\beta\neq\frac{1}{2}\right)\\ f(z)=k+l\log\left(1-ze^{i\theta}\right) & \left(\beta=\frac{1}{2}\right)\end{array}\right. \tag{5}$$

for some $k,l\in\mathbb{C}$ *and* $\theta\in\mathbb{R}$, *then the following statements hold:*

a: *There exists* $\delta=\delta(f)>0$ *such that* $f'\in\mathcal{H}^{\delta+\frac{1}{2(1-\beta)}}$.
b: *If* $\beta\in\left[0,\frac{1}{2}\right)$, *then there exists* $\tau=\tau(f)>0$ *such that* $f\in\mathcal{H}^{\tau+\frac{1}{1-2\beta}}$. *Note that this Hardy space is included in* $\mathcal{H}^{\frac{1}{1-2\beta}}$ *by* [3].
c: *If* $\beta\geq\frac{1}{2}$, *then* $f\in\mathcal{H}^{\infty}$.

Lemma 2.4. *(Stankiewich and Stankiewich, [13]).* $\mathcal{P}_0(\lambda)*\mathcal{P}_0(\mu)\subset\mathcal{P}_0(\gamma)$, *where* $\gamma=1-2(1-\lambda)(1-\mu)$. *The value of* γ *is the best possible.*

3. Main Results

In this section, we present our main results related to some geometric properties and Hardy classes of the normalized Ramanujan-type entire function $R_{\alpha,q}(a;z)$. We easily see that

$$\bigcap_{k\geq 1}\left\{a\in\mathbb{C}:\left|1-aq^k\right|\leq\left|1-a\right|,\ q\in(0,1)\right\}=\left\{a\in\mathbb{C}:1\leq\left|a-1\right|\right\}.$$

Therefore, we have $\left|(a;q)_n\right|\leq\left|1-a\right|^n$ for $a\in\mathbb{C}$, $1\leq\left|a-1\right|$ and $n\in\mathbb{N}$.

Theorem 3.1. *Let* $\beta\in[0,1),\ \alpha>0,\ a\in\mathbb{C},\ q\in(0,1)$. *The following assertions are true:*

a: *Suppose that above numbers satisfy* $1\leq\left|a-1\right|<\frac{(1-q)q^{-2\alpha}}{(1+q^{-\alpha})}$ *and the following inequality*

$$\frac{1}{1-\beta}\leq\frac{\left(1-q-q^{2\alpha}\left|a-1\right|\right)\left(1-q-q^{2\alpha}\left|a-1\right|\left(1+q^{-\alpha}\right)\right)}{q^{\alpha}(1-q)\left|a-1\right|}. \tag{6}$$

Then the normalized Ramanujan-type entire function $R_{\alpha,q}(a;z)$ *is starlike of order* β *in* $\mathbb{D}$.

b: *Suppose that above numbers satisfy*

$$\left(1-q-q^{2\alpha}\left|a-1\right|\right)^3 - q^{\alpha}\left|a-1\right|\left(q^{2\alpha}\left|a-1\right|\left(q^{2\alpha}\left|a-1\right|-3(1-q)\right)+2(1-q)^2\right)>0$$

and the following inequality

(7)

$$\frac{1}{1-\beta} \leq \frac{\left(1-q-q^{2\alpha}\left|a-1\right|\right)^{3}-q^{\alpha}\left|a-1\right|\left(q^{2\alpha}\left|a-1\right|\left(q^{2\alpha}\left|a-1\right|-3(1-q)\right)+2(1-q)^{2}\right)}{2q^{\alpha}(1-q)^{2}\left|a-1\right|}.$$

Then the normalized Ramanujan-type entire function $R_{\alpha,q}(a;z)$ is convex of order β in $\mathbb{D}$.

Proof. **a.** By virtue of Silverman's result which is given in Lemma 2.1, in order to prove the starlikeness of order β of the function $R_{\alpha,q}(a;z)$, it is enough to show that the inequality

$$\sum_{n\geq 2}(n-\beta)\left|\frac{(a;q)_{n-1}q^{\alpha(n-1)^{2}}}{(q;q)_{n-1}}\right| \leq 1-\beta \tag{8}$$

holds true under the hypothesis. According to the hypothesis of the theorem, by using the inequalities

$$(q;q)_{n-1} \geq (1-q)^{n-1},\ \left|(a;q)_{n-1}\right| \leq \left|a-1\right|^{n-1} \tag{9}$$

and

$$(n-1)^{2} \geq 2(n-1)-1 \quad (n \geq 2) \tag{10}$$

together with the sums

$$\sum_{n\geq 2} r^{n-1} = \frac{r}{1-r} \quad \text{and} \quad \sum_{n\geq 2} nr^{n-1} = \frac{r(2-r)}{(1-r)^{2}} \tag{11}$$

for $|r| < 1$, we have

$$\begin{aligned}
&\sum_{n\geq 2}(n-\beta)\left|\frac{(a;q)_{n-1}q^{\alpha(n-1)^2}}{(q;q)_{n-1}}\right| \\
\leq\ & \sum_{n\geq 2}(n-\beta)\frac{|(a;q)_{n-1}|\,q^{\alpha(n-1)^2}}{(q;q)_{n-1}} \\
\leq\ & \sum_{n\geq 2}(n-\beta)\frac{|a-1|^{n-1}\,q^{\alpha(2(n-1)-1)}}{(1-q)^{n-1}} \\
=\ & \frac{1}{q^\alpha}\sum_{n\geq 2}(n-\beta)\left(\frac{|a-1|\,q^{2\alpha}}{1-q}\right)^{n-1} \\
=\ & \frac{1}{q^\alpha}\left(\frac{q^{2\alpha}\,|a-1|\left(2(1-q)-q^{2\alpha}\,|a-1|\right)}{\left(1-q-q^{2\alpha}\,|a-1|\right)^2}-\beta\frac{q^{2\alpha}\,|a-1|}{1-q-q^{2\alpha}\,|a-1|}\right) \\
=\ & \frac{q^\alpha\,|a-1|\left((1-\beta)\left(1-q-q^{2\alpha}\,|a-1|\right)+1-q\right)}{\left(1-q-q^{2\alpha}\,|a-1|\right)^2}.
\end{aligned}$$

The inequality (6) implies that the last sum is bounded above by $1-\beta$. Therefore the inequality (8) is satisfied, that is, $R_{\alpha,q}(a;z)$ is starlike of order β in $\mathbb{D}$.

b. Similarly, from the Lemma 2.1 that to prove the convexity of order β of the function $R_{\alpha,q}(a;z)$, it is enough to show that the inequality

$$\sum_{n\geq 2}n(n-\beta)\left|\frac{(a;q)_{n-1}q^{\alpha(n-1)^2}}{(q;q)_{n-1}}\right|\leq 1-\beta \tag{12}$$

is satisfied under our assumptions. Now, if we consider the inequalities (9) and (10) together with the sums (11) and

$$\sum_{n\geq 2}n^2r^{n-1}=\frac{r(r^2-3r+4)}{(1-r)^3}$$

then we can write that

$$\begin{aligned}
&\sum_{n\geq 2} n(n-\beta)\left|\frac{(a;q)_{n-1}q^{\alpha(n-1)^2}}{(q;q)_{n-1}}\right| \\
\leq\ & \sum_{n\geq 2} n(n-\beta)\frac{|1-a|^{n-1}\, q^{\alpha(2(n-1)-1)}}{(1-q)^{n-1}} \\
=\ & \frac{1}{q^{\alpha}}\sum_{n\geq 2} n(n-\beta)\left(\frac{|1-a|\, q^{2\alpha}}{1-q}\right)^{n-1} \\
=\ & \frac{1}{q^{\alpha}}\left(\frac{q^{2\alpha}\,|a-1|\left(q^{2\alpha}\,|a-1|\left(q^{2\alpha}\,|a-1|-3(1-q)\right)+4(1-q)^2\right)}{\left(1-q-q^{2\alpha}\,|a-1|\right)^3}\right) \\
& -\frac{\beta}{q^{\alpha}}\left(\frac{q^{2\alpha}\,|a-1|\left(2(1-q)-q^{2\alpha}\,|a-1|\right)}{\left(1-q-q^{2\alpha}\,|a-1|\right)^2}\right).
\end{aligned}$$

The inequality (7) implies that the last sum is bounded above by $1-\beta$. Therefore the inequality (12) is satisfied, that is, $R_{\alpha,q}(a;z)$ is convex of order β in $\mathbb{D}$. □

Theorem 3.2. *Let $\beta\in[0,1)$, $\alpha>0$, $a\in\mathbb{C}$, $q\in(0,1)$ and $1\leq|a-1|<\frac{1-q}{q^{2\alpha}}$. If the inequality*

$$\beta<1-\frac{q^{\alpha}\,|a-1|}{1-q-q^{2\alpha}\,|a-1|} \tag{13}$$

holds, then $\frac{R_{\alpha,q}(a;z)}{z}\in\mathcal{P}_0(\beta)$.

Proof. In order to prove $\frac{R_{\alpha,q}(a;z)}{z}\in\mathcal{P}_0(\beta)$, it is enough to show that $\mathrm{Re}\left(\frac{R_{\alpha,q}(a;z)}{z}\right)>\beta$. For this purpose, consider the function $p(z)=\frac{1}{1-\beta}\left(\frac{R_{\alpha,q}(a;z)}{z}-\beta\right)$. It can be easily seen that $|p(z)-1|<1$ implies $\mathrm{Re}\left(\frac{R_{\alpha,q}(a;z)}{z}\right)>\beta$. Now, using the inequalities (9), (10) and the well known geometric series sum (11), we have

$$\begin{aligned}
|p(z)-1| &= \left|\frac{1}{1-\beta}\left(1+\sum_{n\geq 2}\frac{(a;q)_{n-1}q^{\alpha(n-1)^2}}{(q;q)_{n-1}}z^{n-1}-\beta\right)-1\right| \\
&\leq \frac{1}{1-\beta}\sum_{n\geq 2}\frac{|(a;q)_{n-1}|\, q^{\alpha(n-1)^2}}{(q;q)_{n-1}} \\
&\leq \frac{1}{(1-\beta)\, q^{\alpha}}\sum_{n\geq 2}\left(\frac{|1-a|\, q^{2\alpha}}{1-q}\right)^{n-1} \\
&= \frac{q^{\alpha}\,|a-1|}{(1-\beta)\left(1-q-q^{2\alpha}\,|a-1|\right)}.
\end{aligned}$$

Consequently, from (13), $\frac{R_{\alpha,q}(a;z)}{z}$ is in the class $\mathcal{P}_0(\beta)$, and the proof is completed. □

Setting $\alpha - 1 = a = 0$ in Theorem 3.1 and Theorem 3.2, we have the following results.

Corollary 3.3. *The following assertions are true:*

a: If the inequality

$$0 \leq \beta \leq 1 - \frac{q(1-q)}{(1-q-q^2)(1-2q-q^2)} < 1$$

holds for $q \in (0, q_0 \approx 0.292) \cup (q_1 \approx 0.712, 1)$, where q_0, q_1 are real roots of the equation $1-4q+q^2+3q^3+q^4 = 0$, then the normalized Ramanujan entire function $R_q(z)$ is starlike of order β in $\mathbb{D}$.

b: If the inequality

$$0 \leq \beta \leq 1 - \frac{2q(1-q)^2}{1-5q+4q^2+6q^3-3q^4-4q^5-q^6} < 1$$

holds for $q \in (0, q_2 \approx 0.185)$, where q_2 are real root of the equation $-1+7q-8q^2-4q^3+3q^4+4q^5+q^6 = 0$, then the normalized Ramanujan entire function $R_q(z)$ is convex of order β in $\mathbb{D}$.

c: If the inequality

$$0 \leq \beta < 1 - \frac{q}{1-q-q^2} < 1$$

holds for $q \in \left(0, \sqrt{2}-1\right)$, then the function $\frac{R_q(z)}{z}$ is in the class $\mathcal{P}_0(\beta)$.

Setting $\beta = 0$ and $\beta = 1/2$ in Corollary 3.3, we have the following interesting results:

$$\begin{aligned}
q &\in q \in (0, q_0 \approx 0.292) \cup (q_1 \approx 0.712, 1) \Rightarrow R_q(z) \in \mathcal{S}^* \\
q &\in (0, 0.228\,62] \cup (0.758845, 1) \Rightarrow R_q(z) \in \mathcal{S}^*\left(\frac{1}{2}\right) \\
q &\in (0, 0.18492] \Rightarrow R_q(z) \in \mathcal{C} \\
q &\in (0, 0.136\,31] \Rightarrow R_q(z) \in \mathcal{C}\left(\frac{1}{2}\right) \\
q &\in (0, 0.414\,21) \Rightarrow \frac{R_q(z)}{z} \in \mathcal{P}_0 \\
q &\in (0, 0.302\,776) \Rightarrow \frac{R_q(z)}{z} \in \mathcal{P}_0\left(\frac{1}{2}\right).
\end{aligned}$$

Theorem 3.4. *Suppose that the assertions of the Theorem 3.1-b are satisfied. Then* $R_{\alpha,q}(a;z) \in \mathcal{H}^{\frac{1}{1-2\beta}}$ *for* $\beta \in \left[0, \frac{1}{2}\right)$ *and* $R_{\alpha,q}(a;z) \in \mathcal{H}^{\infty}$ *for* $\beta \in \left[\frac{1}{2}, 1\right)$.

Proof. By the definitions of the standard binomial expansion and the standard Maclaurin series for the logarithmic function, we have

$$k + lz\left(1 - ze^{i\theta}\right)^{2\beta-1} = k + l\sum_{n\geq 0}\frac{(1-2\beta)_n}{n!}e^{in\theta}z^{n+1} \tag{14}$$

for $k, l \in \mathbb{C}$ and $\theta \in \mathbb{R}$. On the other hand

$$k + l\log\left(1 - ze^{i\theta}\right) = k - l\sum_{n\geq 0}\frac{1}{n+1}e^{in\theta}z^{n+1}. \tag{15}$$

If we consider the series representation of the function $R_{\alpha,q}(a;z)$ which is given by (4), then we see that the function $R_{\alpha,q}(a;z)$ is not of the forms (14) for $\beta \neq \frac{1}{2}$ and (15) for $\beta = \frac{1}{2}$, respectively. On the other hand, part **b.** of Theorem 3.1 states that the function $R_{\alpha,q}(a;z)$ is convex of order β under our hypothesis. Therefore, the proof is completed by applying Lemma 2.3. □

If we take $\alpha - 1 = a = 0$ in Theorem 3.4, we obtain the following result.

Corollary 3.5. *Let $\beta \in [0,1)$ and $q \in (0, q_2 \approx 0.185)$. If the inequality*

$$0 \leq \beta \leq 1 - \frac{2q(1-q)^2}{1-5q+4q^2+6q^3-3q^4-4q^5-q^6} < 1$$

is satisfied, then $R_q(z) \in \mathcal{H}^{\frac{1}{1-2\beta}}$ for $\beta \in \left[0, \frac{1}{2}\right)$ and $R_q(z) \in \mathcal{H}^{\infty}$ for $\beta \in \left[\frac{1}{2}, 1\right)$.

For $\beta = 0$ and $\beta = 1/2$ in Corollary 3.5, we have the following interesting results:

$$\begin{aligned} q &\in (0, 0.184\,92] \Rightarrow R_q(z) \in \mathcal{H} \\ q &\in (0, 0.136\,31] \Rightarrow R_q(z) \in \mathcal{H}^{\infty}. \end{aligned}$$

Theorem 3.6. *Let $\alpha > 0$, $a \in \mathbb{C}$, $\lambda \in [0,1)$, $\mu < 1$, $\gamma = 1 - 2(1-\lambda)(1-\mu)$ and $q \in (0,1)$. Suppose that the function $f(z) \in \mathcal{R}_0(\mu)$. If the inequalities $1 \leq |a-1| < \frac{1-q}{q^{2\alpha}}$ and*

$$\lambda < 1 - \frac{q^{\alpha}|a-1|}{1-q-q^{2\alpha}|a-1|}$$

hold, then $u(z) = R_{\alpha,q}(a;z) * f(z) \in \mathcal{R}_0(\gamma)$.

Proof. If $f(z) \in \mathcal{R}_0(\mu)$, then this implies that $f'(z) \in \mathcal{P}_0(\mu)$. We know from Theorem 3.2 that the function $\frac{R_{\alpha,q}(a;z)}{z} \in \mathcal{P}_0(\lambda)$. Since $u'(z) = \frac{R_{\alpha,q}(a;z)}{z} * f'(z)$, taking into account Lemma 2.4 we may write that $u'(z) \in \mathcal{P}_0(\gamma)$. This implies that $u(z) \in \mathcal{R}_0(\gamma)$. □

References

[1] Á. Baricz, *Bessel transforms and Hardy space of generalized Bessel functions*, Mathematica, **48** (2006), 127–136.

[2] E. Deniz, *Geometric and monotonic properties of Ramanujan type entire functions*, Ramanujan J. **55** (2021), 103–130

[3] P. L. Duren, *Theory of H_p Spaces. A series of Monographs and Textbooks in Pure and Applied Mathematics*, vol. 38, Academic Press, New York and London, 1970.

[4] P. J. Eenigenburg, and F. R. Keogh, *The Hardy class of some univalent functions and their derivatives*, Michigan Math. J. **17** (1970), 335–346.

[5] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge, 2005.

[6] M. E. H. Ismail, and R. Zhang, *$q-$Bessel functions and Rogers-Ramanujan type identities*, Proc. Am. Math. Soc. **146** (2018) 3633–3646.

[7] I. B. Jung, Y. C. Kim, and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176** (1993), 138–147.

[8] Y. C. Kim, and H. M. Srivastava, *Some families of generalized hypergeometric functions associated with the Hardy space of analytic functions*, Proc. Japan Acad., Seri A, **70** (1994), no. 2, 41–46.

[9] S. Ponnusamy, *The Hardy space of hypergeometric functions*, Complex Var. Elliptic Equ. **29** (1996), 83–96.

[10] J. K. Prajapat, S. Maharana, and D. Bansal, *Radius of starlikeness and Hardy space of Mittag-Leffer functions*, Filomat, **32** (2018), no. 18, 6475–6486.

[11] M. Raza, M. U Din, and S. N. Malik, *Certain geometric properties of normalized Wright functions*, J. Func. Spaces, **2016** Article ID 1896154 (2016), 1–8.

[12] H. Silverman, *Univalent functions with negative coefficients*, Proc. Am. Math. Soc. **51** (1975), no. 1, 109–116.

[13] J. Stankiewich, and Z. Stankiewich, *Some applications of Hadamard convolutions in the theory of functions*, Ann. Univ. Mariae Curie-Sklodowska, **40** (1986), 251–265.

[14] N. Yağmur, *Hardy space of Lommel functions*, Bull. Korean Math. Soc. **52** (2015), no. 3, 1035–1046.

[15] N. Yağmur, and H. Orhan, *Hardy space of generalized Struve functions*, Complex Var. Elliptic Equ. **59** (2014), no. 7, 929–936.

[16] R. Zhang, *Zeros of Ramanujan type entire functions*, Proc. Am. Math. Soc. **145** (2017), 241–250.

Erhan Deniz
Department of Mathematics,
Faculty of Science and Letters,
Kafkas University,
Kars 36100, Türkiye.
E-mail: edeniz36@gmail.com

Murat Çağlar
Department of Mathematics,
Faculty of Science,
Erzurum Technical University,

Erzurum 25100, Türkiye.
E-mail: mcaglar25@gmail.com, murat.caglar@erzurum.edu.tr