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RICCI-BOURGUIGNON SOLITONS AND FISCHER-MARSDEN CONJECTURE ON GENERALIZED SASAKIAN-SPACE-FORMS WITH β-KENMOTSU STRUCTURE

Sudhakar Kumar Chaubey and Young Jin Suh

Abstract. Our aim is to study the properties of Fischer-Marsden conjecture and Ricci-Bourguignon solitons within the framework of generalized Sasakian-space-forms with β-Kenmotsu structure. It is proven that a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure satisfying the Fischer-Marsden equation is a conformal gradient soliton. Also, it is shown that a generalized Sasakian-space-form with β-Kenmotsu structure admitting a gradient Ricci-Bourguignon soliton is either $\Psi\backslash T^k \times M^{2n+1-k}$ or gradient η-Yamabe soliton.

1. Introduction

The contact geometry plays a crucial role in science, technology and medical science. It has broad applications in physics, likes quantization, control theory, geometric optics, thermodynamics, classical mechanics and to the integrable systems. Due to its wide applications in different era, it becomes the center of attraction for researchers. In 1958, Boothby and Wang [9] studied the odd dimensional differentiable manifolds endowed with contact and almost contact metric structures from topological point of view, although the same manifolds have been studied by Sasaki using tensor calculus [41]. They called such manifolds as the contact metric manifolds. Since then, many researchers have studied the properties (geometrically and physically) of contact metric manifolds.

Motivated by the different space-forms, likes real space-forms, Sasakian-space-forms, Kenmotsu space-forms and cosymplectic space-forms, Alegre, Blair and Carriazo [1] defined a new space-form, which is the generalization of above said space-forms, called as a generalized Sasakian-space-form. That

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is, a $(2n+1)$-dimensional almost contact metric manifold M with the global contact form η $(\eta \wedge (d\eta)^n \neq 0\ )$, the structure tensor ϕ and the unit vector field ξ satisfying the following curvature identity

$$\begin{aligned}R(U,V)Z = {} & f_1\{g(V,Z)U - g(U,Z)V\} + f_2\{g(U,\phi Z)\phi V \\ & - g(V,\phi Z)\phi U + 2g(U,\phi V)\phi Z\} + f_3\{\eta(U)\eta(Z)V \\ & - \eta(V)\eta(Z)U + \eta(V)g(U,Z)\xi - \eta(U)g(V,Z)\xi\} \end{aligned} \tag{1.1}$$

for all U, V, $Z \in \mathfrak{X}(M)$, is known as a generalized Sasakian-space-form of dimension $(2n+1)$, where g is the Riemannian metric and f_1, f_2 and f_3 are the smooth functions on M. Here R and $\mathfrak{X}(M)$ denote the curvature tensor with respect to the Levi-Civita connection ∇ and the collection of all smooth vector fields of M, respectively. Throughout the paper, we represent $M(f_1, f_2, f_3)$ as a generalized Sasakian-space-form. Particularly, if we choose $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ and $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then the generalized Sasakian-space-form becomes Sasakian-space-form and Kenmotsu-space-form, respectively. It is observed that the smooth function f_2 vanishes on $M(f_1, f_2, f_3)$ if and only if the manifold is conformally flat [35]. The properties of $M(f_1, f_2, f_3)$ in different contexts have been studied by several geometers but few are ([2–4], [12], [13], [22], [27], [44]).

Let M denote a Riemannian manifold of dimension m and g be the associated Riemannian metric of M. If $\mathfrak{g}$ and g^* represent the collection of all Riemannian metrics of unit volume and the $(0,2)$-type symmetric bilinear tensor on M, respectively, then the linearization of scalar curvature $\mathfrak{L}_g g^*$ is given by

$$\mathfrak{L}_g g^* + \triangle_g(tr_g g^*) + g(g^*, S_g) = div(div(g^*)),$$

where $\triangle_g$ and S, respectively, denote the negative Laplacian of Riemannian metric g and non-zero Ricci tensor of M. Here 'tr' and 'div' stand for trace and divergence. Throughout the manuscript, we assume that $S \neq 0$. If we denote the formal L^2-adjoint of linearized scalar curvature operator $\mathfrak{L}_g$ by $\mathfrak{L}_g^*$, then it can be defined as

$$\mathfrak{L}_g^*(\lambda) + \lambda S_g + (\triangle_g \lambda)g = Hess_g \lambda, \tag{1.2}$$

where $Hess_g$ is the Hessian operator corresponding to the Riemannian metric g and it is defined as $Hess_g\lambda(U,V) = g(\nabla_U D\lambda, V)$, $\forall\ U$, $V\ \in \mathfrak{X}(M)$ and λ is a smooth function on M. Here D denotes the gradient operator of g satisfies $g(U, D\lambda) = U(\lambda)$ for all $U \in \mathfrak{X}(M)$. We call the equation $\mathfrak{L}_g^*(\lambda) = 0$ as the Fischer-Marsden equation and the doublet (g, λ) for which $\mathfrak{L}_g^*(\lambda) = 0$ on M is known as the solution of Fischer-Marsden equation. In particular, for $\lambda = 0$ we get the trivial solution of $\mathfrak{L}_g^*(\lambda) = 0$. This manuscript deals with the study of non-trivial solution of $\mathfrak{L}_g^*(\lambda) = 0$. Bourguignon [10] and Fischer and Marsden [29] considered the complete Riemannian manifolds M satisfying equation $\mathfrak{L}_g^*(\lambda) = 0$ and proved that if (g, λ) is the non-trivial solution of $\mathfrak{L}_g^*(\lambda) = 0$, then M possesses the constant scalar curvature of g. Corvino [23] showed that the doublet (g, λ) is the non-trivial solution of $\mathfrak{L}_g^*(\lambda) = 0$ on a compact Riemannian manifold M of dimensional n if and only if the warped product metric $g^* = g - \lambda^2 dt^2$ is Einstein. We also recall the following Fischer-Marsden conjecture [29] as:

"A compact Riemannian manifold that admits a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$ is necessarily an Einstein manifold."

A non-trivial counterexample of the Fischer-Marsden conjecture has been given by Kobayashi in [36]. In [37], Lafontain studied the same conjecture on conformally flat compact Riemannian manifolds. Cernea and Guan [17] proved that if an n-dimensional closed homogeneous Riemannian manifold M satisfies the equation $\mathfrak{L}_g^*(\lambda) = 0$ with non-trivial solution (g, λ), then M is locally isometric to the product space $S^m \times N$, where S^m and N denote the Euclidean sphere and the Einstein manifold, respectively. Shen [42], in 1997, studied the properties of Fischer-Marsden conjecture and showed that a three dimensional closed manifold with the positive constant scalar curvature satisfying the Fischer-Marsden conjecture possesses a totally geodesic 2-sphere. In this series, Patra and Ghosh [39], in 2017, proved that if the K-contact and the (κ, μ)-contact metric manifolds of dimension $(2n+1)$ satisfy the equation $\mathfrak{L}_g^*(\lambda) = 0$ with the non-trivial solution (g, λ), then the manifolds are Einstein and locally isometric to the unit sphere S^{2n+1}, respectively. Prakasha, Veeresha and Venkatesha [40] showed that a $(\kappa, \mu)'$-almost Kenmotsu manifold (M, g) of dimension $(2n+1)$ is locally isometric to the warped product $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$ or $\mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$ if g satisfies the equation $\mathfrak{L}_g^*(\lambda) = 0$. In [19], Chaubey et al. studied the non-trivial solution of Fischer-Marsden equation within the framework of Kenmotsu manifolds.

A Ricci-Bourguignon flow:

$$\frac{\partial}{\partial t} g = -2(S - \rho r g), \quad g(0) = g_0$$

on an m-dimensional Riemannian manifold M was introduced by Bourguignon [11], where S, r, g, t and ρ represent the Ricci tensor, scalar curvature, Riemannian metric, time and real constant, respectively, on M. This family of geometric flow with $\rho = 0$ becomes the Ricci flow ($\frac{\partial}{\partial t} g = -2S,\ g(0) = g_0$), introduced by Hamilton [31].

A Riemannian metric g on M is said to be a Ricci-Bourguignon soliton $(g, \mathcal{W}, \mu, \rho)$ if there exists a vector field $\mathcal{W}$ such that

$$\frac{1}{2}\mathfrak{L}_{\mathcal{W}} g + S = (\mu + \rho r) g, \tag{1.3}$$

where $\mathfrak{L}_{\mathcal{W}} g$ represents the Lie derivative of g with respect to the vector field $\mathcal{W}$ (called as the potential vector field of Ricci-Bourguignon soliton). A Ricci-Bourguignon soliton with $\mu = 0$, $\mu > 0$ or $\mu < 0$ is said to be steady, shrinking or expanding, respectively, on M. Remark that the Ricci-Bourguignon soliton is trivial if the potential vector field $\mathcal{W}$ is Killing. The conformal version of this problem with constant scalar curvature has been studied by Fischer [28]. If we choose $\mathcal{W} = Dh$ for some smooth function h on M, then equation (1.3) takes the form

$$\nabla^2 h + S = (\mu + \rho r)g, \tag{1.4}$$

where $\nabla^2 h = Hessh$ (Hessian of h). The Riemannian metric g satisfies equation (1.4) is known as the gradient Ricci-Bourguignon soliton (Catino and Mazzieri [16] called it as a gradient ρ-Einstein soliton). In [16], authors proved that every gradient ρ-Einstein soliton is rectifiable. In this series, some remarkable results of Ricci-Bourguignon solitons have been studied in [14, 18, 20, 21, 24–26, 32].

As far as our knowledge goes, the non-trivial solutions of Fischer-Marsden equation and the properties of (gradient) Ricci-Bourguignon soliton on generalized Sasakian-space-forms are not studied by the researchers. This manuscript will fill these gaps. The sufficient condition for an $M(f_1, f_2, f_3)$ to be a conformal gradient soliton is proved in the following:

Theorem 1.1. *Let $M(f_1, f_2, f_3)$ be a complete connected generalized Sasakian-space-form with β-Kenmotsu structure. If $M(f_1, f_2, f_3)$ satisfies the Fischer-Marsden equation, then either $M(f_1, f_2, f_3)$ is a generalized Sasakian-space-form with cosymplectic structure or a conformal gradient soliton.*

We also prove the following:

Theorem 1.2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure. If $M(f_1, f_2, f_3)$ admits a gradient Ricci-Bourguignon soliton (g, Dh, μ, ρ), then either $M(f_1, f_2, f_3)$ is $\Psi\backslash T^k \times M^{2n+1-k}$ or gradient η-Yamabe soliton.*

2. Almost contact metric manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold of class C^∞ with a global differentiable 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Then M is said to be an almost contact metric manifold if

$$\eta(\xi) = 1, \quad \phi^2 + I = \eta \otimes \xi \tag{2.1}$$

and

$$g(U, V) = g(\phi U, \phi V) + \eta(U)\eta(V), \quad g(U, \xi) = \eta(U) \tag{2.2}$$

hold for all $U, V \in \mathfrak{X}(M)$, where I denotes the identity transformation, ϕ is the structure tensor of type $(1, 1)$, ξ is the unit vector field of type $(1, 0)$ and g, a compatible Riemannian metric of M [7]. The structure (ϕ, ξ, η, g) on M is known as an almost contact metric structure to M. From (2.1), it can be easily seen that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad rank\,(\phi) = 2n. \tag{2.3}$$

Also, equations (2.1)-(2.3) infer that

$$g(\phi U, V) + g(U, \phi V) = 0$$

for all U, $V \in \mathfrak{X}(M)$. An almost contact metric manifold M with $d\eta(U,V) = g(U,\phi V)$, $\forall\, U$, $V \in \mathfrak{X}(M)$, is known as a contact metric manifold, where d denotes the exterior derivative operator. If M satisfies the expression $[\phi,\phi] = -2d\eta \otimes \eta$, where $[\phi,\phi]$ represents the Nijenhuis tensor of ϕ, then it is called a normal contact metric manifold. A contact metric manifold M is Sasakian if and only if

$$R(U,V)\xi = \eta(V)U - \eta(U)V$$

for all U, $V \in \mathfrak{X}(M)$ ([7], [8]). It is noticed, from (1.1), that $M(f_1,f_2,f_3)$ satisfies

$$S(U,V) = (2nf_1 + 3f_2 - f_3)g(U,V) - (3f_2 + (2n-1)f_3)\eta(U)\eta(V), \tag{2.4}$$

which is equivalent to

$$QU = (2nf_1 + 3f_2 - f_3)U - (3f_2 + (2n-1)f_3)\eta(U)\xi. \tag{2.5}$$

Setting $Z = \xi$ in (1.1) and then using equations (2.1)-(2.3), we find

$$\begin{aligned} R(U,V)\xi &= (f_1 - f_3)\{\eta(V)U - \eta(U)V\}, \\ R(\xi,U)V &= (f_1 - f_3)\{g(U,V)\xi - \eta(V)U\}. \end{aligned} \tag{2.6}$$

It is also observed that $M(f_1,f_2,f_3)$ satisfies the following identities:

$$S(U,\xi) = 2n(f_1 - f_3)\eta(U), \tag{2.7}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \tag{2.8}$$

for all $U \in \mathfrak{X}(M)$.

Let $M(f_1,f_2,f_3)$ be a generalized Sasakian-space-form of dimension $(2n+1)$ with β-Kenmotsu structure. Then we have

$$\nabla_U\xi = \beta(U - \eta(U)\xi) \iff (\nabla_U\eta)(V) = \beta(g(U,V) - \eta(U)\eta(V)), \tag{2.9}$$

which gives $(\nabla_\xi\eta)(U) = (\nabla_U\eta)(\xi) = 0$. Particularily if we choose $\beta = 0$ or $\beta = 1$ on $M(f_1,f_2,f_3)$, then $M(f_1,f_2,f_3)$ reduces to the generalized Sasakian-space-form with cosymplectic and Kenmotsu structures, respectively. In [2], Alegre and Carriazo proved that a generalized Sasakian-space-form with β-Kenmotsu structure satisfies the relation $f_1 - f_3 + \xi(\beta) + \beta^2 = 0$. Throughout the manuscript, we suppose that β is constant on $M(f_1,f_2,f_3)$.

An almost contact metric manifold M is said to be $\mathfrak{T}$-semisymmetric if $R(U,V)\cdot\mathfrak{T} = 0$ for all U, $V \in \mathfrak{X}(M)$, where $\mathfrak{T}$ is an arbitrary tensor of type (p,q) and $R(U,V)$ represents the derivation of tensor algebra at each point of M for tangent vectors U and V. In particular, if we replace $\mathfrak{T}$ by R (resp., S and C), then we obtain the semisymmetric (resp., Ricci semisymmetric and Weyl semisymmetric) almost contact metric manifold, where C denotes the Weyl conformal curvature tensor.

3. Fischer-Marsden conjecture on generalized-Sasakian-space-forms with β-Kenmotsu structure

This section deals with the main results of $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfying the equation $\mathfrak{L}_g^*(\lambda) = 0$. To achieve our goal, first we prove the following proposition.

Proposition 3.1. *An $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfies*

(i) $(\nabla_\xi Q)(V) - (\nabla_V Q)(\xi) = [\xi((2n-1)f_3 + 3f_2) + \beta(3f_2 + (2n-1)f_3)](V - \eta(V)\xi)$,

(ii) $dr(\xi) = -n\beta[(2n-1)f_3 + 3f_2]$

for all $V \in \mathfrak{X}(M)$.

Proof. From (2.7), it is obvious that $Q\xi = 2n(f_1 - f_3)\xi$. Differentiating it covariantly with respect to ∇ along the vector field V and then using equation (2.9), we find

$$\begin{aligned}(\nabla_V Q)(\xi) &= 2n\beta(f_1 - f_3)\eta(V)\xi - \beta QV + 2nV(f_1 - f_3)\xi \\ &\quad + 2n(f_1 - f_3)\beta V - 2n(f_1 - f_3)\beta\eta(V)\xi.\end{aligned} \tag{3.1}$$

Again, taking the covariant derivative of equation (2.5) with respect to ∇ along the vector field ξ and then using equations (2.2) and (2.9), we get

$$(\nabla_\xi Q)(V) = \xi(2nf_1 + 3f_2 - f_3)V - \xi(3f_2 + (2n-1)f_3)\eta(V)\xi. \tag{3.2}$$

Since β is constant, by hypothesis, and therefore $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfies the identity $f_1 - f_3 = -\beta^2$ (see [2], Proposition 4.3). Thus,

$$\begin{aligned}U(2nf_1 + 3f_2 - f_3) &= U(3f_2 + (2n-1)f_3) + 2nU(f_1 - f_3) \\ &= U(3f_2 + (2n-1)f_3),\ \forall\, U \in \mathfrak{X}(M).\end{aligned} \tag{3.3}$$

The first result of Proposition 3.1 follows from equations (2.5), (3.1), (3.2) and (3.3). Contracting equation (3.1) along the vector field V and using equations (2.8) and (3.3), we get the second part of Proposition 3.1. □

Proof of Theorem 1.1. Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form with β-Kenmotsu structure. Suppose $M(f_1, f_2, f_3)$ satisfies the Fischer-Marsden equation, that is, the metric g satisfies equation $\mathfrak{L}_g^*(\lambda) = 0$. With this fact, equation (1.2) takes the form

$$(\triangle_g\lambda)g + \lambda S_g - Hess_g\lambda = 0, \tag{3.4}$$

which leads $\triangle_g\lambda = -\frac{r\lambda}{2n}$. From equation (3.4), we conclude that

$$\nabla_U D\lambda = \lambda QU + fU \tag{3.5}$$

for all vector field U on M, where $f = -\frac{r\lambda}{2n}$. Taking covariant derivative of equation (3.5) with respect to ∇ along the vector field V, and then the foregoing equation, (3.5) and identity $R(U,V)D\lambda = \nabla_U\nabla_V D\lambda - \nabla_V\nabla_U D\lambda - \nabla_{[U,V]}D\lambda$ give

$$R(U,V)D\lambda = U(\lambda)QV - V(\lambda)QU + U(f)V - V(f)U$$

$$+\lambda\{(\nabla_U Q)(V)-(\nabla_V Q)(U)\}. \tag{3.6}$$

Replacing U by ξ in equation (3.6) and then taking inner product of the foregoing equation with U, we get

$$\begin{aligned} & g(R(\xi,V)D\lambda,U) \\ &= \xi(\lambda)S(V,U)-V(\lambda)S(\xi,U)+\xi(f)g(V,U)-V(f)\eta(U) \\ &+\lambda[\xi((2n-1)f_3+3f_2)+\beta((2n-1)f_3+3f_2)]\{g(U,V)-\eta(U)\eta(V)\}. \end{aligned} \tag{3.7}$$

From equation (2.6), we conclude that

$$g(R(\xi,V)U,D\lambda)=(f_1-f_3)\{\xi(\lambda)g(V,U)-V(\lambda)\eta(U)\}, \tag{3.8}$$

where $V(\lambda)=g(V,D\lambda)$. In consequence of equations (2.7), (3.7) and (3.8), we lead

$$\begin{aligned} & \lambda[\xi((2n-1)f_3+3f_2)+\beta((2n-1)f_3+3f_2)]\{g(U,V)-\eta(U)\eta(V)\} \\ & +\xi(\lambda)S(V,U)-2n(f_1-f_3)V(\lambda)\eta(U)+\xi(f)g(V,U)-V(f)\eta(U) \\ &= (f_1-f_3)\{V(\lambda)\eta(U)-\xi(\lambda)g(V,U)\}. \end{aligned} \tag{3.9}$$

Setting $U=V=e_i$ in equation (3.9), where $\{e_i,\ i=1,2,\ldots,(2n+1)\}$ denotes a set of orthonormal vector fields of the tangent space of $M(f_1,f_2,f_3)$, and then taking summation over i, $1\le i\le(2n+1)$, we get

$$r\xi(\lambda)+2n\xi(f)+2n\lambda\,[\xi((2n-1)f_3+3f_2)+\beta((2n-1)f_3+3f_2)]=0. \tag{3.10}$$

It is obvious that $2nf=-r\lambda$. Differentiating this equation covariantly with respect to ∇ along the vector field ξ, and then following Proposition 3.1(ii), we conclude that

$$2n\xi(f)=-r\xi(\lambda)-\lambda\xi(r). \tag{3.11}$$

From equations (3.10) and (3.11), we infer

$$-\lambda\xi(r)+2n\lambda\,[\xi((2n-1)f_3+3f_2)+\beta((2n-1)f_3+3f_2)]=0,$$

which becomes

$$-\xi(r)+2n\,[\xi((2n-1)f_3+3f_2)+\beta((2n-1)f_3+3f_2)]=0, \tag{3.12}$$

since $\lambda\neq0$. Now,

$$3f_2+(2n-1)f_3=\frac{r}{2n}-(2n+1)(f_1-f_3), \tag{3.13}$$

and

$$3f_2+(2n-1)f_3=2nf_1+3f_2-f_3-2n(f_1-f_3),$$

where equation (2.8) is used. From equations (3.12), (3.13) and Proposition 3.1(ii) we infer that $\xi(r)=2n\xi(3f_2+(2n-1)f_3)$. Using this relation in (3.12), we find

$$\beta(3f_2+(2n-1)f_3)=0. \tag{3.14}$$

This infers that either $3f_2+(2n-1)f_3=0$ or $\beta=0$. If $\beta=0$, then $M(f_1,f_2,f_3)$ becomes a generalized Sasakian-space-form with cosymplectic structure. On the other hand, we consider that $3f_2 + (2n-1)f_3 = 0$ and $\beta \neq 0$. Thus, from (3.13) we immediately get $\xi(r) = 0$, that is, r is locally constant along the Reeb vector field ξ. Using $3f_2 + (2n-1)f_3 = 0$ in (3.13), we notice that

$$r = 2n(2n+1)(f_1 - f_3), \tag{3.15}$$

which shows that the scalar curvature of $M(f_1, f_2, f_3)$ is constant. Equations (3.9), (3.13), (3.14) and (3.15) give

$$\begin{aligned} &\xi(\lambda)S(V,U) - 2n(f_1 - f_3)V(\lambda)\eta(U) + \xi(f)g(V,U) - V(f)\eta(U) \\ &= (f_1 - f_3)\{V(\lambda)\eta(U) - \xi(\lambda)g(V,U)\}. \end{aligned}$$

Again from (3.11) and (3.15), we conclude that $2nV(f) = -rV(\lambda) \Longrightarrow 2n\xi(\lambda) = -r\xi(\lambda)$. Using these relations and equation (3.15) in the above equation, we find

$$\xi(\lambda)\{S(V,U) - 2n(f_1 - f_3)g(V,U)\} = 0.$$

This infers that either $\xi(\lambda) = 0$ or $S = 2n(f_1 - f_3)$. If possible, we suppose that $\xi(\lambda) = 0$ and $S \neq 2n(f_1 - f_3)g$. The covariant derivative of $\xi(\lambda) = 0 \Longleftrightarrow g(\xi, D\lambda) = 0$ along the vector field U gives

$$g(\nabla_U \xi, D\lambda) + g(\xi, \nabla_U D\lambda) = 0,$$

which in view of equations (2.9) and (3.5) becomes

$$\beta U(\lambda) + \lambda S(U, \xi) + f\eta(U) = 0, \tag{3.16}$$

since $\xi(\lambda) = 0$. Next using $f = -\frac{r}{2n}\lambda$, equations (2.7) and (3.15) in (3.16), we find

$$\beta U(\lambda) - \lambda(f_1 - f_3)\eta(U) = 0,$$

which gives $f_1 = f_3$. Thus from (3.15) we have $f =$ constant. This fact with equation (3.4) infers that $\lambda = 0$ because $S \neq 0$. Since we are interested in the non-trivial solution of Fischer-Marsden equation, therefore the hypothesis $\xi(\lambda) = 0$ is inadmissible. Thus $S = 2n(f_1 - f_3)g$. That is the manifold under consideration is Einstein. Now, from (3.5) we have

$$\nabla_U D\lambda = -\lambda(f_1 - f_3)U, \tag{3.17}$$

which reduces to

$$\triangle\lambda = -\lambda(f_1 - f_3)(2n+1). \tag{3.18}$$

From equation (3.17), we conclude that

$$\nabla\nabla\lambda = \beta^2\lambda g, \tag{3.19}$$

where $\beta^2 = f_3 - f_1 \neq 0$.

A connected, complete Riemannian manifold M of dimension m is said to be a conformal gradient soliton if $\nabla^2\mathfrak{f} = \varepsilon g$ for some nonconstant smooth function $\mathfrak{f}$, also called potential function of the conformal gradient soliton [15]. Here $\varepsilon : M \to \mathbb{R}$ is a function.

This definition together with equation (3.19) prove the statement of Theorem 1.1. □

From Theorem 1.1, we conclude the following:

Corollary 3.2. *A generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfying the Fischer-Marsden equation is Einstein.*

Corollary 3.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfying Fischer-Marsden equation possesses the constant scalar curvature.*

Corollary 3.4. *If a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfies the Fischer-Marsden equation, then $3f_2 + (2n-1)f_3 = 0$.*

Remark 3.5. In [38], Obata proved that a complete and connected Riemannian manifold M of $\dim M > 1$ admits a non-trivial solution ϱ of the system of partial differential equation

$$\nabla\nabla\varrho = -\mathfrak{c}\varrho g \tag{3.20}$$

if and only if M is isometric to a sphere of radius $\frac{1}{\sqrt{\mathfrak{c}}}$, $\mathfrak{c} > 0$.

The characterization of equation (3.20) with $\mathfrak{c} \leq 0$ has been given in [33]. Kanai [33] proved that if a complete connected Riemannian manifold M of dimension n satisfies equation (3.20), $\mathfrak{c} < 0$, if and only if M is the warped product $\bar{M} \times_{\mathfrak{h}} \mathbb{R}$ of a complete Riemannian manifold $\bar{M}$ and the real line $\mathbb{R}$, warped by a function $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\mathfrak{h}} + \mathfrak{c}\mathfrak{h} = 0$, $\mathfrak{h} > 0$ (see [33], Corollary E). This fact with equation (3.19) state that a $(2n+1)$-dimensional complete connected generalized Sasakian-space-form with β-Kenmotsu structure satisfying the Fischer-Marsden equation is a warped product $\bar{M} \times_{\mathfrak{h}} \mathbb{R}$ of a complete Kähler manifold $\bar{M}$ of dimension $2n$ and the real line $\mathbb{R}$, where the warping function $\mathfrak{h}$ satisfies the equation $\ddot{\mathfrak{h}} - \beta^2\mathfrak{h} = 0$.

Let $N(c)$ be a complex space form with constant holomorphic sectional curvature c. We know that a complete and simply connected complex space form consists of a complex projective space (CP^n), a complex Euclidean space (C^n) or a complex hyperbolic space (CH^n) according as $c > 0$, $c = 0$, or $c < 0$, respectively. For instance, see [5, 34, 43]. Alegre, Blair and Carriazo [1] proved that the warped product $M = N(c) \times_\gamma \mathbb{R}$, $\gamma = \gamma(t)(\neq 0)$, endowed with an almost contact metric structure $(\phi, \xi, \eta, g_\gamma)$ is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure, where $\beta = \frac{\dot{\gamma}}{\gamma}$ and the smooth functions f_1, f_2, f_3 are given by

$$f_1 = \frac{c - 4(\dot{\gamma})^2}{4\gamma^2}, \quad f_2 = \frac{c}{4\gamma^2}, \quad f_3 = \frac{c - 4(\dot{\gamma})^2}{4\gamma^2} + \frac{\ddot{\gamma}}{\gamma}. \tag{3.21}$$

Here $\dot{\gamma}$ and $\ddot{\gamma}$ are, respectively, the first and second derivatives of γ with respect to t and $\mathbb{R}$ denotes the real line.

Let $M(f_1, f_2, f_3) = N(c) \times_\gamma \mathbb{R}$ satisfy the Fischer-Marsden equation. Then from Theorem 1.1, we conclude that $f_1 - f_3 = -\beta^2$ and $3f_2 + (2n-1)f_3 = 0$. Now, equation (3.21) gives $\ddot{\gamma} = \beta^2\gamma$ and

$$3f_2 + (2n-1)f_3 = \frac{(n+1)c}{2\gamma^2}.$$

This shows that $3f_2 + (2n-1)f_3 = 0$ if and only if $c = 0$, that is, $N(c)$ is a complex Euclidean space. Thus, we can state:

Proposition 3.6. *A complete connected generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfying the Fischer-Marsden equation is a warped product $C^n \times_\gamma \mathbb{R}$ of a complete complex Euclidean space C^n and the real line $\mathbb{R}$, warped by a function $\gamma : \mathbb{R} \to \mathbb{R}$.*

Suppose that the generalized Sasakian-space-form with β-Kenmotsu structure admits the Fischer-Marsden conjecture. Then from Theorem 1.1, we have $3f_2 + (2n-1)f_3 = 0$, and consequently equation (2.4) takes the form

$$S = 2n(f_1 - f_3)g.$$

It is well-known that

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U, R(X,Y)V),\ \forall\ X,Y,U,V \in \mathfrak{X}(M).$$

The last two equations infer that $R \cdot S = 0$. That is, the manifold under consideration is Ricci semisymmetric. Thus, we can state:

Corollary 3.7. *If the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure admits the Fischer-Marsden conjecture, then it is Ricci semisymmetric.*

Remark 3.8. By the straight forward calculations (as like Corollary 3.7), we can easily show that if $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfies the Fischer-Marsden equation, then M is Ricci concircular semisymmetric ($\hat{C}(U,V)\cdot S = 0$), Ricci projective semisymmetric ($P(U,V) \cdot S = 0$) and Ricci conformal semisymmetric ($C(U,V) \cdot S = 0$), where $\hat{C}$, P and C denote the concircular, projective and conformal curvature tensors, respectively.

4. Ricci-Bourguignon solitons on generalized Sasakian-space-forms with β-Kenmotsu structure

Let $M(f_1, f_2, f_3)$ admit a Ricci-Bourguignon soliton $(g, \mathcal{W}, \mu, \rho)$. Then from equation (1.3) we have

$$\frac{1}{2}[g(\nabla_U \mathcal{W}, V) + g(U, \nabla_V \mathcal{W})] + S(U,V) = (\mu + \rho r)g(U,V),$$

where S is the non-vanishing Ricci tensor of M. Let the potential vector field $\mathcal{W}$ of Ricci-Bourguignon soliton be pointwise collinear with the Reeb vector field ξ, that is, $\mathcal{W} = \mathfrak{a}\xi$ for some smooth function $\mathfrak{a}$. Then we have

$$\nabla_U \mathcal{W} = U(\mathfrak{a})\xi + \mathfrak{a}\nabla_U \xi = U(\mathfrak{a})\xi + \mathfrak{a}\beta(U - \eta(U)\xi),$$

where equation (2.9) is used. The last two equations together with equation (2.4) give

$$\begin{aligned} &2(\mu+\rho r)g(U,V) \\ &= U(\mathfrak{a})\eta(V)+V(\mathfrak{a})\eta(U)+2\mathfrak{a}\beta\{g(U,V)-\eta(U)\eta(V)\} \\ &\quad +2\{(2nf_1+3f_2-f_3)g(U,V)-(3f_2+(2n-1)f_3)\eta(U)\eta(V)\}. \end{aligned} \tag{4.1}$$

Substitute ξ in lieu of V in the above equation, we find

$$U(\mathfrak{a})=\{2(\mu+\rho r)-\xi(\mathfrak{a})-4n(f_1-f_3)\}\eta(U), \tag{4.2}$$

where equations (2.1) and (2.2) are used. Again, setting $U=\xi$ in the above equation we obtain

$$\mu+\rho r=\xi(\mathfrak{a})+2n(f_1-f_3). \tag{4.3}$$

Using equation (4.3) in equation (4.2), we conclude that

$$D\mathfrak{a}=\xi(\mathfrak{a})\xi. \tag{4.4}$$

In consequence of equations (4.3) and (4.4), equation (4.1) assumes the following form

$$\begin{aligned} &\xi(\mathfrak{a})\eta\otimes\eta+\mathfrak{a}\beta\{g-\eta\otimes\eta\}+(2nf_1+3f_2-f_3)g-(3f_2+(2n-1)f_3)\eta\otimes\eta \\ &=\{\xi(\mathfrak{a})+2n(f_1-f_3)\}g. \end{aligned}$$

Let $\{e_i : i=1,2,\dots,(2n+1)\}$ denote an orthonormal basis of $M(f_1,f_2,f_3)$. Then contraction of the above equation gives

$$\xi(\mathfrak{a})=\mathfrak{a}\beta+3f_2+(2n-1)f_3. \tag{4.5}$$

By considering above facts, we can state:

Proposition 4.1. *Let a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure admit a Ricci-Bourguignon soliton $(g,\mathcal{W},\mu,\rho)$. If the potential vector field $\mathcal{W}$ of $(g,\mathcal{W},\mu,\rho)$ is pointwise collinear with the Reeb vector field ξ, that is, $\mathcal{W}=\mathfrak{a}\xi$ for some smooth function $\mathfrak{a}$, then the gradient of '$\mathfrak{a}$' is pointwise collinear with the Reeb vector field ξ.*

Particularly, we take $\xi=\frac{\partial}{\partial t}$ then (4.5) reduces to

$$\frac{\partial\mathfrak{a}}{\partial t}-\mathfrak{a}\beta=3f_2+(2n-1)f_3. \tag{4.6}$$

The straightforward calculations show that $\mathfrak{a}=C_1e^{\beta t}+e^{\beta t}\int e^{-\beta t}(3f_2+(2n-1)f_3)dt$ is a solution of the partial differential equation (4.6). Here C_1 is a smooth function, which is independent of t. Thus we conclude our finding as:

Corollary 4.2. *Let the metric of a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure be a Ricci-Bourguignon soliton $(g,\mathcal{W},\mu,\rho)$. If the potential vector field $\mathcal{W}$ of $(g,\mathcal{W},\mu,\rho)$ is pointwise collinear with the Reeb vector field ξ, then $\mathfrak{a}$ satisfies PDE (4.6) and determined by $\mathfrak{a}=C_1e^{\beta t}+e^{\beta t}\int e^{-\beta t}(3f_2+(2n-1)f_3)dt$.*

Proof of Theorem 1.2. Let $M(f_1, f_2, f_3)$ admit a gradient Ricci-Bourguignon soliton. Then from equation (1.4), we have

$$\nabla_U Dh + QU = (\mu + \rho r)U, \ \forall\, U \in \mathfrak{X}(M). \tag{4.7}$$

Differentiating equation (4.7) covariantly with respect to the vector field V, we find

$$\nabla_V \nabla_U Dh = -(\nabla_V Q)(U) - Q(\nabla_V U) + \rho V(r)U + (\mu + \rho r)\nabla_V U. \tag{4.8}$$

Using equations (4.7) and (4.8) in the curvature identity

$$R(U, V)Dh = \nabla_U \nabla_V Dh - \nabla_V \nabla_U Dh - \nabla_{[U,V]} Dh,$$

we lead to

$$R(U, V)Dh = (\nabla_V Q)(U) - (\nabla_U Q)(V) + \rho\{U(r)V - V(r)U\}. \tag{4.9}$$

From equation (2.7), we have $Q\xi = 2n(f_1 - f_3)\xi$. The covariant derivative of this equation along the vector field U gives

$$(\nabla_U Q)(\xi) = -\beta QU + 2n\beta(f_1 - f_3)U,$$

where equation (2.9) has been used. From equations (2.5) and (2.9), it is clear that

$$\begin{aligned}(\nabla_U Q)(V) &= U(3f_2 + (2n-1)f_3)\{V - \eta(V)\xi\} \\ &\quad - \beta(3f_2 + (2n-1)f_3)[g(U, V)\xi + \eta(V)U - 2\eta(U)\eta(V)\xi],\end{aligned} \tag{4.10}$$

since $f_1 - f_3 = -\beta^2$. Interchanging U and V in equation (4.10) and then subtracting foregoing equation with (4.10), we infer

$$\begin{aligned}(\nabla_U Q)(V) - (\nabla_V Q)(U) &= U(3f_2 + (2n-1)f_3)\{V - \eta(V)\xi\} \\ &\quad - V(3f_2 + (2n-1)f_3)\{U - \eta(U)\xi\} \\ &\quad - \beta(3f_2 + (2n-1)f_3)[\eta(V)U - \eta(U)V].\end{aligned} \tag{4.11}$$

Contracting equation (4.9) over U, we find

$$S(V, Dh) = \left[\frac{1}{2} - 2n\rho\right] V(r).$$

Also, from (2.8) we get

$$V(r) = 2nV(3f_2 + (2n-1)f_3), \tag{4.12}$$

since $f_1 - f_3$ is constant. The last two equations give

$$S(V, Dh) = n(1 - 4n\rho)V(3f_2 + (2n-1)f_3). \tag{4.13}$$

We have from (2.4)

$$S(V, Dh) = (2nf_1 + 3f_2 - f_3)V(h) - (3f_2 + (2n-1)f_3)\xi(h)\eta(V). \tag{4.14}$$

Comparing right hand sides of (4.13) and (4.14), we find

$$\begin{aligned}&n(1 - 4n\rho)V(3f_2 + (2n-1)f_3) \\ &= (2nf_1 + 3f_2 - f_3)V(h) - (3f_2 + (2n-1)f_3)\xi(h)\eta(V).\end{aligned} \tag{4.15}$$

Contracting (4.11) over U, we find

$$V(r) = 2(2n-1)V(3f_2 + (2n-1)f_3) + 2\xi(3f_2 + (2n-1)f_3)\eta(V) \\ + 4n\beta(3f_2 + (2n-1)f_3)\eta(V). \tag{4.16}$$

In view of (4.12) and (4.16), we obtain

$$(n-1)V(3f_2 + (2n-1)f_3) \\ = -\{\xi(3f_2 + (2n-1)f_3) + 2n\beta(3f_2 + (2n-1)f_3)\}\eta(V). \tag{4.17}$$

Replacing V with ξ in the above equation, we lead to $\xi(3f_2 + (2n-1)f_3) = -2\beta(3f_2 + (2n-1)f_3)$. Using this result in (4.17), we find

$$V(3f_2 + (2n-1)f_3) = -2\beta(3f_2 + (2n-1)f_3)\eta(V). \tag{4.18}$$

In consequence of (4.18), equation (4.15) reduces to

$$n(4n\rho - 1)2\beta(3f_2 + (2n-1)f_3)\eta(V) \\ = 2n(f_1 - f_3)V(h) + (3f_2 + (2n-1)f_3)V(h) \\ - (3f_2 + (2n-1)f_3)\xi(h)\eta(V). \tag{4.19}$$

Setting $V = \xi$ in the above equation, we find

$$\beta(4n\rho - 1)(3f_2 + (2n-1)f_3) = (f_1 - f_3)\xi(h). \tag{4.20}$$

Equations (4.19) and (4.20) infer that

$$[V(h) - \xi(h)\eta(V)]\{2n(f_1 - f_3) + 3f_2 + (2n-1)f_3\} = 0.$$

This shows that either $3f_2 + (2n-1)f_3 = 2n\beta^2$ or $V(h) = \xi(h)\eta(V)$. Now we divide our study in two cases as:

Case I. We suppose that $3f_2 + (2n-1)f_3 = 2n\beta^2 =$ constant and $V(h) \neq \xi(h)\eta(V)$. This fact together with (4.18) conclude that $3f_2 + (2n-1)f_3 = 0$ and $\beta = 0$. These relations and equation (2.4) reveal that the manifold $M(f_1, f_2, f_3)$ is Ricci flat. In [30], Fischer and Wolf have studied the properties of compact Ricci-flat Riemannian manifolds and established several interesting results. They proved that a compact connected Ricci-flat m-manifold M^m has the expression $M^m = \Psi\backslash T^k \times M^{m-k}$, where k is the first Betti number $b_1(M^m)$, T^k is a flat Riemannian k-torus, M^{m-k} is a compact connected Ricci-flat $(m-k)$-manifold, and Ψ is a finite group of fixed point free isometries of $T^k \times M^{m-k}$ of a certain sort (see Theorem 4.1, [30]).

Since $\beta = 0$. Thus $M(f_1, f_2, f_3)$ becomes a generalized Sasakian-space-form with cosymplectic structure and it can be expressed as:

$$M = \Psi\backslash T^k \times M^{2n+1-k},$$

provided $V(h) \neq \xi(h)\eta(V)$.

Case II. We consider that $2nf_1 + 3f_2 - f_3 \neq 0$ and $V(h) = \xi(h)\eta(V) \iff dh = \xi(h) \wedge \eta$. The exterior derivative of this expression gives $d^2h = d(\xi(h)) \wedge \eta = 0$, since $d^2 = 0$ and η is closed. This infers that $\xi(h)$ is constant. The covariant derivative of $Dh = \xi(h)\xi$ along the vector field U gives

$$\begin{aligned}\nabla_U Dh &= U(\xi(h))\xi + \xi(h)\beta(U - \eta(U)\xi)\\ &= \xi(h)\beta(U - \eta(U)\xi),\end{aligned}$$

since $\xi(h)$ is constant. This equation takes the form

$$Hess(h) = \xi(h)\beta(g - \eta \otimes \eta). \tag{4.21}$$

A Riemannian manifold M of dimension m is said to be a gradient generalized soliton [6] if there exist $\aleph_1$, $\aleph_2$, $\aleph_3 \in \mathbb{R}$ such that

$$\nabla D\mathfrak{p} + \aleph_1 Q = \aleph_2 I + \aleph_3 \eta \otimes \xi,$$

where Q denotes the Ricci operator, I is the identity map, $\mathfrak{p}$ is a smooth function on M and η is a 1-form associated to ξ, that is, $g(U, \xi) = \eta(U)$ for all U. Particularly, if we choose $\aleph_1 = 0$ in the above equation then the gradient generalized soliton reduces to the gradient η-Yamabe soliton. From last two equations, we conclude that $M(f_1, f_2, f_3)$ with β-Kenmotsu structure is a gradient η-Yamabe soliton. This completes the proof of Theorem 1.2. □

We suppose that $M(f_1, f_2, f_3)$ admits a gradient Ricci-Bourguignon soliton and $Dh \neq \xi(h)\xi$. Then from Theorem 1.2 we have $3f_2 + (2n-1)f_3 = 0$, and $\beta = 0$. Now, from (2.9), we notice that $\nabla\xi = 0$ and hence $M(f_1, f_2, f_3)$ reduces to a generalized Sasakian-space-form with cosymplectic structure. We have the following corollary as:

Corollary 4.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure satisfying a gradient Ricci-Bourguignon soliton is a generalized Sasakian-space-form with cosymplectic structure, provided $Dh \neq \xi(h)\xi$.*

Again, we consider that an $M(f_1, f_2, f_3)$ with Kenmotsu structure admits a Ricci-Bourguignon soliton and $Dh \neq \xi(h)\xi$. Then we have $r = 0$, that is the scalar curvature of $M(f_1, f_2, f_3)$ is vanishes identically. In view of this fact, equation (1.4) reduces to

$$\nabla^2 h = \mu g, \tag{4.22}$$

which is the conformal gradient soliton [15]. Now, we state:

Corollary 4.4. *A gradient Ricci-Bourguignon soliton on $M(f_1, f_2, f_3)$ with β-Kenmotsu structure is a conformal gradient soliton, provided $Dh \neq \xi(h)\xi$.*

Let $M(f_1, f_2, f_3)$ with β-Kenmotsu structure admit a gradient Ricci-Bourguignon soliton and $Dh \neq \xi(h)\xi$. Then we have $2nf_1 + 3f_2 - f_3 = 0$. That is the smooth functions f_1, f_2 and f_3 are linearly dependent. Thus we state:

Corollary 4.5. *Let $M(f_1, f_2, f_3)$ with β-Kenmotsu structure admit a gradient Ricci-Bourguignon soliton. Then the smooth functions f_1, f_2 and f_3 are linearly dependent, provided $Dh \neq \xi(h)\xi$.*

Let $M(f_1, f_2, f_3)$ with β-Kenmotsu structure, $\beta \neq 0$, admit a gradient Ricci-Bourguignon soliton (g, Dh, μ, ρ). Then from Theorem 1.2 we have $Dh = \xi(h)\xi$. Now, taking covariant derivative of $Dh = \xi(h)\xi$ along U we find

$$\nabla_U Dh = g(\nabla_U \xi, Dh)\xi + g(\xi, \nabla_U Dh)\xi + g(\xi, Dh)\nabla_U \xi.$$

In consequence of equations (2.7), (2.9), (4.7) and $U(h) = \xi(h)\eta(U)\xi$, the above equation assumes the form

$$\nabla_U Dh = \beta\xi(h)(U - \eta(U)\xi) + (\mu + \rho r - 2n(f_1 - f_3))\eta(U)\xi. \tag{4.23}$$

Contracting equation (4.23) over U, we find

$$\triangle h = \lambda_1,$$

where $\lambda_1 = 2n[\beta\xi(h) - (f_1 - f_3)] + \mu + \rho r$ and $\triangle$ denotes the Laplace operator.

A partial differential equation $\triangle\psi = \varphi$ on a Riemannian manifold M is termed as the Poisson's equation for smooth functions φ and ψ.
Thus, we can state our result as:

Corollary 4.6. *Let the metric of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with β-Kenmotsu structure, $\beta \neq 0$, be a gradient Ricci-Bourguignon soliton (g, Dh, μ, ρ). Then the potential function h of (g, Dh, μ, ρ) satisfies the Poisson's equation $\triangle h = 2n[\beta\xi(h) - (f_1 - f_3)] + \mu + \rho r$.*

Data Availability. Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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Sudhakar Kumar Chaubey
Section of Mathematics
IT Department
University of Technology and Applied Sciences-Shinas
P.O. Box 77, Postal Code 324, Oman
Email address: `sudhakar.chaubey@shct.edu.om`

Young Jin Suh
Department of Mathematics and RIRCM
Kyungpook National University
Daegu 41566, Korea
Email address: `yjsuh@knu.ac.kr`