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ON S-MULTIPLICATION RINGS

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Abstract. Let R be a commutative ring with identity and S be a multiplicatively closed subset of R. In this article we introduce a new class of ring, called S-multiplication rings which are S-versions of multiplication rings. An R-module M is said to be S-multiplication if for each submodule N of M, $sN \subseteq JM \subseteq N$ for some $s \in S$ and ideal J of R (see for instance [4, Definition 1]). An ideal I of R is called S-multiplication if I is an S-multiplication R-module. A commutative ring R is called an S-multiplication ring if each ideal of R is S-multiplication. We characterize some special rings such as multiplication rings, almost multiplication rings, arithmetical ring, and S-PIR. Moreover, we generalize some properties of multiplication rings to S-multiplication rings and we study the transfer of this notion to various context of commutative ring extensions such as trivial ring extensions and amalgamated algebras along an ideal.

1. Introduction

Throughout this paper all rings are commutative with 1 and all modules are unital. Recall that R is said to be a PIR (principal ideal ring) if every ideal of R is principal. Anderson and Dumitrescu introduced later in their study of S-Noetherian rings in [5], the concept of S-PIR. An ideal I of R is said to be S-finite (resp. S-principal) if there are $s \in S$ and a finitely generated (resp. a principal) ideal K of R such that $sI \subseteq K \subseteq I$. We say that R is said to be an S-Noetherian ring (resp. an S-PIR) if each ideal of R is S-finite (resp. S-principal). Clearly, every PIR is an S-PIR for any multiplicatively closed subset S. Recall that an ideal I of R is called a multiplication ideal if for each ideal J of R contained in I, there is an ideal I' of R such that $J = II'$. We say that R is said to be a multiplication ring if every ideal of R is multiplication and R is said to be an almost multiplication ring if R_P is a multiplication ring for all maximal ideal P of R (for instance see [3]). It is well known that every localization of a multiplication ring is still a multiplication ring. Consequently, it is easy to show that every multiplication ring is an

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almost multiplication ring but the converse is not true in general (for instance see [3, p. 765]). Recall from [12, 13], that R is said to be arithmetical if every finitely generated ideal of R is locally principal equivalently that every finitely generated ideal is multiplication (cf. [3, Theorem 3]). It is proved in [8, Lemma 2.6] that every almost multiplication ring is arithmetical but the converse is not true in general (see for instance [8, Example 2.7(2)]. The following diagram of implications summarizes the relation between the prementioned class of rings

$$\text{multiplication} \Rightarrow \text{almost multiplication} \Rightarrow \text{arithmetical.}$$

Recently, in [4], the authors introduced and studied the concept of S-multiplication modules. An R-module M is said to be S-multiplication if for each submodule N of M there are $s \in S$ and an ideal I' of R such that $sN \subseteq MI' \subseteq N$, in this case we can take $I' := (N : M)$. We say that an ideal I of R is an S-multiplication ideal if I is an S-multiplication R-module. We say that R is an S-multiplication ring if each ideal of R is S-multiplication. Our goal is to study S-multiplication rings. Moreover, to examine conditions under which an S-multiplication ring R is a multiplication ring or an S-PIR for some multiplicatively closed subset S of R, we study the transfer of the S-multiplication property in the trivial ring extension and amalgamated algebras along an ideal, respectively. In Section 3, we provides some original class of rings satisfying the S-multiplication property.

2. Main results

We begin this section by the definition of our S-version.

Definition 2.1. Let R be a ring and S be a multiplicatively closed subset of R. We say that an ideal I of R is an S-multiplication ideal if I is an S-multiplication R-module. We say that R is an S-multiplication ring if each ideal of R is S-multiplication. If P is a prime ideal of R, we say that R is a P-multiplication ring if R is an $(R - P)$-multiplication ring.

Example 2.2. Every multiplication ring R is an S-multiplication ring for any multiplicatively closed subset S of R. The converse is true if $S \subseteq U(R)$, where $U(R)$ is the group of all units of R.

The fact that multiplication rings are S-multiplication rings for any multiplicatively closed subset S is not reversible in general, see for instance Example 3.1.

Example 2.3. Let R be a ring and S be a multiplicatively closed subset of R. If R is an S-PIR, then R is an S-multiplication ring.

Proof. Let R be an S-PIR and let I be an ideal of R. Then there are $s \in S$ and a principal ideal K of R such that $sI \subseteq K \subseteq I$. Let $J \subseteq I$ an ideal of R. Then $sJ \subseteq sI \subseteq K$. Thus there is an ideal I' of R such that $sJ = I'K$ since K is multiplication (since K is principal by [3, Theorem 3]). Then $s^2J = sI'K \subseteq sI'I \subseteq I'K \subseteq J$. Then I is an S-multiplication ideal, as desired. □

Let S be a multiplicatively closed subset of a ring R. The saturation of S denoted by S^* is defined as follows: $S^* := \{x \in R : \exists x_0 \in R, xx_0 \in S\}$. S is said to be saturated if $S = S^*$. Notice that we always have $S \subseteq S^*$. We say that S satisfies the maximal multiple condition if there exists an $s_0 \in S$ such that $s \mid s_0$ for each $s \in S$. Notice that, if $S := \{1_R\}$, then the definitions of S-multiplication and multiplication rings coincide in R and if zero is an element of S, then R is obviously an S-multiplication ring. To avoid this trivial case, in the rest of this paper, it is assumed that all multiplicatively closed subset don't include the zero element.

Proposition 2.4. *Let R be a ring and S be a multiplicatively closed subset of R. Then*:

(1) *If $S \subseteq S'$ are multiplicatively closed subsets of R and R is an S-multiplication ring, then R is an S'-multiplication ring.*
(2) *R is an S-multiplication ring if and only if R is an S^*-multiplication ring.*

Proof. It is straightforward. □

Next, we study the transfer of the S-multiplication property in the ring of fractions.

Proposition 2.5. *Let R be a ring and S be a multiplicatively closed subset of R. If R is an S-multiplication ring, then $S^{-1}R$ is a multiplication ring. The converse is true if S satisfies the maximal multiple condition.*

To prove Proposition 2.5, we establish the following lemma, which is a direct consequence of [4, Proposition 3].

Lemma 2.6. *Let R be a ring and S_1 and S_2 be two multiplicatively closed subsets of R. Let $\overline{S_1} := \{\frac{s}{1} \in S_2^{-1}R : s \in S_1\}$ be a multiplicatively closed subset of $S_2^{-1}R$. Assume that R is an S_1-multiplication ring. Then the following statements hold*:

(1) *$S_2^{-1}R$ is an $\overline{S_1}$-multiplication ring.*
(2) *If $S_1 \subseteq (S_2)^*$, then $S_2^{-1}R$ is a multiplication ring.*

Proof of Proposition 2.5. Assume that R is an S-multiplication ring. Put $\overline{S} := \{\frac{s}{1} \in S^{-1}R : s \in S\}$. Then $S^{-1}R$ is an $\overline{S}$-multiplication ring by Lemma 2.6(1). Then $S^{-1}R$ is a multiplication ring by Lemma 2.6(2). Conversely, assume that S satisfies the maximal multiple condition. Let $J \subseteq I$ be ideals of R. Then $S^{-1}J \subseteq S^{-1}I$ are ideals of $S^{-1}R$, therefore there is an ideal I' of R such that $S^{-1}J = S^{-1}(II')$. Let $x \in J$. Then there is $s_1 \in S$ such that $s_1x \in I'I$ and so $s_1J \subseteq I'I$. For the same reasoning, we prove that there is $s_2 \in S$ such that $s_2I'I \subseteq J$. Put $s = s_2s_1 \in S$. Then $sJ \subseteq s_2I'I \subseteq J$ and hence I is an S-multiplication ideal, as desired. □

It is clear that if S is a multiplicatively closed subset of a ring R, then $\overline{S} := S + I$ is a multiplicatively closed subset of R/I for every ideal I of R. The next result investigates the S-multiplication property in quotient rings.

Proposition 2.7. *Let R be a ring, I be an ideal of R and let S be a multiplicatively closed subset of R. If R is an S-multiplication ring, then R/I is an $\overline{S}$-multiplication ring, where $\overline{S} = S + I$. The converse is true if there is $s_0 \in S$ such that $s_0 I = 0$.*

Proof. Assume that R is an S-multiplication ring. Let $f : R \to R/I$ defined by $f(r) = r + I$ for all $r \in R$. It is clear that f is a surjective ring homomorphism. Let $K \subseteq J$ be ideals of R/I. Then $f^{-1}(K) \subseteq f^{-1}(J)$ are ideals of R, so there exist $s \in S$ and an ideal I' of R such that $sf^{-1}(K) \subseteq f^{-1}(J)I' \subseteq f^{-1}(K)$. Then $f(s)K \subseteq Jf(I') \subseteq K$. Then J is an $\overline{S}$-multiplication ideal and hence R/I is an $\overline{S}$-multiplication ring. Conversely, assume that there is $s_0 \in S$ such that $s_0 I = 0$. Let $K \subseteq J$ be ideals of R. Then $f(K) \subseteq f(J)$ are ideals of R/I, so there are $s \in S$ and an ideal L of R/I such that $f(s)f(K) \subseteq f(J)L \subseteq f(K)$. Therefore, $f(sK) \subseteq f(I'J) \subseteq f(K)$ with I' an ideal of R containing I. Then $(sK + I) \subseteq (I'J + I) \subseteq (K + I)$, that is, $s_0 sK \subseteq s_0 I'J \subseteq K$. Thus J is an S-multiplication ideal and hence R is an S-multiplication ring. □

Next, we study the transfer of the S-multiplication property in the direct product. It is clear that if S_i is a multiplicatively closed subsets of a ring R_i for all $i = 1, \ldots, n$, then $S = \prod_{i=1}^{n} S_i$ is a multiplicatively closed subset of $R = \prod_{i=1}^{n} R_i$.

The following result, which is a direct consequence of [4, Theorem 5], is important enough to be designated a proposition, therefore we will remove the proof.

Proposition 2.8. *Let $R_1, \ldots, R_n$ be rings and $S_1, \ldots, S_n$ be multiplicatively closed subsets of $R_1, \ldots, R_n$, respectively. Put $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} S_i$ a multiplicatively closed subset of R. The following statements are equivalent*:

(1) *R is an S-multiplication ring.*
(2) *R_i is an S_i-multiplication ring for each $i = 1, 2, \ldots, n$.*

Next, we examine conditions under which an S-multiplication ring R is a multiplication ring for some multiplicatively closed subset S of R.

Theorem 2.9. *For a ring R, the following statements are equivalent*:

(1) *R is a multiplication ring.*
(2) *R is a P-multiplication ring for each prime ideal P of R.*
(3) *R is an M-multiplication ring for each maximal ideal M of R.*

Proof. (1) $\Rightarrow$ (2) It follows from Example 2.2.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Assume that R is an M-multiplication ring for each maximal ideal M of R. Let I be an ideal of R. Then I is an M-multiplication ideal for each

maximal ideal M of R. Thus I is a multiplication ideal by [4, Theorem 1] and hence R is a multiplication ring. $\square$

Recall from [3, p. 761], that a quasi-local multiplication ring is a PIR. It is also clear that every Noetherian ring R is an S-Noetherian ring for any multiplicatively closed subset S of R.

Proposition 2.10. *Let R be a Noetherian ring. Then R is a P-multiplication ring if and only if R is an $(R-P)$-PIR for each prime ideal P of R.*

Proof. By Example 2.3, we only need prove that R is an $(R-P)$-PIR if R is a P-multiplication ring for each prime ideal P of R. Assume that R is a P-multiplication ring for each prime ideal P of R. Then R is a multiplication ring by Theorem 2.9. Therefore, R_P is a quasi-local multiplication ring. Hence R is an $(R-P)$-PIR by [5, Proposition 2(g)]. $\square$

In [5], the authors proved that a ring R is a ZPI-ring if and only if R is an M-PIR for each maximal ideal M of R and that a domain D is a Dedekind domain if and only if D is an M-PID for each maximal ideal M of D. It is well known that every ZPI-ring is a multiplication ring. The following is a consequence of Proposition 2.10.

Corollary 2.11. *The following statements hold*:

(1) *Every Noetherian multiplication ring is a ZPI ring.*
(2) *Every Noetherian multiplication domain is a Dedekind domain.*

Proof. (1) Let R be a Noetherian multiplication ring. Then by Theorem 2.9, R is a Noetherian M-multiplication ring for each maximal ideal M of R. Then by Proposition 2.10, R is an M-PIR for each maximal ideal M of R. Thus R is a ZPI-ring by [5, Corollary 13].

(2) Let R be a Noetherian multiplication domain. Then by Theorem 2.9 and Proposition 2.10, R is an M-PID for each maximal ideal M of R and hence R is a Dedekind domain by [5, Corollary 13]. $\square$

Recall from [4, Definition 2], that a module M over a ring R is called S-cyclic, where S is a multiplicatively closed subset of R, if there exist $s \in S$ and $m \in M$ with $sM \subseteq Rm \subseteq M$. If $S := R - P$ with P a prime ideal of R, then M is called a P-cyclic R-module. They also proved by [4, Proposition 6] that if an R-module M is a P-multiplication R-module for a prime ideal P of R with $M_P \neq 0_P$, then M is P-cyclic.

Proposition 2.12. *Let R be a ring and P be a prime ideal of R such that $I_P \neq 0_P$ for each ideal I of R. Then R is a P-multiplication ring if and only if R is an $(R-P)$-PIR.*

Proof. By Example 2.3, we only need prove that R is an $(R-P)$-PIR if R is a P-multiplication ring for a prime ideal P of R such that $I_P \neq 0_P$ for each ideal I of R. Let I be an ideal of R. Then I is a P-multiplication ideal and

$I_P \neq 0_P$. Thus by [4, Proposition 6], I is P-cyclic, so there exist $x_0 \notin P$ and $y \in I$ such that $x_0 I \subseteq Ry \subseteq I$. Therefore I is an $(R-P)$-principal ideal and hence R is an $(R-P)$-PIR. $\square$

Let R be a non-Noetherian von Neumann regular ring. Then by [3, Theorem 6], $R[X]$ is an almost multiplication ring (resp. an arithmetical ring). On the other hand, assume that $R[X]$ is an S-multiplication ring for $S := \{1_R\}$, then $R[X]$ is a multiplication ring by Example 2.2(1), a contradiction by [3, p. 765]. Hence, almost multiplication rings and arithmetical rings are not S-multiplication rings for an arbitrary multiplicatively closed subset S. In what follows we characterize almost multiplication rings and arithmetical rings which are S-multiplication rings for an arbitrary multiplicatively closed subset S of R.

Theorem 2.13. *Let R be an S-Noetherian ring* (*not necessary Noetherian for instance see* [7, Remark 3.4(2)]). *If R is an almost multiplication* (*resp. an arithmetical*) *ring, then R is an S-multiplication ring.*

Proof. Assume that R is an S-Noetherian almost multiplication ring. Then by [3, Theorem 1], every ideal of R is locally principal. Let I be an ideal of R. Then there are $s \in S$ and a finitely generated ideal K of R such that $sI \subseteq K \subseteq I$. Then K is a multiplication ideal by [3, Theorem 3]. Let $J \subseteq I$ an ideal of R, then $sJ \subseteq sI \subseteq K$. Then there is an ideal I' of R such that $sJ = I'K$. Then $s^2J = sI'K \subseteq sI'I \subseteq I'K \subseteq J$ and so $s^2J \subseteq (sI')I \subseteq J$. Then I is an S-multiplication ideal and hence R is an S-multiplication ring. Respectively, assume that R is an S-Noetherian arithmetical ring. Let I be an ideal of R, then there exist $s \in S$ and a finitely generated ideal K of R such that $sI \subseteq K \subseteq I$. Let $J \subseteq I$ an ideal of R, then $sJ \subseteq sI \subseteq K$. Then there exists an ideal I' of R such that $sJ = I'K$ by [13, Theorem 2]. Therefore $s^2J = sI'K \subseteq sI'I \subseteq I'K \subseteq J$. Then I is an S-multiplication ideal and hence R is an S-multiplication ring. $\square$

Recall from [5, p. 4412], that each domain D is D^*-Noetherian, where $D^* := D - \{0\}$. Then the following result is a direct consequence of Theorem 2.13.

Corollary 2.14. *Every almost multiplication* (*resp. arithmetical*) *domain D is a D^*-multiplication domain.*

It is proved in [8, Lemma 2.6], that every Noetherian almost multiplication (resp. arithmetical) ring is a multiplication ring. In what follows we give a new proof of this result using the S-concept.

Corollary 2.15. *Every Noetherian almost multiplication* (*resp. arithmetical*) *ring R is a multiplication ring.*

Proof. Assume that R is a Noetherian almost multiplication (resp. arithmetical) ring. Then by [5, Proposition 12], R is an M-Noetherian almost multiplication (resp. arithmetical) ring for all maximal ideal M of R. Then R is an

M-multiplication ring for each maximal ideal M of R by Theorem 2.13. Thus by Theorem 2.9, R is a multiplication ring. $\square$

Let I be an ideal of a ring R, denotes by $sat_M(I) := IR_M \cap R$ the $(R-M)$-saturation of I.

Theorem 2.16. *Let R be an almost multiplication ring. Assume that for every finitely generated ideal J of R, $sat_M(J) \neq 0_M$ for all maximal ideal M of R. Then R is a P-multiplication ring for all prime ideal P of R if and only if R is Noetherian.*

Proof. For every finitely generated ideal J of R, put $K := sat_M(J)$ for all maximal ideal M of R. Assume that R is a P-multiplication ring for all prime ideal P of R. Then by Theorem 2.9, R is a multiplication ring. Then K is a multiplication ideal. Since $K_M \neq 0_M$ for all maximal ideal M of R, then by [4, Corollary 3], K is M-cyclic for all maximal ideal M of R. Then there exist $x_0 \notin M$ and $y \in K$ such that $x_0K \subseteq Ry \subseteq K$. Then K is M-finite for all maximal ideal M of R. Then by [5, Proposition 2(b)], $K = (J : t)$ for some $t \notin M$. On the other hand, since R is an almost multiplication ring, then by [3, Theorem 1], R_M is a principal ideal ring, hence Noetherian for all maximal ideal M of R. Then by [5, Proposition 2(f)], R is M-Noetherian for all maximal ideal M of R and hence by [5, Proposition 12], R is Noetherian. The converse follows from Theorem 2.13. $\square$

Definition 2.17. Let R be a ring and S be a multiplicatively closed subset of R. We say that R is an S-arithmetical ring if every finitely generated ideal of R is an S-multiplication ideal. If P is a prime ideal of R, we say that R is a P-arithmetical ring if R is an $(R-P)$-arithmetical ring.

Example 2.18. Every arithmetical ring R is S-arithmetical for any multiplicatively closed subset S of R. The converse is true if $S \subseteq U(R)$.

The fact that if R is an arithmetical ring, then R is an S-arithmetical ring for any multiplicatively closed subset S of R is of course, not reversible in general, for instance see Examples 3.3 and 3.4.

Proposition 2.19. *Let R be a ring and S be a multiplicatively closed subset of R. If R is an S-multiplication ring, then R is an S-arithmetical ring.*

Proof. The proof is straightforward. $\square$

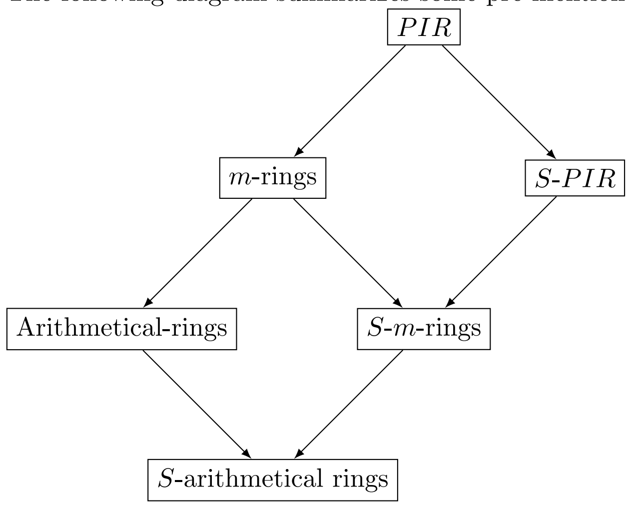
The converse of Proposition 2.19 is not true in general, for instance see Section 3. Next, we examine conditions under which an S-arithmetical ring R is an S-multiplication for some multiplicatively closed subset of R.

Theorem 2.20. *Let R be a ring and S be a multiplicatively closed subset of R. Suppose that R is S-Noetherian not necessary Noetherian. Then R is an S-arithmetical ring if and only if R is an S-multiplication ring.*

Proof. The necessary condition is given by Example 2.18. Assume that R is an S-arithmetical ring. Let I be an ideal of R. Then there exist $s \in S$ and a finitely generated ideal K of R such that $sI \subseteq K \subseteq I$. Let $J \subseteq I$ be an ideal of R. Then $sJ \subseteq sI \subseteq K$, so there exist $s' \in S$ and an ideal I' of R such that if $ss'J \subseteq KI' \subseteq J$, then $s's^2J \subseteq sKI' \subseteq sI'I \subseteq KI' \subseteq J$.

Put $t = s's^2 \in S$, then $tJ \subseteq sI'I \subseteq J$. Then I is an S-multiplication ideal and hence R is an S-multiplication ring. $\square$

The following diagram summarizes some pre-mentioned implications.



- m-rings: multiplication rings.
- PIR: principal ideal rings.
- S-PIR: S-principal ideal rings.
- S-m-rings: S-multiplication rings.
- Black arrows are direct implications.

Let A be a ring and E be an A-module, the idealization $A \propto E$ (also called the trivial extension), introduced by Nagata in 1956 (cf. [17]) is defined as the A-module $A \oplus E$ with multiplication defined by $(a, e)(b, f) := (ab, af + be)$. It is clear that if S is a multiplicatively closed subset of A, then $S \propto F$ is a multiplicatively closed subset of $A \propto E$ for each submodule F of E. It is said in [6, p. 19] that for any submodule F of E, $(S \propto F)^* = (S \propto 0)^*$ and that $(A \propto E)_{S \propto F} \cong (A \propto E)_{S \propto 0}$. It is easy to show that $S \propto 0$ satisfies the maximal multiple condition if S satisfies the maximal multiple condition. Recall from [6, Theorem 3.2(2)] that every prime (resp. maximal) ideal of $A \propto E$ has the form $P \propto E$, where P is a prime (resp. maximal) ideal of A. In what follows, we study the transfer of the S-multiplication property from the trivial rings extension to their components.

Theorem 2.21. *Under the above notations.*

(A) *If* $A\propto E$ *is an* $(S\propto E)$*-multiplication ring, then* A *is an* S*-multiplication ring and* E *is an* S*-multiplication module.*

(B) *Assume that* $ann(P)+(PE:E)=A$ *for each prime ideal* P *of* A*. Then the following statements are equivalent*:

1. $A\propto E$ *is a multiplication ring.*
2. $A\propto E$ *is a* $P\propto E$*-multiplication ring for each prime ideal* P *of* A*.*
3. A *is a* P*-multiplication ring and* E *is a* P*-multiplication module for each prime ideal* P *of* A*.*
4. A *is a multiplication ring and* E *is a multiplication module.*

Proof. (A) Assume that $A\propto E$ is an $(S\propto E)$-multiplication ring. Then $0\propto E$ is an $(S\propto E)$-multiplication ideal of $A\propto E$. Therefore E is an S-multiplication module by [4, Theorem 3]. On the other hand, let $J\subseteq I$ be ideals of A, then $J\propto E\subseteq I\propto E$ are ideals of $A\propto E$. Then there exists $(s,e)\in S\propto E$ such that $(s,e)J\propto E\subseteq (J\propto E:I\propto E)I\propto E$. Or by [1, Lemma 1], $(J\propto E:I\propto E)=(J:I)\propto E$. Then $sJ\subseteq (J:I)I$. Thus I is an S-multiplication ideal and hence A is an S-multiplication ring.

(B) (1) $\Rightarrow$ (2) It follows from example 2.2.

(2) $\Rightarrow$ (3) It follows from (A).

(3) $\Rightarrow$ (4) It follows from Theorem 2.9 and [4, Theorem 1].

(4) $\Rightarrow$ (1) It follows from [2, Theorem 11(1)]. □

The converse of Theorem 2.21(A) is not true in general as shown by the following example.

Example 2.22. Let (A,P) be a local multiplication ring (P proper) and $E=A/P$ be an A-module such that $EP=0$. Then clearly A is a $\{1_R\}$-multiplication ring and by [8, Proposition 2.1], E is a $\{1_A\}$-multiplication module. Suppose that $A\propto E$ is a $\{1_A\}\propto E$-multiplication ring. Then by Proposition 2.4(2), $A\propto E$ is a $\{1_A\}\propto 0$-multiplication ring. Therefore $A\propto E$ is multiplication, a contradiction by [15, Example 2.6].

Let $f:A\to B$ be a ring homomorphism and let J be an ideal of B. The amalgamated algebra of A with B along J with respect to f is the subring of $A\times B$ given by: $A\bowtie^f J:=\{(a,f(a)+j):a\in A, j\in J\}$. This construction is introduced and studied by D'Anna, Finocchiaro and Fontana in [9,10]. Notice that if $B:=A$, $f:=id_A$ and $J:=I$ an ideal of A, then $A\bowtie^f J=A\bowtie I$. It is clear that if S is a multiplicatively closed subset of A, then $S':=\{(s,f(s)):s\in S\}$ is a multiplicatively closed subset of $A\bowtie^f J$ and $f(S)$ is a multiplicatively closed subset of B. We examine conditions under which $A\bowtie^f J$ is an S'-multiplication ring.

Theorem 2.23. *Under the above notation. Assume that* J *is a nonzero proper ideal of* B*. If* $A\bowtie^f J$ *is an* S'*-multiplication ring, then* A *is an* S*-multiplication ring and* $f(A)+J$ *is an* $f(S)$*-multiplication ring. The converse is true if* J *is generated by an idempotent.*

Before proving Theorem 2.23, we establish the following lemma.

Lemma 2.24. *Let $f : A \to B$ be a ring homomorphism and let J be a nonzero proper ideal of B. Assume that J is finitely generated by an idempotent $J = \langle e \rangle$, $e^2 = e$). Then*

$$\alpha : A \bowtie^f J \to A \times \frac{f(A)+J}{ann(J)}$$
$$(a, f(a)+j) \mapsto (a, \overline{f(a)+j})$$

is a ring isomorphism.

Proof. It is easy to show that α is well defined and is a ring homomorphism. Let $(a, \overline{f(b)+k}) \in A \times \frac{f(A)+J}{ann(J)}$, we have $\alpha(a, f(a) + e(f(b) + k - f(a))) = (a, \overline{f(b)+k})$. Indeed, $\overline{f(a)+e(f(b)+k-f(a))} = \overline{f(a)} + \overline{e}(\overline{f(b)+k}) - \overline{e}\overline{f(a)}$. Since, $(e-1)J = eBe - Be = 0$, then $\overline{e} = \overline{1}$. Therefore, $\overline{f(a)} + \overline{e}\overline{f(b)+k} - \overline{e}\overline{f(a)} = \overline{f(b)+k}$, and hence α is surjective. Let $(a, f(a)+j) \in Ker(\alpha)$. Then $(a, \overline{f(a)+j}) = 0$ so $a = 0 = \overline{f(a)+j}$, therefore $j \in J \cap ann(J) = (0)$ and hence α is injective. Thus α is a ring isomorphism. □

Proof of Theorem 2.23. (1) Let $P_A : A \bowtie^f J \to A$ be the natural projection of $A \bowtie^f J \subseteq A \times B$ into A and $p : A \bowtie^f J \to f(A) + J$ be the surjective ring homomorphism defined by $p((a, f(a)+j)) = f(a) + j$ for all $a \in A$ and $j \in J$. Assume that $A \bowtie^f J$ is an S'-multiplication ring. Then $P_A(A \bowtie^f J) = A$ is an S-multiplication ring and $p(A \bowtie^f J) = f(A) + J$ is an $f(S)$-multiplication ring. Conversely, assume that J is generated by an idempotent. By Proposition 2.7, $(f(A) + J)/ann(J)$ is an $\overline{f(S)}$-multiplication ring, where $\overline{f(S)} := f(S) + ann(J)$. Therefore $A \times (f(A) + J)/ann(J)$ is an $S \times \overline{f(S)}$-multiplication ring by Proposition 2.8 and hence $A \bowtie^f J$ is an S'-multiplication ring by Proposition 2.4 and Lemma 2.24. □

For a commutative ring A and an ideal I of A, the amalgamated duplication of A along I is the subring of $A \times A$ given by

$$A \bowtie I := \{(a, a+i) : a \in A, i \in I\}.$$

This ring was introduced and studied by D'Anna and Fontana in [11]. Notice that if $B := A$, $f := id_A$ and $J := I$ is an ideal of A, then $A \bowtie^f J = A \bowtie I$. The following result is a direct consequence of Theorem 2.23. Notice that $S' := \{(s, s) : s \in S\}$ is a multiplicatively closed subset of $A \bowtie I$ for each multiplicatively closed subset S of A.

Corollary 2.25. *Let A be a ring, I a nonzero proper ideal of A and S a multiplicatively closed subset of A. If $A \bowtie I$ is an S'-multiplication ring, then A is an S-multiplication ring. The converse is true if I is generated by an idempotent.*

Example 2.26. We keep the notation of Corollary 2.25. Let $A := \mathbb{Z}_6$, $S := \{\overline{1}, \overline{3}\}$ a multiplicatively closed subset of A. It is well known that if A is a multiplication ring, then A is an S-multiplication ring by Example 2.2. Let

$I := (\overline{4})$ be a proper ideal of A generated by an idempotent of A. Then by Corollary 2.25, $A \bowtie I$ is an S'-multiplication ring, where $S' := \{(s, s) : s \in S\}$.

Proposition 2.27. *We keep the notations of Theorem 2.23. Assume there exists $s \in S$ such that $f(s)J = 0$ for example $S \cap Ker(f) \neq \emptyset$. Then $A \bowtie^f J$ is an S'-multiplication ring if A is a multiplication ring.*

Proof. Assume that A is a multiplication ring. Then by [9, Proposition 5.1(3)], $A \bowtie^f J/0 \times J \cong A$ is a multiplication ring. Then by Example 2.2, $A \bowtie^f J/0 \times J$ is an $(S' + 0 \times J)$-multiplication ring. Let $s \in S$ such that $f(s)J = 0$. Then $(s, f(s))0 \times J = 0$ and by Proposition 2.7, $A \bowtie^f J$ is an S'-multiplication ring. □

3. More examples

In this section, the main objective is to provide some original examples to illustrate some of the results previously stated. We begin by providing an example of an S-multiplication ring that is not multiplication.

Example 3.1. Let (A, P) be a local multiplication ring, $E \neq 0$ be an A-module such that $EP = 0$ (for instance $E = A/P$) and S be a multiplicatively closed subset of A such that $S \cap P \neq \emptyset$. Then:

(1) $A \propto E$ is an $S \propto E$-multiplication ring.
(2) $A \propto E$ is not a multiplication ring.

Proof. (1) By [6, Theorem 3.1], we have $(A \propto E/0 \propto E) \cong A$ is a multiplication ring and hence an $(S \propto E + 0 \propto E)$-multiplication ring. Let $s \in S \cap P$. Then $(s, 0)0 \propto E = 0$. Therefore $A \propto E$ is an $S \propto E$-multiplication ring by Proposition 2.7.

(2) It follows from [15, Example 2.6]. □

Next, we give an example of an S-arithmetical ring that is not arithmetical.

Example 3.2. Let A be an arithmetical domain, let B be a domain, let J be a nonzero proper ideal of B, let $f : A \to B$ be a non injective ring homomorphism, let S be a multiplicatively closed subset of A such that $\emptyset \neq S \cap Ker(f)$, and let $R := A \bowtie^f J$ and $S' := \{(s, f(s)) : s \in S\}$. Then:

(1) R is an S'-arithmetical ring.
(2) R is not an arithmetical ring.

Proof. (1) Notice that it is easy to show that Proposition 2.7 is true for the S-arithmetical property. By [9, Proposition 5.1(3)], $R/0 \times J \cong A$ is an arithmetical ring. Then by Example 2.18, $R/0 \times J$ is an $S' + 0 \times J$-arithmetical ring. Let $s \in S \cap Ker(f)$. Then $(s, f(s))0 \times J = 0$. So by Proposition 2.7 for the S-arithmetical property, R is an S'-arithmetical ring.

(2) It follows from [14, Theorem 2.9]. □

Next, we give some examples of S-arithmetical rings that are not S-multiplication.

Example 3.3. Let A_0 be a non Noetherian von Neumann regular ring, let $A := A_0[X]$, let S be a multiplicatively closed subset of A, let $R := A \times A$ and let $S' := \{1_A\} \times S$. Then:

(1) R is an S'-arithmetical ring.
(2) R is not an S'-multiplication ring.

Proof. (1) By [3, Theorem 6], A is arithmetical. Then by Example 2.18 and Proposition 2.7 for the S-arithmetical property, R is an S'-arithmetical ring.

(2) Assume that R is an S'-multiplication ring. Then A is a multiplication ring by Proposition 2.8, a contradiction by [3, p. 765]. □

Example 3.4. Let $A := \mathbb{Z}$, the ring of integers, let $E := \mathbb{Q}$ the field of rational numbers, let $R := A \propto E$ and $S := \{1_A\} \propto E$ be a multiplicatively closed subset of R. Then:

(1) R is an S-arithmetical ring.
(2) R is not an S-multiplication ring.

Proof. (1) By [16, Theorem 9], R is an arithmetical ring and hence an S-arithmetical ring by Example 2.18.

(2) Assume that R is an S-multiplication ring. Then by Proposition 2.4(2), R is a $\{1_A\} \propto 0$-multiplication ring. Then $0 \propto \mathbb{Q}$ is a $\{1_A\} \propto 0$-multiplication ideal. Then Q is a multiplication $\mathbb{Z}$-module by [4, Theorem 3], a contradiction. □

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